15-251

Great Theoretical Ideas in Computer Science

This is The Big Oh!

Lecture 21, November 4, 2008

Counting

The Power of One

Algebra

The Power of X

Asymptotics

The Power of O

How to add 2 n-bit numbers

```
  * * * * * * * * * * *
+ * * * * * * * * * * *
  ---------------------
```

How to add 2 n-bit numbers

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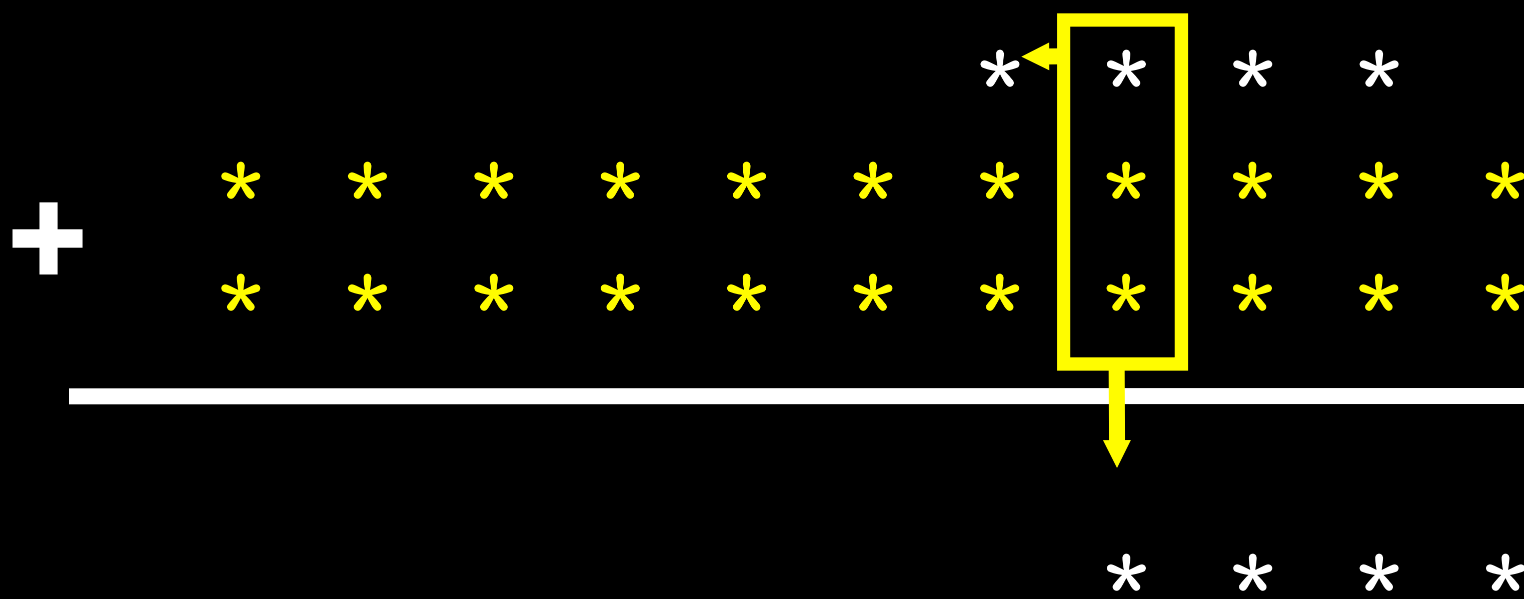

How to add 2 n-bit numbers

"Grade school addition"

