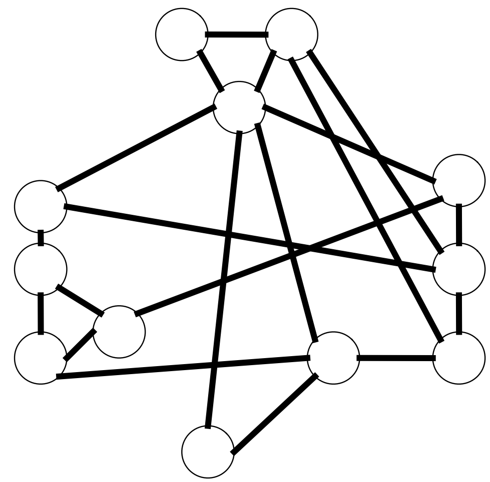
15-251

Great Theoretical Ideas in Computer Science

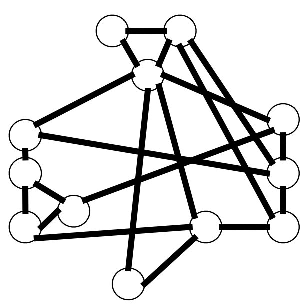
A Graph Named “Gadget”



Complexity Theory:

Efficient Reductions Between Computational Problems

Lecture 27 (November 25, 2008)



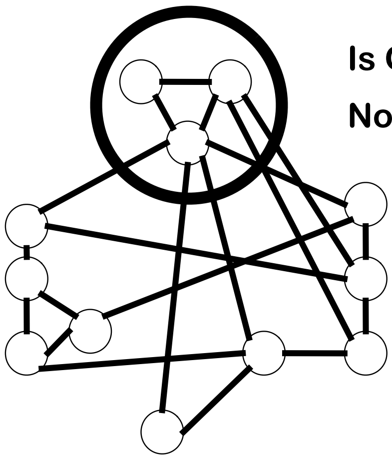
K-Coloring

We define a k-coloring of a graph:

- Each node gets colored with one color
- At most k different colors are used
- If two nodes have an edge between them they must have different colors

A graph is called k-colorable if and only if it has a k-coloring

A 2-CRAYOLA Question!



Is Gadget 2-colorable?

No, it contains a triangle

Theorem: G contains an odd cycle if and only if G is not 2-colorable

Alternate coloring algorithm:

To 2-color a connected graph G, pick an arbitrary node v, and color it white

Color all v's neighbors black

Color all their uncolored neighbors white, and so on

If the algorithm terminates without a color conflict, output the 2-coloring

Else, output an odd cycle

A 2-CRAYOLA Question!

Given a graph G, how can we decide if it is 2-colorable?

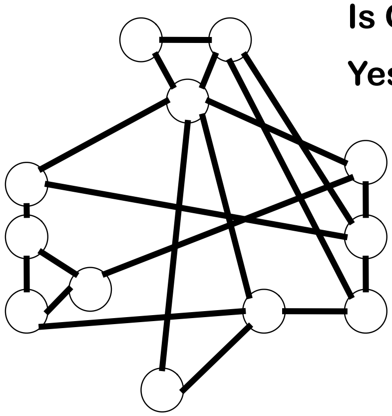
Answer: Enumerate all 2^n possible colorings to look for a valid 2-color

How can we efficiently decide if G is 2-colorable?

A 2-CRAYOLA Question!

Theorem: G contains an odd cycle if and only if G is not 2-colorable

A 3-CRAYOLA Question!



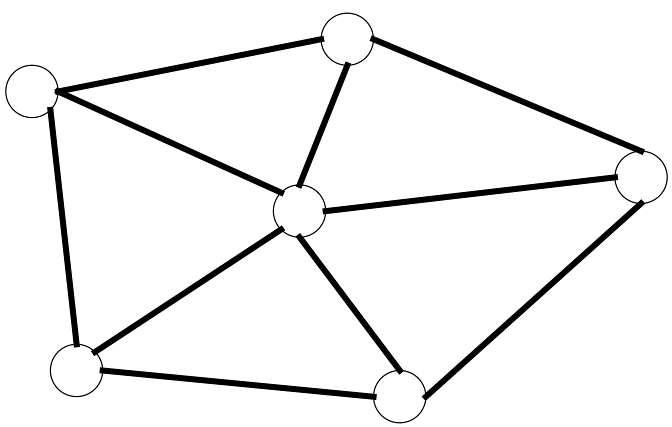
Is Gadget 3-colorable?

