

15-251

Great Theoretical Ideas in Computer Science

What does this do?

```
(_,_,_){_/_<=1?(_,_+1,_
_):!(_%_)?(_,_+1,0):_%_==_
/_
_&&!_?(printf("%d\t",_/_),_(_,_
_+1,0)):_%_>1&&_%_<_/_?(_
_,1+
_,_+!(_/_%(_%_))):_<_*_
?(_,_+1,_):0;}main(){_(100,0,0);}
```

Turing's Legacy: The Limits Of Computation

Lecture 25 (November 18, 2007)



This lecture will change the way you
think about computer programs...

Many questions which appear easy at first
glance are impossible to solve in general

The HELLO assignment

Write a JAVA program to output the word
"HELLO" on the screen and halt.

Space and time are not an issue.
The program is for an ideal computer.

PASS for any working HELLO program, no
partial credit.

Grading Script

The grading script G must be able to take any
Java program P and grade it.

$$G(P) = \begin{cases} \text{Pass, if P prints only the word} \\ \text{"HELLO" and halts.} \\ \text{Fail, otherwise.} \end{cases}$$

How exactly might such a script work?

What does this do?

```

_( , , ) { _ / <= 1 ? _ ( , , + 1 , _
_ ) : ! ( _ % _ ) ? _ ( , , + 1 , 0 ) : _ % _ ==
_ /
_ && ! _ ? ( printf ( " % d \ t " , _ / _ ) , _ ( , ,
+ 1 , 0 ) ) : _ % _ > 1 && _ % _ < _ / _ ? _ (
_ , 1 +
_ , _ + ! ( _ / _ % ( _ % _ ) ) ) : _ < _ *
_ ? _ ( , , + 1 , _ ) : 0 ; } main () { _ ( 100 , 0 , 0 ) ; }

```

What kind of program
could a student who
hated his/her TA
hand in?



Nasty Program

```

n:=0;
while (n is not a counter-example
to the Riemann Hypothesis) {
    n++;
}
print "Hello";

```

The nasty program is a PASS if and only if the Riemann Hypothesis is false.

A TA nightmare: Despite
the simplicity of the
HELLO assignment,
there is no program to
correctly grade it!

And we will prove this.



The theory of what can
and can't be computed
by an ideal computer is
called
Computability Theory
or Recursion Theory.



From the last lecture:

Are all reals describable? NO
Are all reals computable? NO

We saw that
computable $\Rightarrow$ describable
but do we also have
describable $\Rightarrow$ computable?

The "grading function" we just described
is not computable! (We'll see a proof soon.)

Computable Function

Fix a finite set of symbols, Σ

Fix a precise programming language, e.g., Java

A program is any finite string of characters that is syntactically valid.

A function $f: \Sigma^* \rightarrow \Sigma^*$ is computable if there is a program P that when executed on an ideal computer, computes f .

That is, for all strings x in Σ^* , $f(x) = P(x)$.

Hence: countably many computable functions!

There are only countably many Java programs.

Hence, there are only countably many computable functions.



Uncountably Many Functions

The functions $f: \Sigma^* \rightarrow \{0,1\}$ are in 1-1 onto correspondence with the subsets of Σ^* (the powerset of Σ^*).

Subset S of $\Sigma^* \Leftrightarrow$ Function f_S

x in S	$\Leftrightarrow$	$f_S(x) = 1$
x not in S	$\Leftrightarrow$	$f_S(x) = 0$

Hence, the set of all $f: \Sigma^* \rightarrow \{0,1\}$ has the same size as the power set of Σ^* , which is uncountable.

Countably many computable functions.

Uncountably many functions from Σ^* to $\{0,1\}$.

Thus, most functions from Σ^* to $\{0,1\}$ are not computable.



Can we explicitly describe an uncomputable function?

Can we describe an interesting uncomputable function?



Notation And Conventions

Fix a single programming language (Java)

When we write program P we are talking about the text of the source code for P

$\downarrow$
 $P(x)$ means the output that arises from running program P on input x , assuming that P eventually halts.

$P(x) = \perp$ means P did not halt on x

The meaning of $P(P)$

It follows from our conventions that $P(P)$ means the output obtained when we run P on the text of its own source code

The Halting Set K

Definition:

K is the set of all programs P such that $P(P)$ halts.

$$K = \{ \text{Java } P \mid P(P) \text{ halts} \}$$

The Halting Problem

Is there a program HALT such that:

$\text{HALT}(P) = \text{yes, if } P(P) \text{ halts} \Rightarrow P \in K$
 $\text{HALT}(P) = \text{no, if } P(P) \text{ does not halt} \Rightarrow P \notin K$

deciding membership in K

THEOREM: There is no program to solve the halting problem (Alan Turing 1937)

Suppose a program HALT existed that solved the halting problem.

