15-251

Great Theoretical Ideas in Computer Science

Grade School Revisited: How To Multiply Two Numbers

Lecture 22, November 6, 2008

Gauss

(a+bi)

Gauss' Complex Puzzle

Remember how to multiply two complex numbers a + bi and c + di?

(a+bi)(c+di) = [ac –bd] + [ad + bc] i

Input: a,b,c,d
Output: ac-bd, ad+bc

If multiplying two real numbers costs $1 and adding them costs a penny, what is the cheapest way to obtain the output from the input?

Can you do better than $4.02?

Gauss' $3.05 Method

Input: a,b,c,d
Output: ac-bd, ad+bc

c	$X_1 = a + b$	
c	$X_2 = c + d$	
$	$X_3 = X_1 X_2$	$= ac + ad + bc + bd$
$	$X_4 = ac$	
$	$X_5 = bd$	
c	$X_6 = X_4 - X_5$	$= ac - bd$
cc	$X_7 = X_3 - X_4 - X_5$	$= bc + ad$

Time complexity of grade school addition

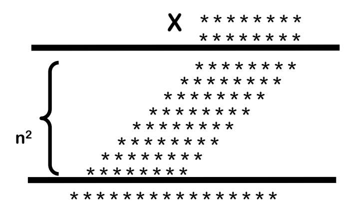
T(n) = amount of time grade school addition uses to add two n-bit numbers

We saw that T(n) was linear

$T(n) = \Theta(n)$

The Gauss optimization saves one multiplication out of four.
It requires 25% less work.

Time complexity of grade school multiplication

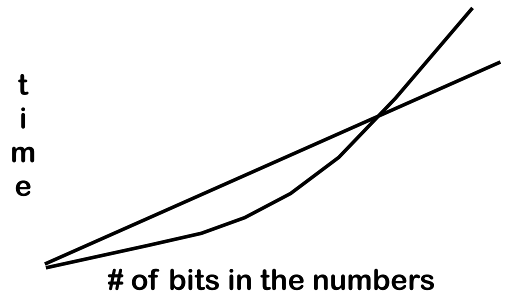


T(n) = The amount of time grade school multiplication uses to add two n-bit numbers

We saw that T(n) was quadratic

$T(n) = \Theta(n^2)$

Grade School Addition: Linear time
Grade School Multiplication: Quadratic time



No matter how dramatic the difference in the constants, the quadratic curve will eventually dominate the linear curve

Any addition algorithm takes Ω(n) time

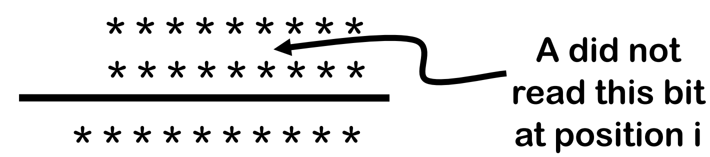
Claim: Any algorithm for addition must read all of the input bits

Proof: Suppose there is a mystery algorithm A that does not examine each bit

Give A a pair of numbers. There must be some unexamined bit position i in one of the numbers

Is there a sub-linear time method for addition?

Any addition algorithm takes Ω(n) time



If A is not correct on the inputs, we found a bug

If A is correct, flip the bit at position i and give A the new pair of numbers. A gives the same answer as before, which is now wrong.

Grade school addition can't be improved upon by more than a constant factor

Can we even break the quadratic time barrier?

In other words, can we do something very different than grade school multiplication?

Grade School Addition: $\Theta(n)$ time.
Furthermore, it is optimal

Grade School Multiplication: $\Theta(n^2)$ time

Is there a clever algorithm to multiply two numbers in linear time?

Despite years of research, no one knows! If you resolve this question, Carnegie Mellon will give you a PhD!