Yes!

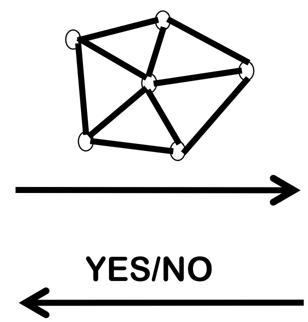
3-Coloring Is Decidable by Brute Force

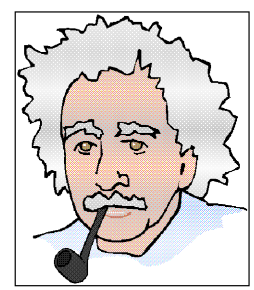
Try out all 3^n colorings until you determine if G has a 3-coloring

A 3-CRAYOLA Question!



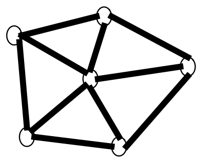
A 3-CRAYOLA Oracle





3-Colorability Oracle

Better 3-CRAYOLA Oracle



NO, or

YES here is how: gives 3-coloring of the nodes

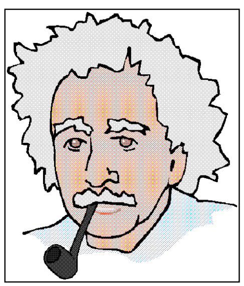
3-Colorability Search Oracle

Christmas Present

BUT I WANTED a SEARCH oracle for Christmas

I am really bummed out

GIVEN: 3-Colorability Decision Oracle

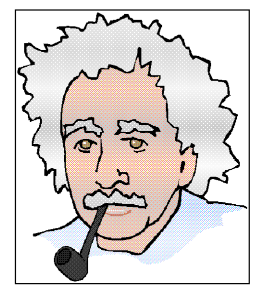


3-Colorability Search Oracle

3-Colorability Decision Oracle

Christmas Present

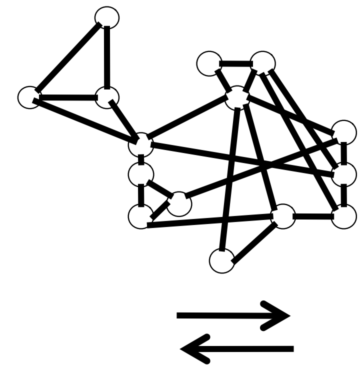
How do I turn a mere decision oracle into a search oracle?

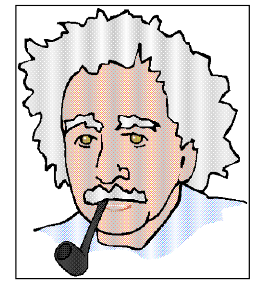


GIVEN: 3-Colorability Decision Oracle

What if I gave the oracle partial colorings of G? For each partial coloring of G, I could pick an uncolored node and try different colors on it until the oracle says “YES”

Beanie’s Fix





GIVEN:
3-Colorability
Decision Oracle

Beanie’s Flawed Idea

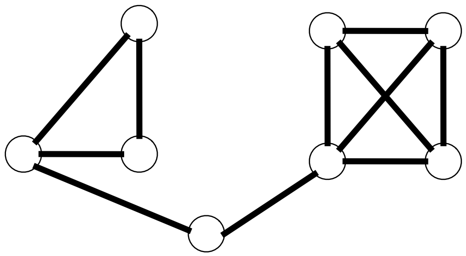
Rats, the oracle does not take partial colorings….

Let’s now look at two other problems:

1. K-Clique
2. K-Independent Set

K-Cliques

A K-clique is a set of K nodes with all K(K-1)/2 possible edges between them

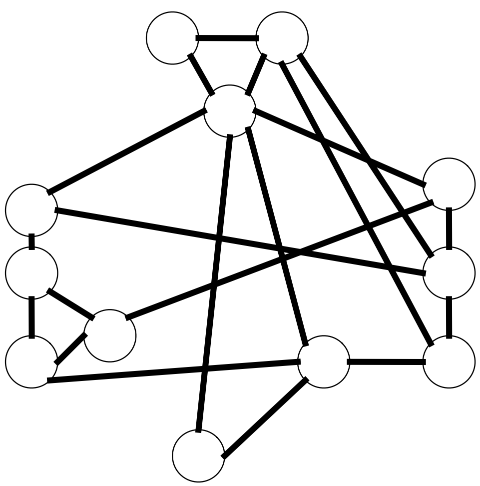


This graph contains a 4-clique

Given: (G, k)
Question: Does G contain a k-clique?

