

# A Circuit for Coloring

15-251: Great Theoretical Ideas in Computer Science

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Suppose that we have a graph  $G = (V, E)$  on  $n$  nodes. We'd like to test colorability by devising a circuit. We'll have a Boolean variable indicating the presence or absence of each possible edge, and three Boolean variables that encode which color is assigned to each node. Specifically,

$$X_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ is an edge} \\ 0 & \text{otherwise;} \end{cases}$$
$$Y_{ik} = \begin{cases} 1 & \text{if node } i \text{ is colored } k \in \{\text{red, blue, green}\} \\ 0 & \text{otherwise.} \end{cases}$$

We ensure that an assignment of Boolean variables  $Y_{ik}$ , for  $i = 1, 2, \dots, n$  and  $k = 1, 2, 3$  is a valid assignment of colors with following circuit  $\mathcal{E}(Y)$ :

$$\mathcal{E}(Y) = \bigwedge_{i=1}^n \{(Y_{i1} \wedge \neg Y_{i2} \wedge \neg Y_{i3}) \vee (Y_{i2} \wedge \neg Y_{i1} \wedge \neg Y_{i3}) \vee (Y_{i3} \wedge \neg Y_{i1} \wedge \neg Y_{i2})\}.$$

In words, if a Boolean variable is true for a color, then the Boolean variables for the other two colors have to be false.

We then test whether an assignment of colors is a valid coloring with the following circuit:

$$\mathcal{C}(X, Y) = \bigwedge_{i,j=1}^n \{\neg X_{ij} \vee \mathcal{K}(Y_i, Y_j)\}.$$

In this expression, if  $(i, j)$  is not an edge, so that  $\neg X_{ij}$  is true, then we don't care about the coloring of the endpoints. However, if it *is* an edge, so that  $\neg X_{ij}$  is false, then we demand that  $\mathcal{K}(Y_i, Y_j)$  is true, where the circuit  $\mathcal{K}(Y_i, Y_j)$  checks that the colors on the two neighboring nodes are not the same:

$$\mathcal{K}(Y_i, Y_j) = \bigwedge_{k=1}^3 \neg (Y_{ik} \wedge Y_{jk})$$

Putting the pieces together, our overall circuit to test colorability of the graph is then

$$V(X, Y) = \mathcal{E}(Y) \wedge \mathcal{C}(X, Y).$$

This can be efficiently constructed from the graph, in  $O(n^2)$  operations. Note that for a given graph, the Boolean variables  $X$  are fixed, and we feed in Boolean variables  $Y$  to test whether or not it encodes a valid coloring of the graph.

Now suppose that we want to test whether a given graph  $G = (V, E)$  has an independent set of size  $k$ . In a similar fashion as above, we introduce Boolean variables  $Z_i$ , for  $i = 1, 2, \dots, n$ , where now

$$Z_i = \begin{cases} 1 & \text{if node } i \text{ is an element of the independent set} \\ 0 & \text{otherwise.} \end{cases}$$

Then a circuit  $\mathcal{I}(X, Z)$  that is true if and only if  $Z$  encodes an independent set for the graph encoded by  $X$  is

$$\mathcal{I}(X, Z) = \bigwedge_{i,j=1}^n \{\neg X_{ij} \vee \neg(Z_i \wedge Z_j)\}.$$

The idea here is that if both  $i$  and  $j$  are in the putative independent set, so that  $Z_i$  and  $Z_j$  are both true, then we require that  $(i, j)$  is not an edge, so that  $\neg X_{ij}$  is true.

But note that we have ignored the size  $k$  of the independent set, which is an input to the problem. Can you figure out how to modify the construction to account for  $k$ ?