Divide And Conquer

An approach to faster algorithms:

- DIVIDE a problem into smaller subproblems
- CONQUER them recursively
- GLUE the answers together so as to obtain the answer to the larger problem

Multiplication of 2 n-bit numbers

n bits

X =	X
Y =	Y

n/2 bits n/2 bits

$$X = a\,2^{n/2} + b \qquad Y = c\,2^{n/2} + d$$

$$X \times Y = ac\,2^{n} + (ad + bc)\,2^{n/2} + bd$$

Same thing for numbers in decimal!

n digits

X =	a	b
Y =	c	d

n/2 digits n/2 digits

$$X = a\,10^{n/2} + b \qquad Y = c\,10^{n/2} + d$$

$$X \times Y = ac\,10^{n} + (ad + bc)\,10^{n/2} + bd$$

Multiplication of 2 n-bit numbers

X =	a	b
Y =	c	d

n/2 bits n/2 bits

$$X \times Y = ac\,2^{n} + (ad + bc)\,2^{n/2} + bd$$

```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
          return MULT(a,c) 2^n + (MULT(a,d)
          + MULT(b,c)) 2^(n/2) + MULT(b,d)
```

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39

1*2 1*1 2*2 2*1

2 1 4 2

Hence: $12*21 = 2*10^2 + (1 + 4)10^1 + 2 = 252$

X =	a	b
Y =	c	d

$$X \times Y = ac\,10^{n} + (ad + bc)\,10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

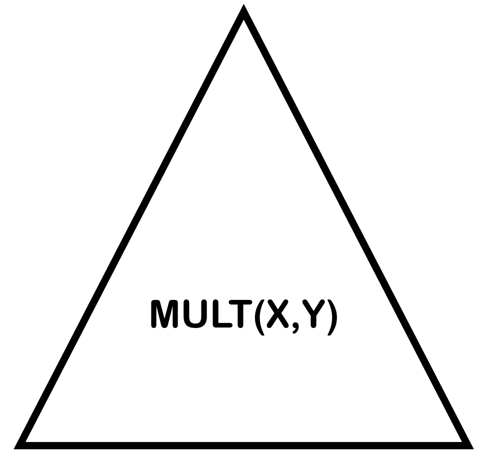
1234*2139 1234*4276 5678*2139 5678*4276

2521 4689 7141 1326
*10⁴ + *10² + *10² + *1 = 2639526

X = a b
Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

Divide, Conquer, and Glue



Multiplying (Divide & Conquer style)

12345678 * 21394276

26395269 52765846 12145242 24279128
*10⁸ + *10⁴ + *10⁴ + *1

= 264126842539128

X = a b
Y = c d

$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$

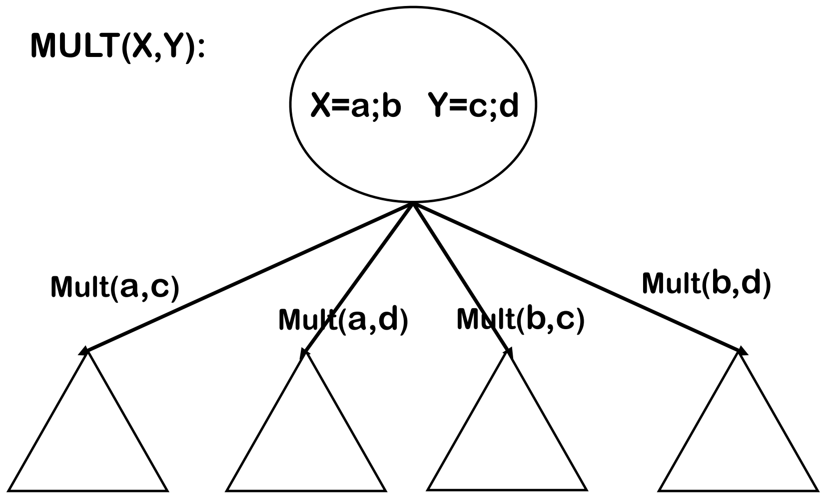
Divide, Conquer, and Glue

MULT(X,Y):