Time complexity of grade school addition

```
  * * * * * * * * * *
  * * * * * * * * * *
+ * * * * * * * * * *
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* * * * * * * * * * *
```
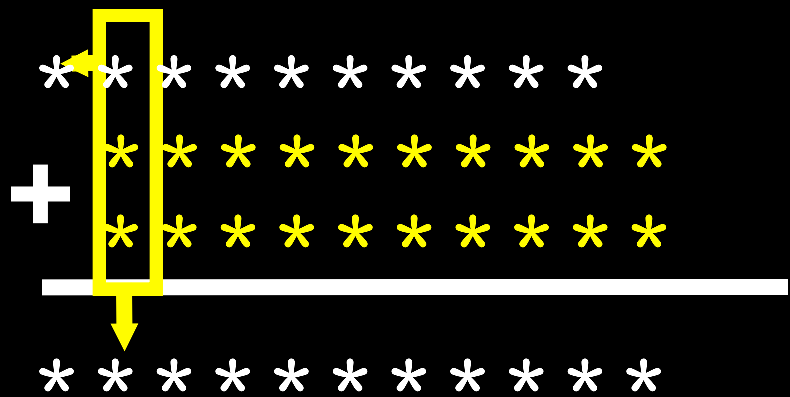

T(n) = amount of time grade school addition uses to add two **n**-bit numbers

Time complexity of grade school addition



T(n) = amount of time grade school addition uses to add two n-bit numbers

What do we mean by "time"?

Our Goal

We want to define "time" in a way that transcends implementation details and allows us to make assertions about grade school addition in a very general yet useful way.

Roadblock ???

A given algorithm will take different amounts of time on the same inputs depending on such factors as:

- Processor speed
- Instruction set
- Disk speed
- Brand of compiler

On any reasonable computer, adding 3 bits and writing down the two bit answer can be done in constant time

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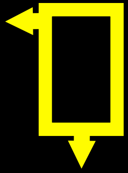
Pick any particular computer M and define c to be the time it takes to perform  on that computer.

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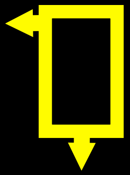
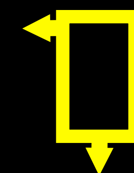
Pick any particular computer M and define c to be the time it takes to perform  on that computer.

Total time to add two n-bit numbers using grade school addition:

On any reasonable computer, adding 3 bits and writing down the two bit answer can be done in constant time

Pick any particular computer M and define c to be the time it takes to perform  on that computer.

Total time to add two n-bit numbers using grade school addition:

cn [i.e., c time for each of n columns]

On another computer M’, the time to perform may be c’.