$\text{HALT}(P) = \text{yes, if } P(P) \text{ halts}$
 $\text{HALT}(P) = \text{no, if } P(P) \text{ does not halt}$

We will call HALT as a subroutine in a new program called CONFUSE .

CONFUSE

```
CONFUSE(P)
{ if (HALT(P))
  then loop forever;    //i.e., we dont halt
  else exit;           //i.e., we halt
  // text of HALT goes here
}
```

Does $\text{CONFUSE}(\text{CONFUSE})$ halt?

CONFUSE

```
CONFUSE(P)
{ if (HALT(P))
  then loop forever;    //i.e., we dont halt
  else exit;           //i.e., we halt
  // text of HALT goes here }
```

Suppose $\text{CONFUSE}(\text{CONFUSE})$ halts:
then $\text{HALT}(\text{CONFUSE}) = \text{TRUE}$
 $\Rightarrow \text{CONFUSE}$ will loop forever on input CONFUSE

Suppose $\text{CONFUSE}(\text{CONFUSE})$ does not halt
then $\text{HALT}(\text{CONFUSE}) = \text{FALSE}$

CONTRADICTION

Alan Turing (1912-1954)

Theorem: [1937]

There is no program to solve the halting problem



Turing's argument is essentially the reincarnation of Cantor's Diagonalization argument that we saw in the previous lecture.



All Programs (the input)

	P_0	P_1	P_2	...	P_j	...
P_0						
P_1						
...						
P_i						
...						

Programs (computable functions) are countable, so we can put them in a (countably long) list

All Programs (the input)

	P_0	P_1	P_2	...	P_j	...
P_0						
P_1						
...						
P_i						
...						

YES, if $P_i(P_j)$ halts
No, otherwise

All Programs (the input)

	P_0	P_1	P_2	...	P_j	...
P_0	d_0					
P_1		d_1				
...			...			
P_i				d_i		
...					...	

Let $d_i = \text{HALT}(P_i)$

$\text{CONFUSE}(P_i)$ halts iff $d_i = \text{no}$
(The CONFUSE function is the negation of the diagonal.)
Hence CONFUSE cannot be on this list.

Is there a real number that can be described, but not computed?



Consider the real number R whose binary expansion has a 1 in the j^{th} position iff the j^{th} program halts.



Proof that R cannot be computed

Suppose it is, and program FRED computes it. then consider the following program:

```

MYSTERY(program text P)
  for j = 0 to forever do {
    if (P == Pj)
      then use FRED to compute jth bit of R
      return YES if (bit == 1), NO if (bit == 0)
  }

```

MYSTERY solves the halting problem!

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We saw that

computable $\Rightarrow$ describable
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Computability

A Vocabulary lesson

Computability Theory: Vocabulary Lesson

We call a set $S \subseteq \Sigma^*$ decidable or recursive if there is a program P such that:

$P(x) = \text{yes}$, if $x \in S$
 $P(x) = \text{no}$, if $x \notin S$

We already know: the halting set K is undecidable

Decidable and Computable

Subset S of Σ^* $\Leftrightarrow$ Function f_S

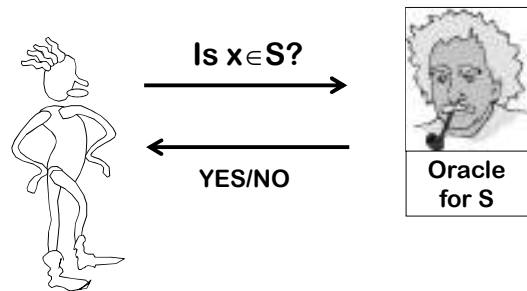
$x \in S$	$\Leftrightarrow$	$f_S(x) = 1$
$x \text{ not in } S$	$\Leftrightarrow$	$f_S(x) = 0$

Set S is decidable $\Leftrightarrow$ function f_S is computable

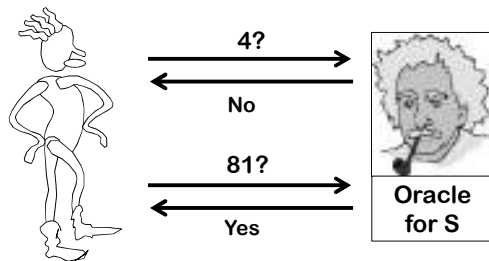
Sets are “decidable” (or undecidable), whereas functions are “computable” (or not)