if |X| = |Y| = 1
then return XY,
else…

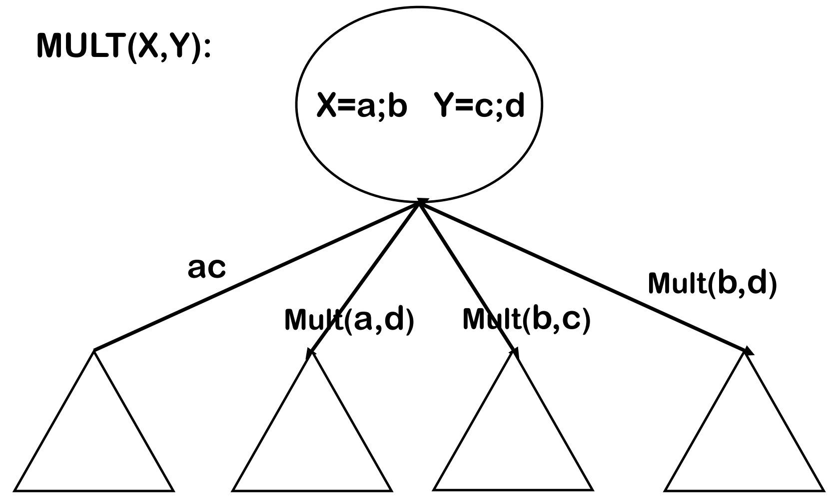
Divide, Conquer, and Glue

MULT(X,Y):



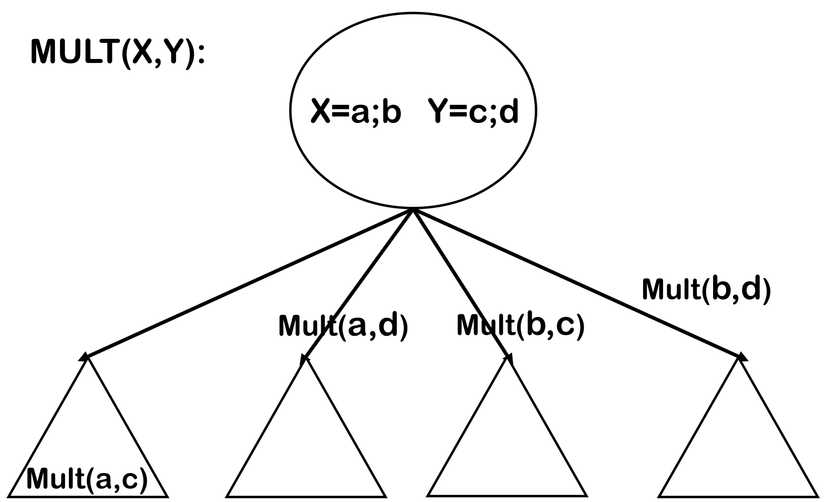
Divide, Conquer, and Glue

MULT(X,Y):



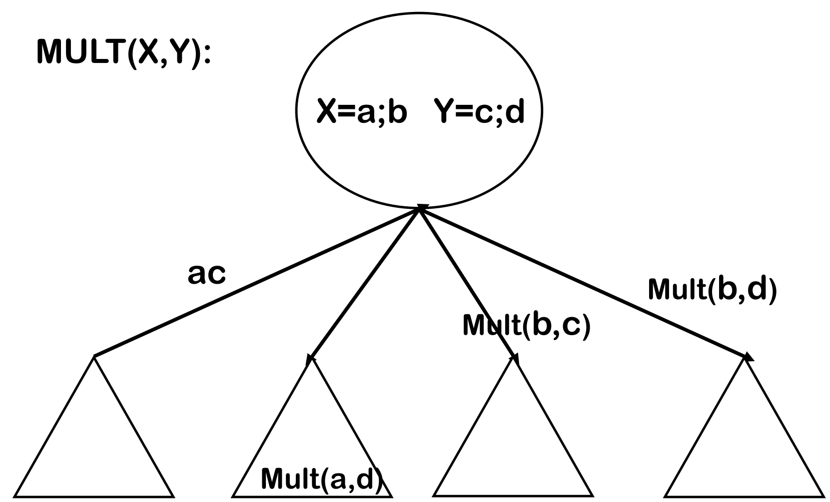
Divide, Conquer, and Glue

MULT(X,Y):