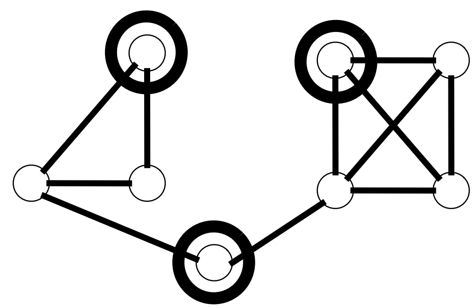
BRUTE FORCE: Try out all n choose k possible locations for the k clique

A Graph Named “Gadget”



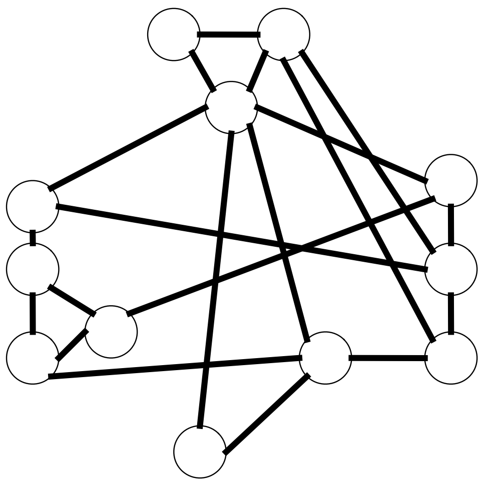
Independent Set

An independent set is a set of nodes with no edges between them



This graph contains an independent set of size 3

A Graph Named “Gadget”



Clique / Independent Set

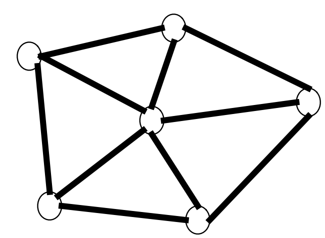
Two problems that are cosmetically different, but substantially the same

Given: (G, k)
Question: Does G contain an independent set of size k?

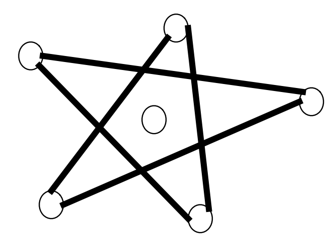
BRUTE FORCE: Try out all n choose k possible locations for the k independent set

Complement of G

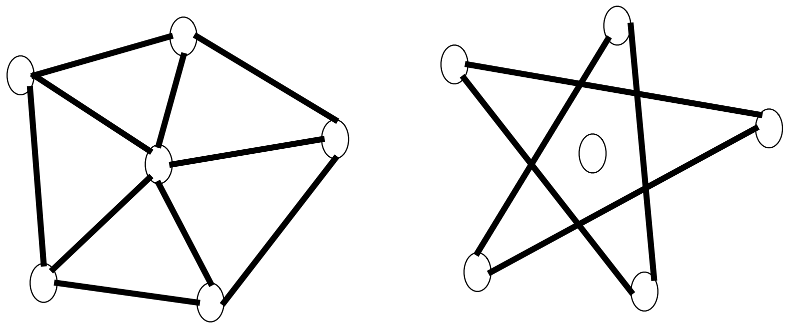
Given a graph G, let G*, the complement of G, be the graph obtained by the rule that two nodes in G* are connected if and only if the corresponding nodes of G are not connected



G



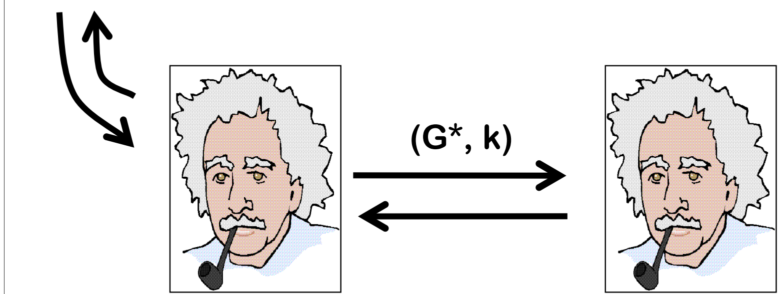
G*



G has a k-clique ⇔ G* has an independent set of size k

Let G be an n-node graph

(G,k)