Oracles and Reductions

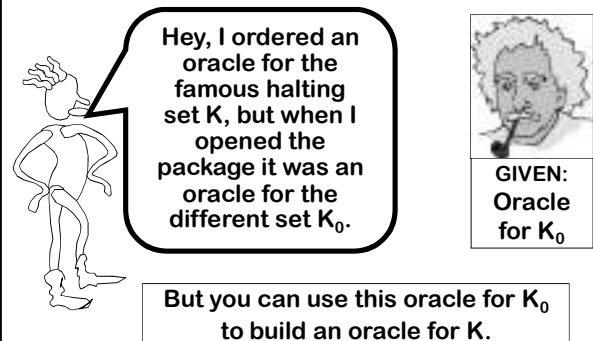
Oracle For Set S



Example Oracle $S = \text{Odd Naturals}$

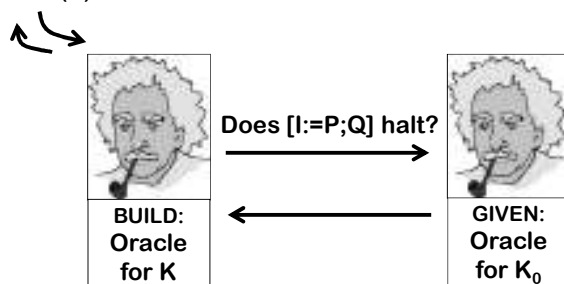


K_0 = the set of programs that take no input and halt



K_0 = the set of programs that take no input and halt

$P = [\text{input } I; Q]$
Does $P(P)$ halt?



We've reduced the problem of deciding membership in K to the problem of deciding membership in K_0 .

Hence, deciding membership for K_0 must be at least as hard as deciding membership for K.

Thus if K_0 were decidable then K would be as well.

We already know K is not decidable, hence K_0 is not decidable.



HELLO = the set of programs that print hello and halt

Does P halt?



Let P' be P with all print statements removed. (assume there are no side effects)

Is $[P'; \text{print HELLO}]$ a hello program?



BUILD:
Oracle
for K_0



GIVEN:
HELLO
Oracle



Hence, the set HELLO is not decidable.

EQUAL = All $\langle P, Q \rangle$ such that P and Q have identical output behavior on all inputs

Is P in set HELLO?



Let $H1 = [\text{print HELLO}]$



BUILD:
HELLO
Oracle



GIVEN:
EQUAL
Oracle

Are P and $H1$ equal?



Halting with input, Halting without input, HELLO, and EQUAL are all undecidable.

Diophantine Equations

Does polynomial $4X^2Y + XY^2 + 1 = 0$ have an integer root? I.e., does it have a zero at a point where all variables are integers?

$D = \{\text{multivariate integer polynomials } P \mid P \text{ has a root where all variables are integers}\}$

Famous Theorem: D is undecidable!
[This is the solution to Hilbert's 10th problem]



Hilbert

Resolution of Hilbert's 10th Problem: Dramatis Personae



Martin Davis, Julia Robinson, Yuri Matiyasevich (1982)

Polynomials can Encode Programs

There is a computable function
 F : Java programs that take no input $\rightarrow$
 Polynomials over the integers

Such that
 program P halts $\Leftrightarrow F(P)$ has an integer root

D = the set of all integer
 polynomials with integer roots

Does program P
 halt?



BUILD:
 HALTING
 Oracle

$F(P)$ has
 integer root?



GIVEN:
 Oracle
 for D



PHILOSOPHICAL
 INTERLUDE



CHURCH-TURING THESIS

Any well-defined procedure that can
 be grasped and performed by the
 human mind and pencil/paper, can be
 performed on a conventional digital
 computer with no bound on memory.



The Church-Turing Thesis is NOT a
 theorem. It is a statement of belief
 concerning the universe we live in.

Your opinion will be influenced by your
 religious, scientific, and philosophical
 beliefs...

...mileage may vary

Empirical Intuition

No one has ever given a counter-example to the Church-Turing thesis. I.e., no one has given a concrete example of something humans compute in a consistent and well defined way, but that can't be programmed on a computer. The thesis is true.

Mechanical Intuition

The brain is a machine. The components of the machine obey fixed physical laws. In principle, an entire brain can be simulated step by step on a digital computer. Thus, any thoughts of such a brain can be computed by a simulating computer. The thesis is true.

Quantum Intuition

The brain is a machine, but not a classical one. It is inherently quantum mechanical in nature and does not reduce to simple particles in motion. Thus, there are inherent barriers to being simulated on a digital computer. The thesis is false. However, the thesis is true if we allow quantum computers.