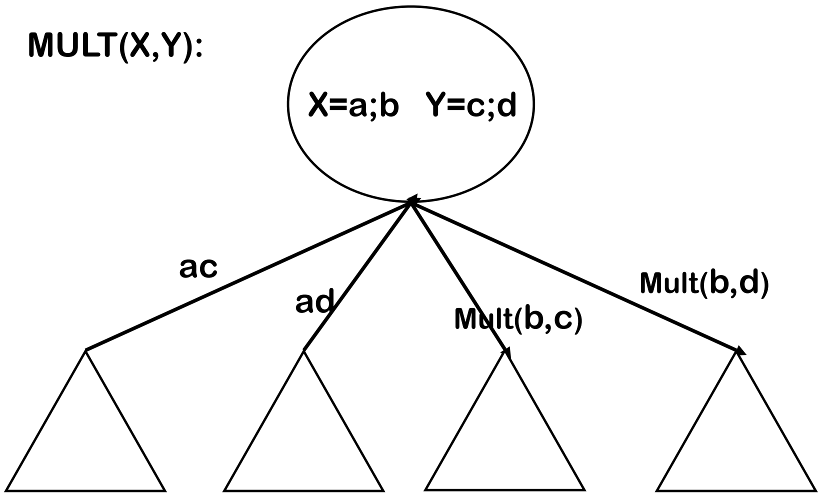
Divide, Conquer, and Glue

MULT(X,Y):



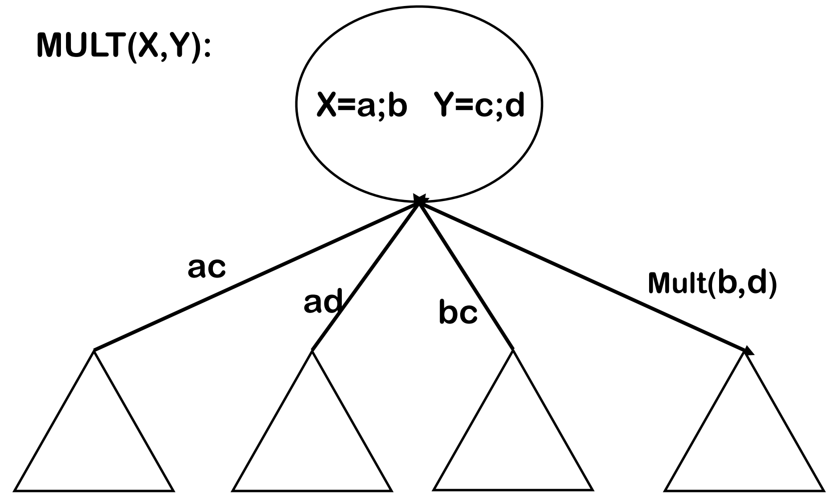
Divide, Conquer, and Glue

MULT(X,Y):