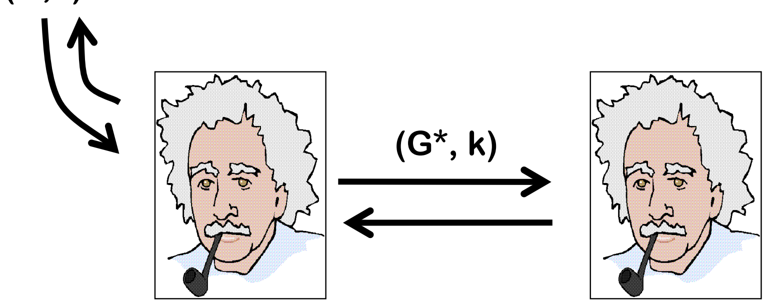
(G*, k)

BUILD: Clique Oracle

GIVEN: Independent Set Oracle

Let G be an n-node graph

(G,k)



(G*, k)

BUILD: Independent Set Oracle

GIVEN: Clique Oracle

Clique / Independent Set

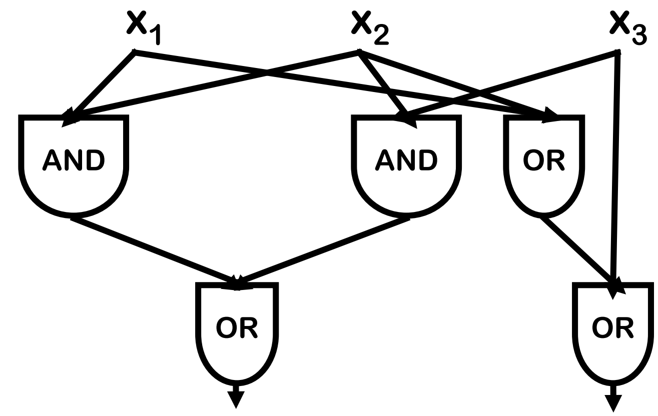
Two problems that are cosmetically different, but substantially the same

Thus, we can quickly reduce a clique problem to an independent set problem and vice versa

There is a fast method for one if and only if there is a fast method for the other

Combinatorial Circuits

AND, OR, NOT, 0, 1 gates wired together with no feedback allowed



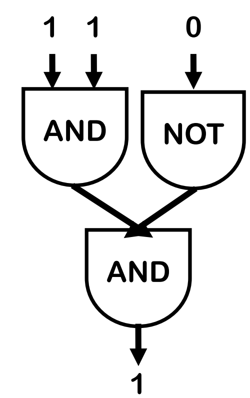
Let's now look at two other problems:

1. Circuit Satisfiability
2. Graph 3-Colorability

Circuit-Satisfiability

Given a circuit with n-inputs and one output, is there a way to assign 0-1 values to the input wires so that the output value is 1 (true)?

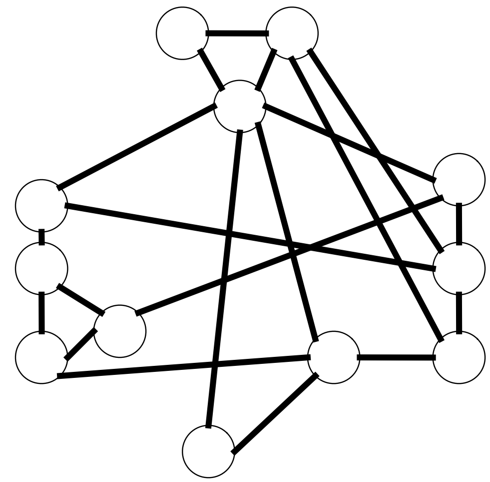
Yes, this circuit is satisfiable: 110



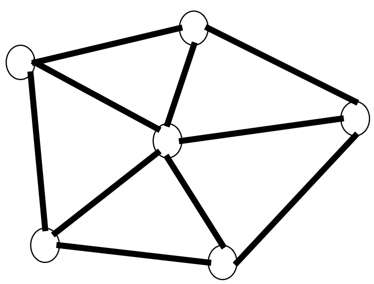
Circuit-Satisfiability

Given: A circuit with n-inputs and one output, is there a way to assign 0-1 values to the input wires so that the output value is 1 (true)?