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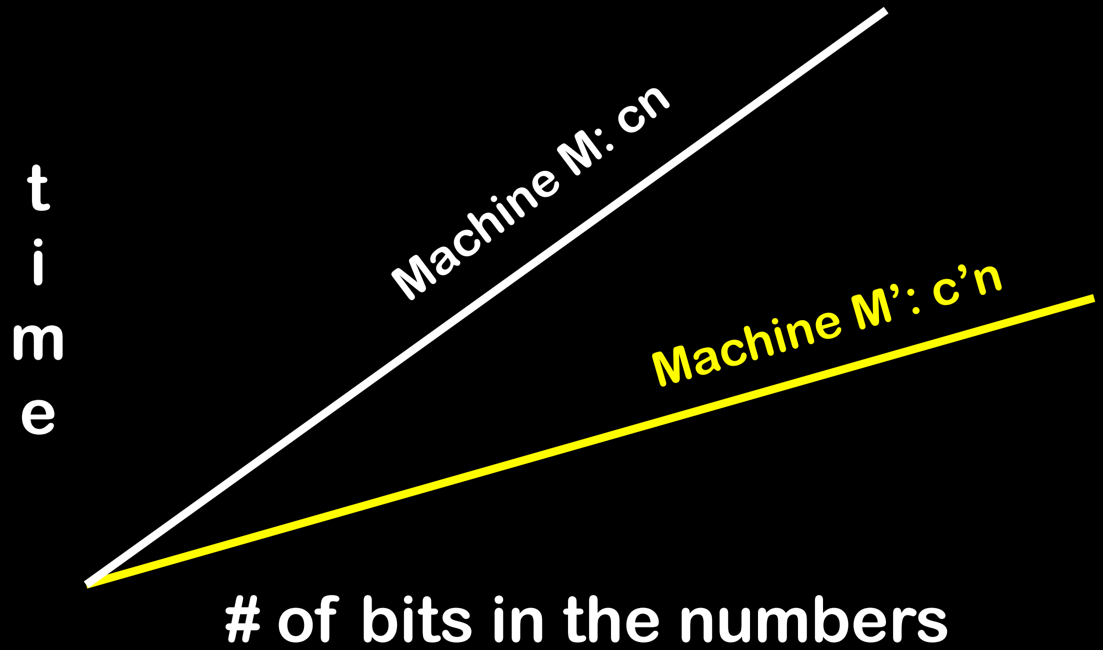
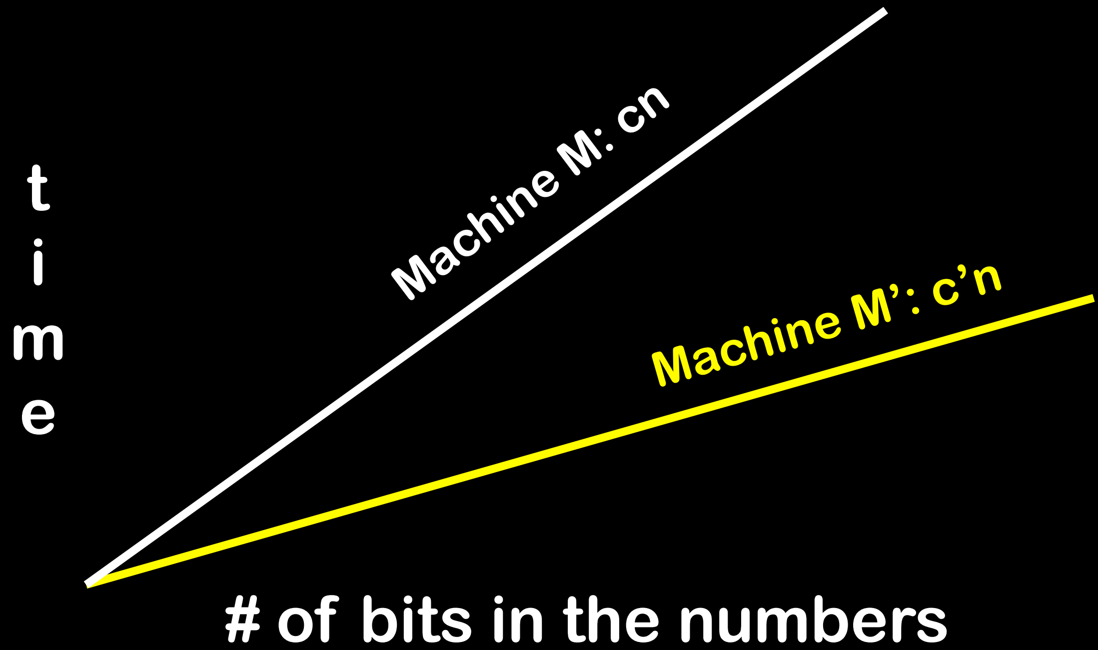
Total time to add two n-bit numbers using grade school addition:

On another computer M', the time to perform [addition-step icon] may be c'.

Total time to add two n-bit numbers using grade school addition:

c'n [c' time for each of n columns]


Machine M: cn
Machine M': c'n
t i m e
# of bits in the numbers



The fact that we get a line is invariant under changes of implementations. Different machines result in different slopes, but the time taken grows linearly as input size increases.

Thus we arrive at an implementation-independent insight:

Grade School Addition is a linear time algorithm

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Grade School Addition is a linear time algorithm

This process of abstracting away details and determining the rate of resource usage in terms of the problem size n is one of the fundamental ideas in computer science.

Time vs Input Size

For any algorithm, define

Input Size = # of bits to specify its inputs.