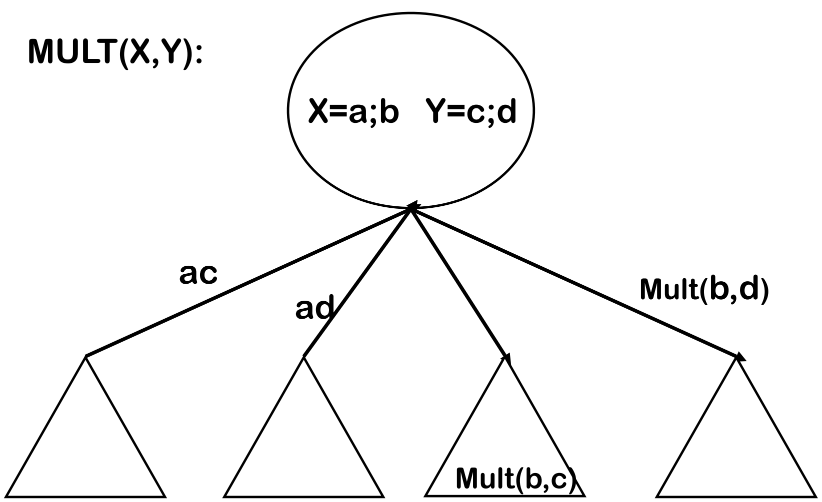
Divide, Conquer, and Glue

MULT(X,Y):



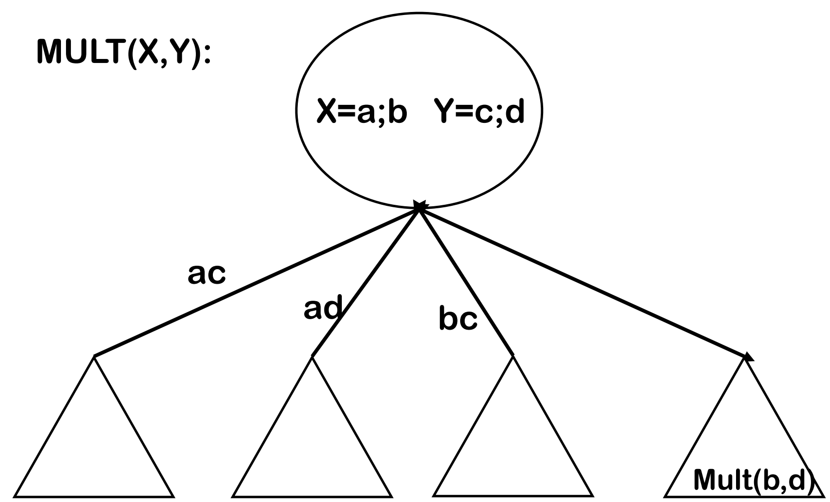
Divide, Conquer, and Glue

MULT(X,Y):

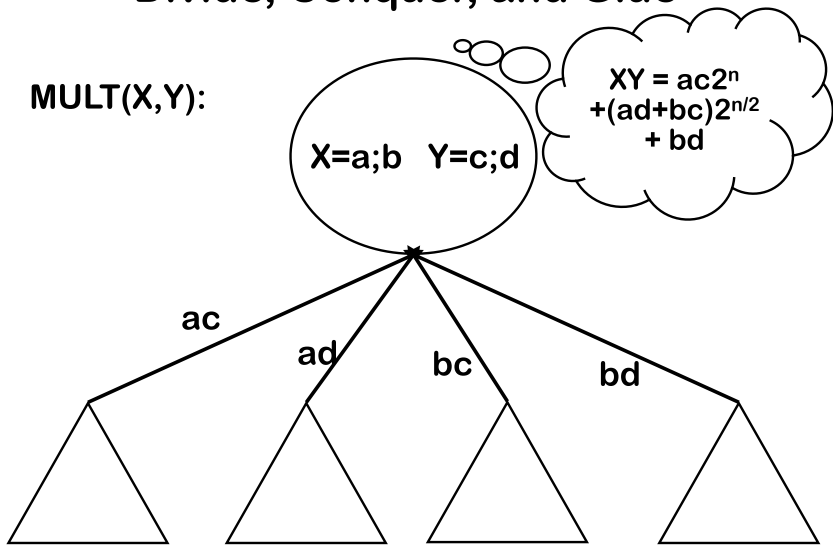


Divide, Conquer, and Glue

MULT(X,Y):



Divide, Conquer, and Glue



Recurrence Relation

$T(1) = k$ for some constant k

$T(n) = 4\,T(n/2) + k'n + k''$ for constants k' and k''

```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
          return MULT(a,c) 2^n + (MULT(a,d)
          + MULT(b,c)) 2^(n/2) + MULT(b,d)
```

Time required by MULT

$T(n)$ = time taken by MULT on two n-bit numbers

What is $T(n)$? What is its growth rate?