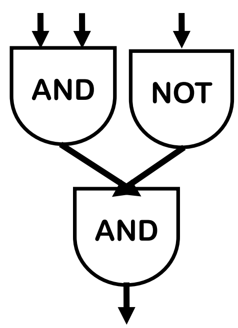
BRUTE FORCE: Try out all 2^n assignments

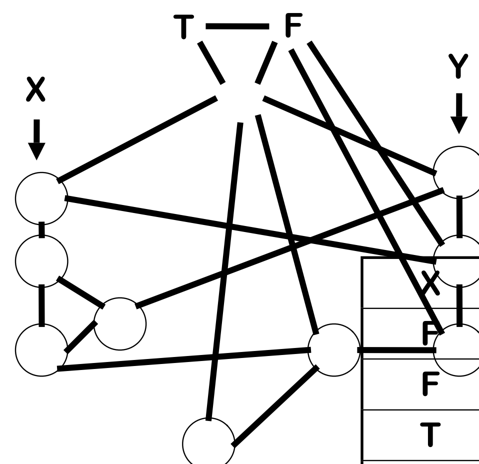


3-Colorability

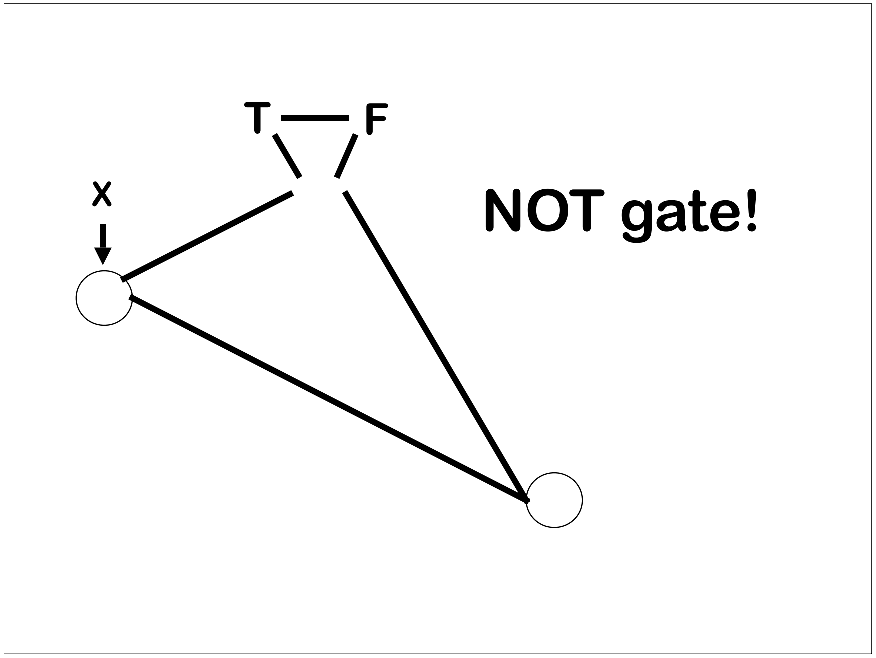