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$TIME_n$ = the worst-case amount of time used by the algorithm on inputs of size n

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Define

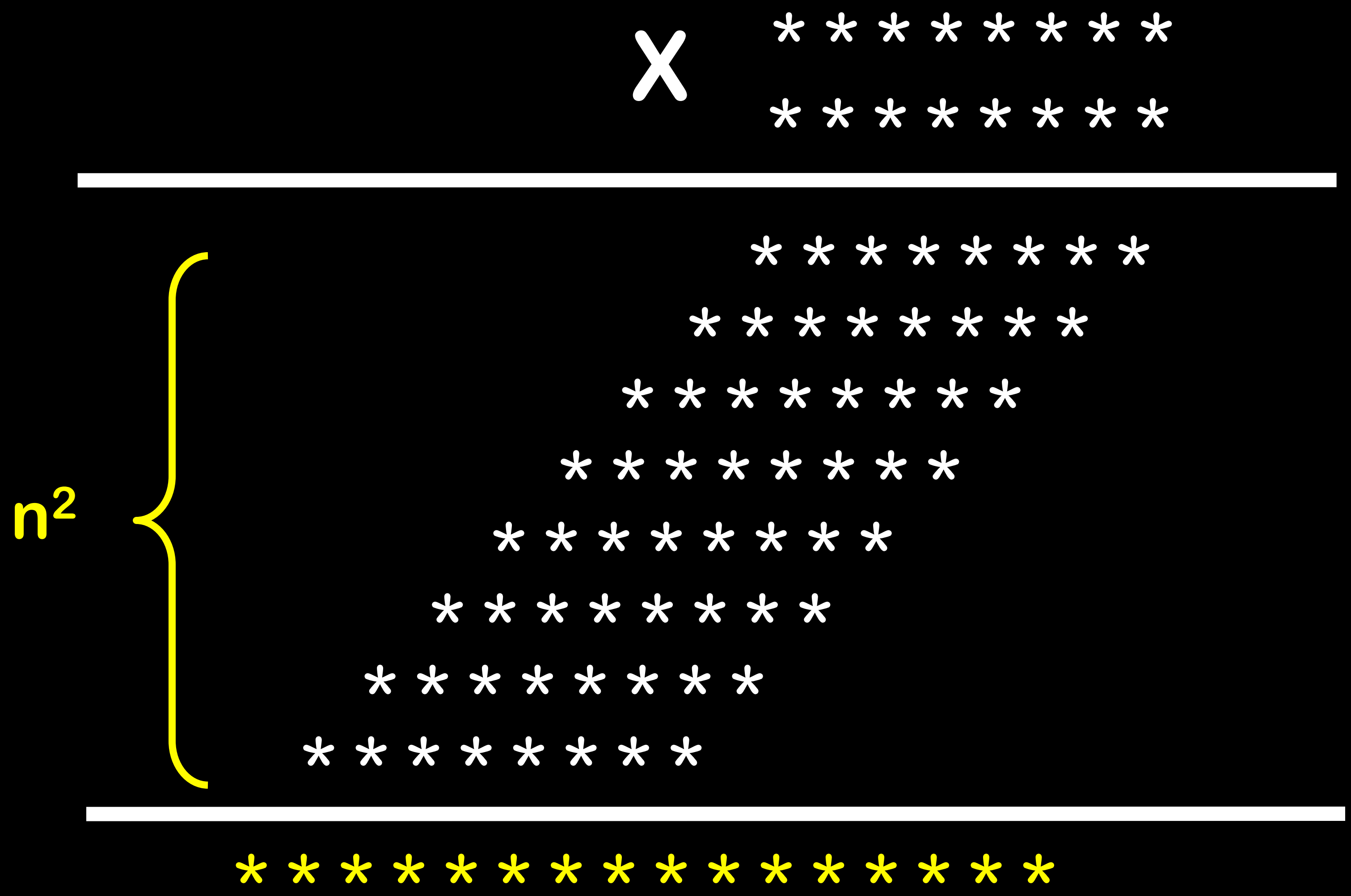
TIME_n = the worst-case amount of time used by the algorithm on inputs of size n

We often ask: **What is the growth rate of Time_n ?**

How to multiply 2 n-bit numbers.

```
                  * * * * * * * *
            X     * * * * * * * *
-----------------------------------
                 * * * * * * * *
               * * * * * * * *
             * * * * * * * *
           * * * * * * * *
         * * * * * * * *
       * * * * * * * *
     * * * * * * * *
   * * * * * * * *
-----------------------------------
   * * * * * * * * * * * * * * * *
```

How to multiply 2 n-bit numbers.