Big Question: Is it $\Theta(n^2)$?

$$T(n) = \underbrace{4}_{\text{conquering time}}\, T(n/2) + (\underbrace{k'n + k''}_{\text{divide and glue}})$$

Recurrence Relation

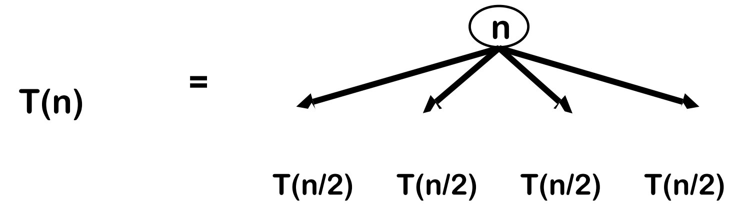
$T(1) = 1$

$T(n) = 4\,T(n/2) + n$

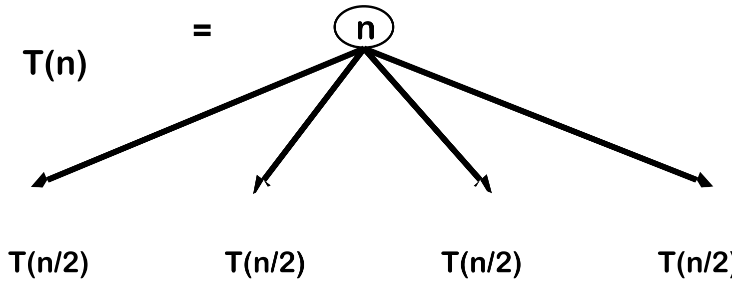
```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
          return MULT(a,c) 2^n + (MULT(a,d)
          + MULT(b,c)) 2^(n/2) + MULT(b,d)
```

Technique: Labeled Tree Representation

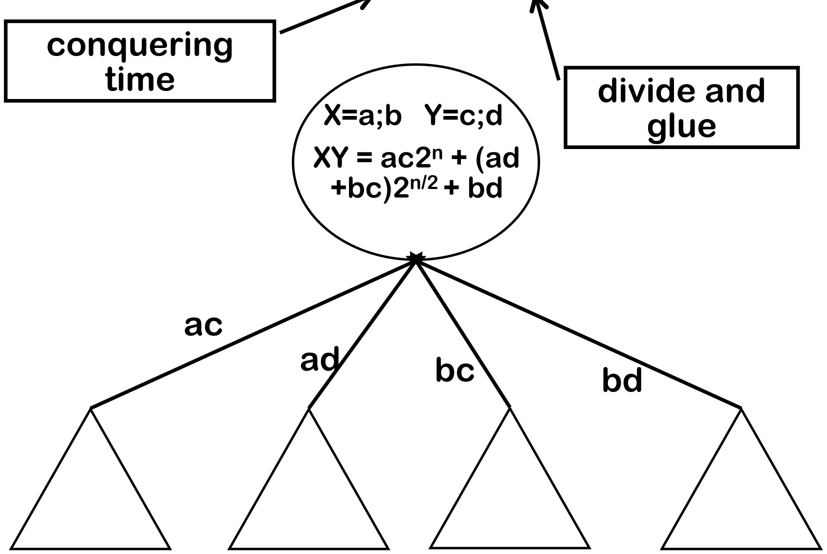
T(n) = n + 4 T(n/2)