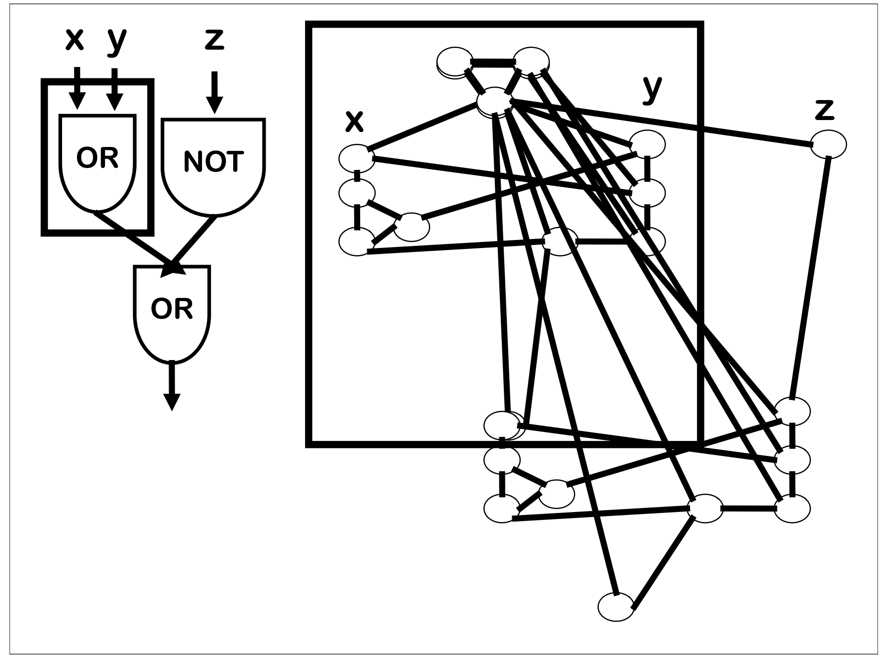
Circuit Satisfiability

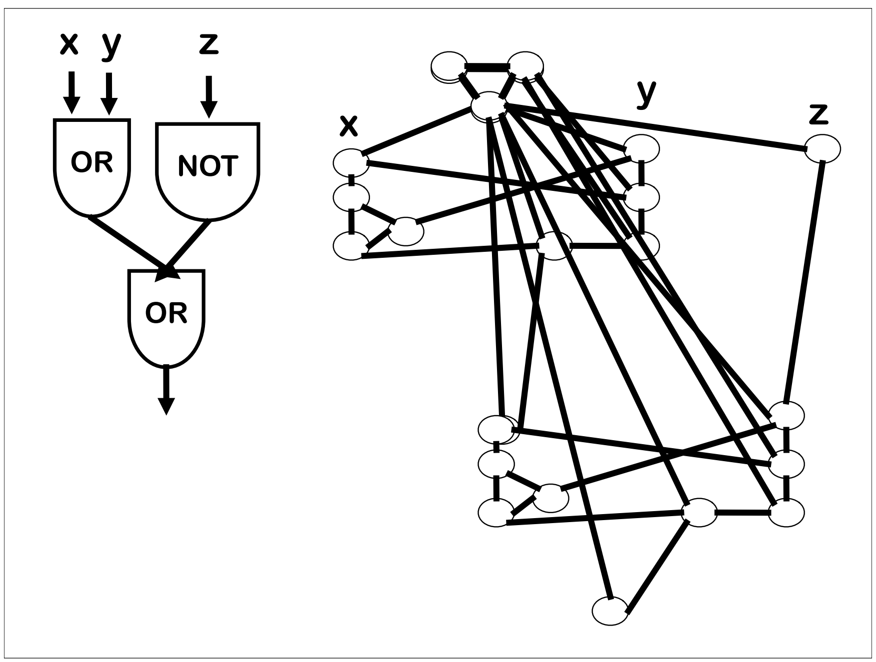


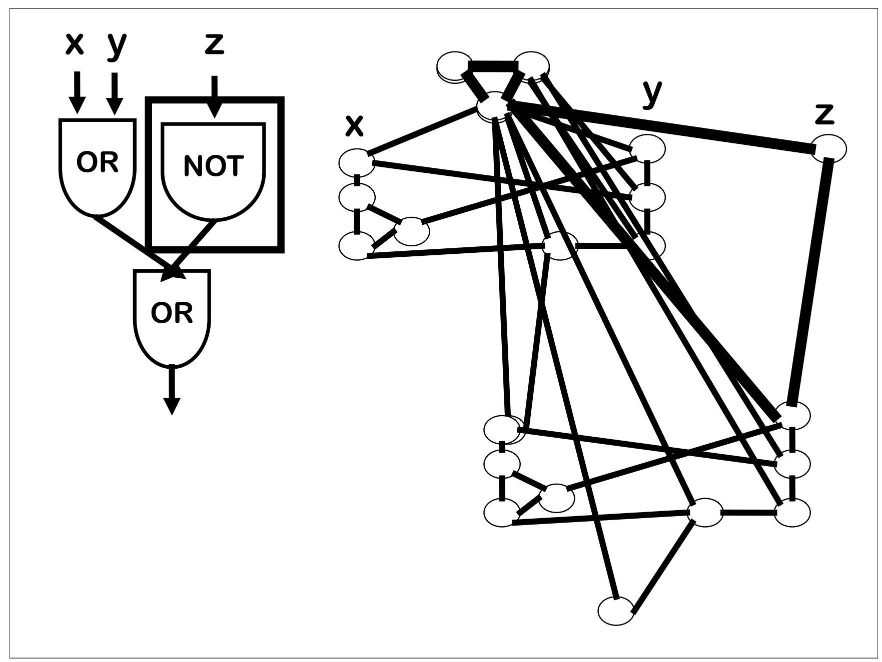


X	Y	OR
F	F	F
F	T	T
T	F	T
T	T	T


T
F
x
NOT gate!