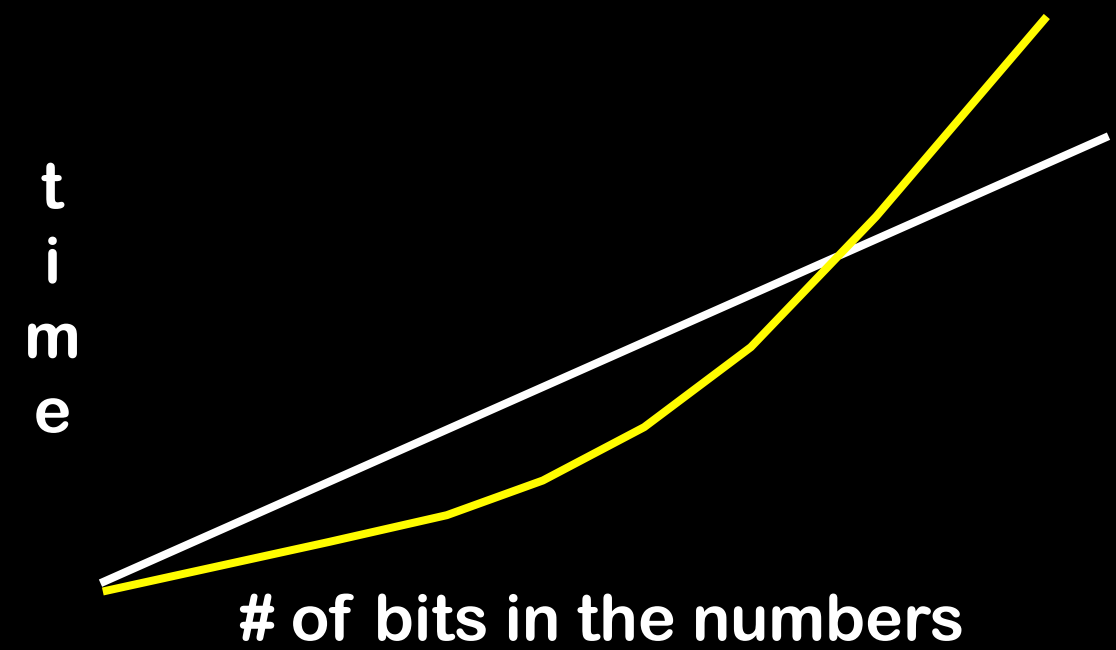


How to multiply 2 n-bit numbers.