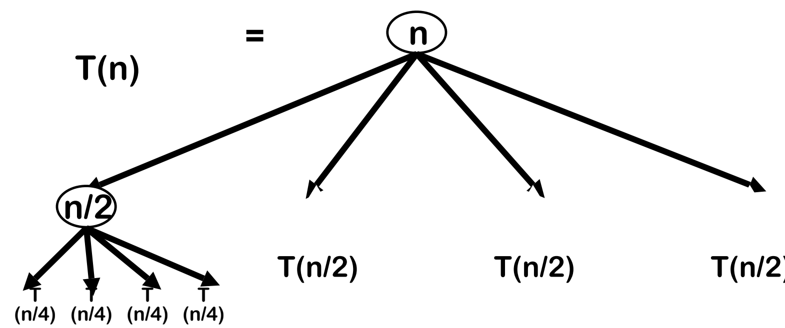


T(1) = 1

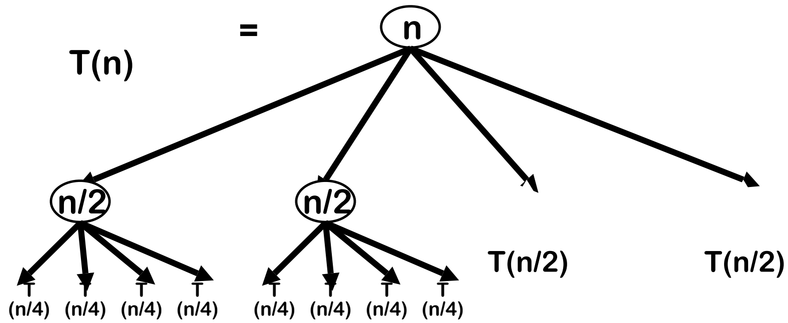


T(n) = 4 T(n/2) + (k'n + k'')





T(n) =



1n =	n
2n =	n/2 + n/2 + n/2 + n/2
4n =	
2^in =	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
(n)n =	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

$$n(1+2+4+8+\ldots+n) = n(2n-1) = 2n^2-n$$

0	n
1	n/2 + n/2 + n/2 + n/2
2	
i	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
$\log_2(n)$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

Divide and Conquer MULT: $\Theta(n^2)$ time
Grade School Multiplication: $\Theta(n^2)$ time

MULT revisited

```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
          return MULT(a,c) 2^n + (MULT(a,d)
          + MULT(b,c)) 2^(n/2) + MULT(b,d)
```

MULT calls itself 4 times. Can you see a way to reduce the number of calls?

Karatsuba, Anatolii Alexeevich (1937-)

Sometime in the late 1950's Karatsuba had formulated the first algorithm to break the n^2 barrier!

Gauss' optimization

Input: a,b,c,d
Output: ac-bd, ad+bc

c	$X_1 = a + b$	
c	$X_2 = c + d$	
\$	$X_3 = X_1 X_2$	$= ac + ad + bc + bd$
\$	$X_4 = ac$	
\$	$X_5 = bd$	
c	$X_6 = X_4 - X_5$	$= ac - bd$
cc	$X_7 = X_3 - X_4 - X_5$	$= bc + ad$

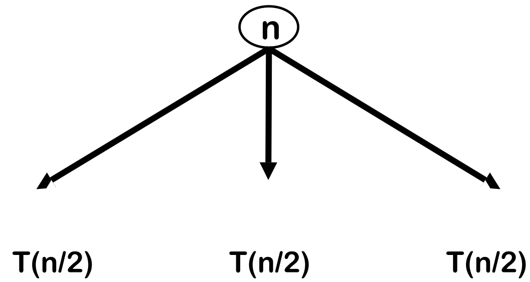
Gaussified MULT (Karatsuba 1962)

```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
                e : = MULT(a,c)
                f  := MULT(b,d)
    return
    e 2^n + (MULT(a+b,c+d) – e - f) 2^(n/2) + f
```

$T(n) = 3\,T(n/2) + n$

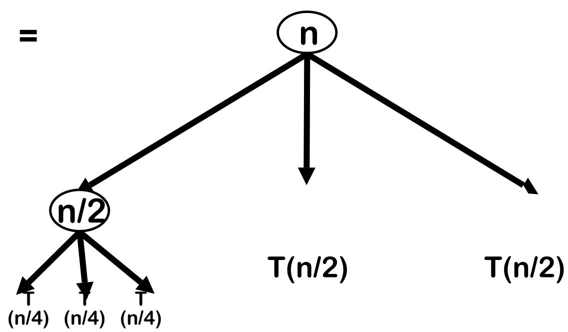
Actually: $T(n) = 2\,T(n/2) + T(n/2 + 1) + kn$