x y
z
OR
NOT
OR
x
y
z


x y
z
OR
NOT
OR
x
y
z


x y
z
OR
NOT
OR
x
y
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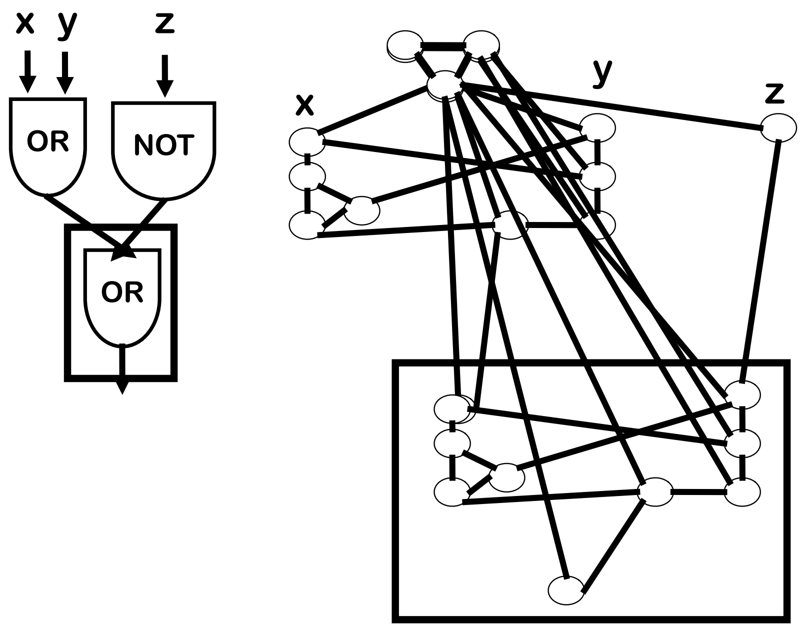
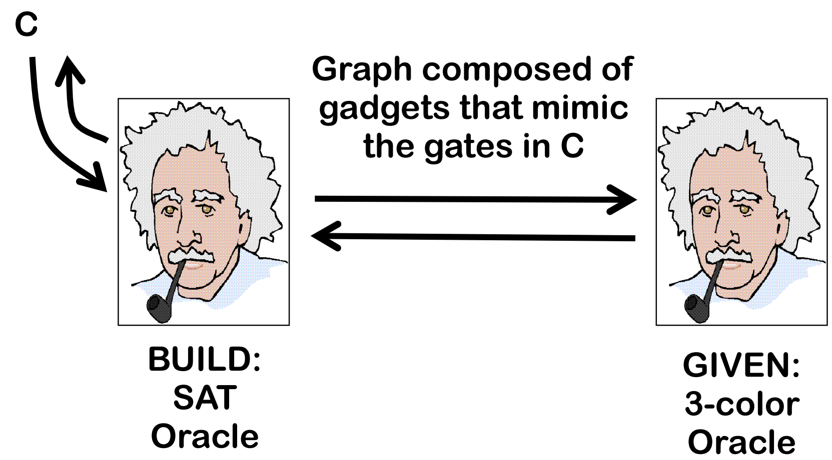
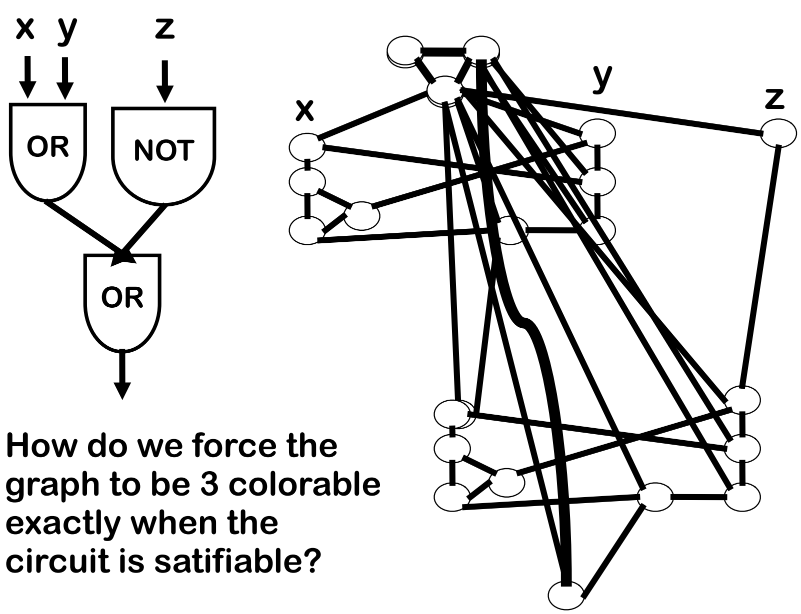