$T(n) =$



0	n
1	n/2 + n/2 + n/2
2	
i	Level i is the sum of 3^i copies of $n/2^i$
	. . . .
$\log_2(n)$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

$T(n) =$



1n =	n
3/2n =	n/2 + n/2 + n/2
9/4n =	
$(3/2)^i n$ =	Level i is the sum of 3^i copies of $n/2^i$
	. . . .
$(3/2)^{\log n} n$ =	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

$$n(1+3/2+(3/2)^2+ \ldots + (3/2)^{\log_2 n}) = 3n^{1.58\ldots} - 2n$$

Dramatic Improvement for Large n

$T(n) = 3n^{\log_2 3} - 2n$

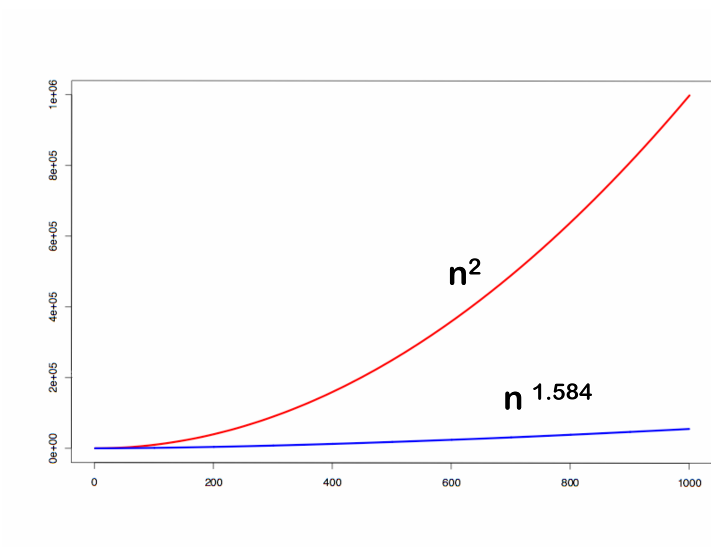
$= \Theta(n^{\log_2 3})$

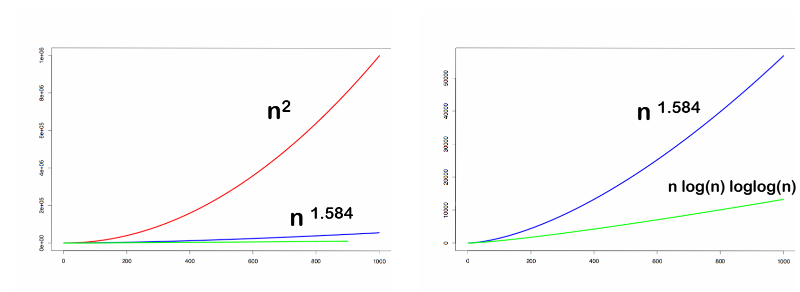
$= \Theta(n^{1.58\ldots})$

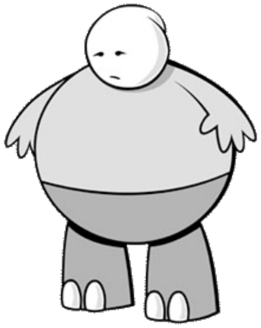
A huge savings over $\Theta(n^2)$ when n gets large.

Multiplication Algorithms

Kindergarten	$n2^n$
Grade School	n^2
Karatsuba	$n^{1.58\ldots}$
Fastest Known	n logn loglogn







Here's What You Need to Know…

- Gauss's Multiplication Trick
- Proof of Lower bound for addition
- Divide and Conquer
- Solving Recurrences
- Karatsuba Multiplication