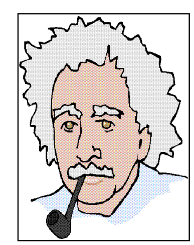
Let C be an n-input circuit



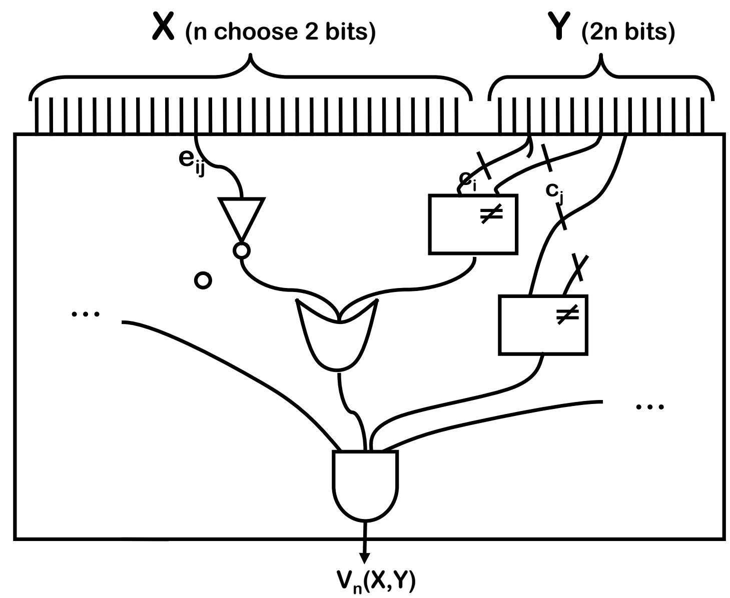


How do we force the graph to be 3 colorable exactly when the circuit is satifiable?

You can quickly transform a method to decide 3-coloring into a method to decide circuit satifiability!



Given an oracle for circuit SAT, how can you quickly solve 3-colorability?



Can you make a circuit that takes a description of a graph and a node coloring, and checks if it is a valid 3-coloring?

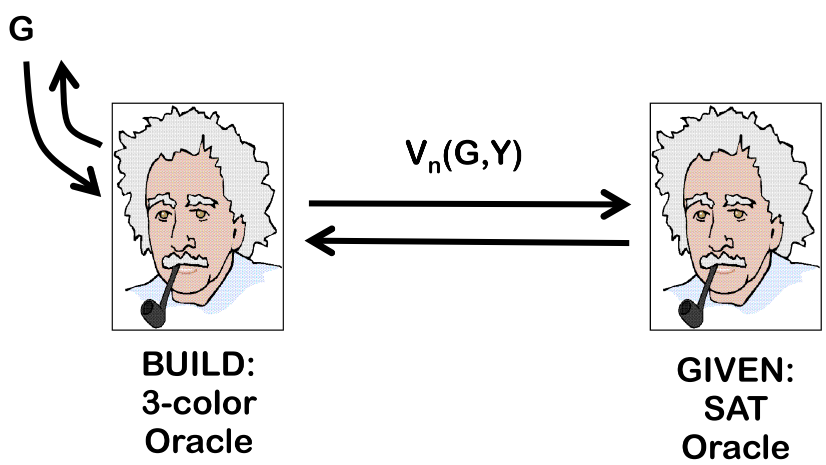
$V_n(X,Y)$

Let V_n be a circuit that takes an n-node graph X and an assignment of colors to nodes Y, and verifies that Y is a valid 3 coloring of X. I.e., $V_n(X,Y) = 1$ iff Y is a 3 coloring of X

X is expressed as an n choose 2 bit sequence. Y is expressed as a 2n bit sequence

Given n, we can construct V_n in time $O(n^2)$

Let G be an n-node graph



Circuit-SAT / 3-Colorability

Clique / Independent Set

Circuit-SAT / 3-Colorability

Two problems that are cosmetically different, but substantially the same

Given an oracle for circuit SAT, how can you quickly solve k-clique?

Circuit-SAT / 3-Colorability

Clique / Independent Set

FACT: No one knows a way to solve any of the 4 problems that is fast on all instances

Four problems that are cosmetically different, but substantially the same

Summary

Many problems that appear different on the surface can be efficiently reduced to each other, revealing a deeper similarity