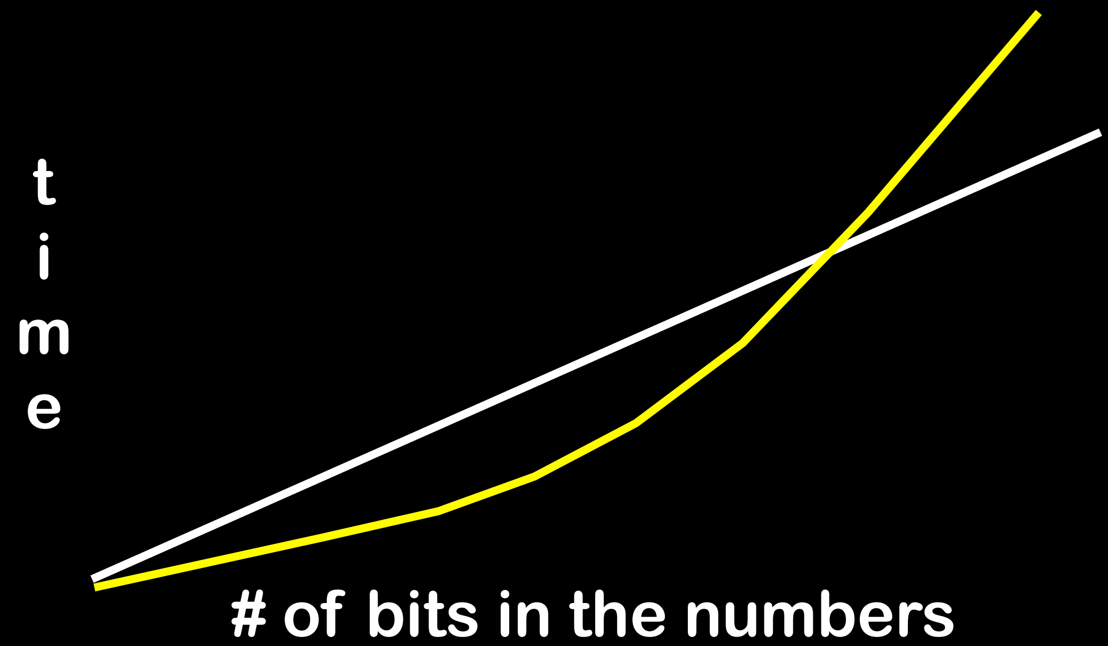
n^2

The total time is bounded by cn^2 (abstracting away the implementation details).

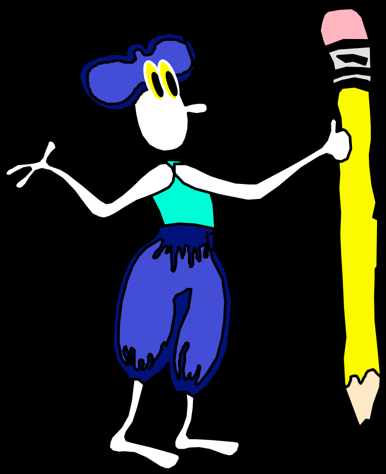
Grade School Addition: Linear time
Grade School Multiplication: Quadratic time



Grade School Addition: Linear time
Grade School Multiplication: Quadratic time

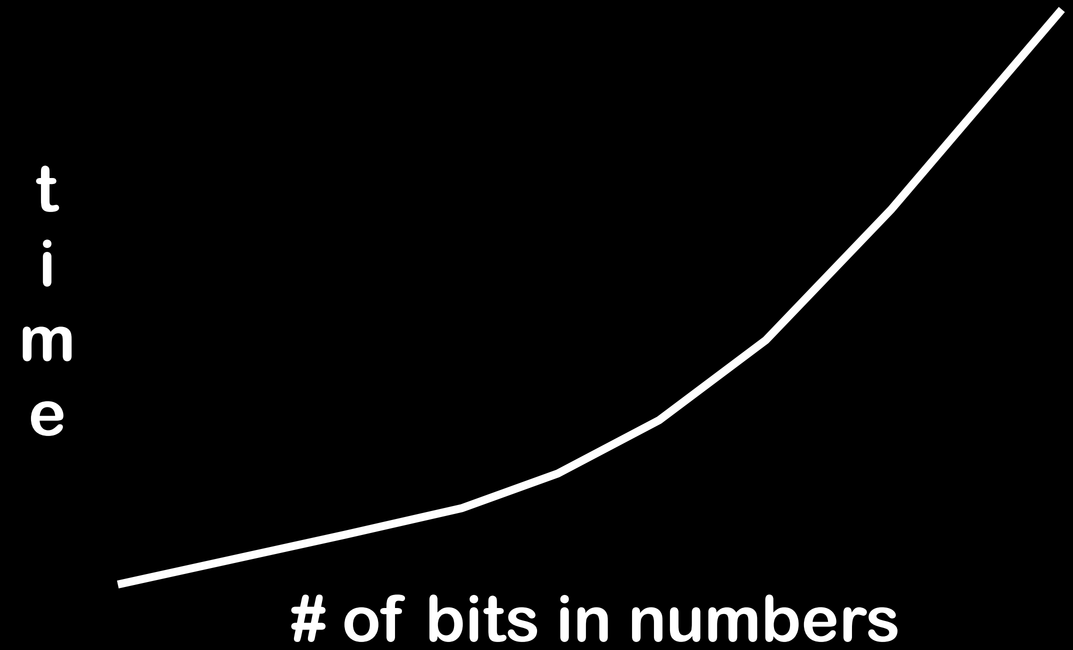


No matter how dramatic the difference in the constants, the quadratic curve will eventually dominate the linear curve



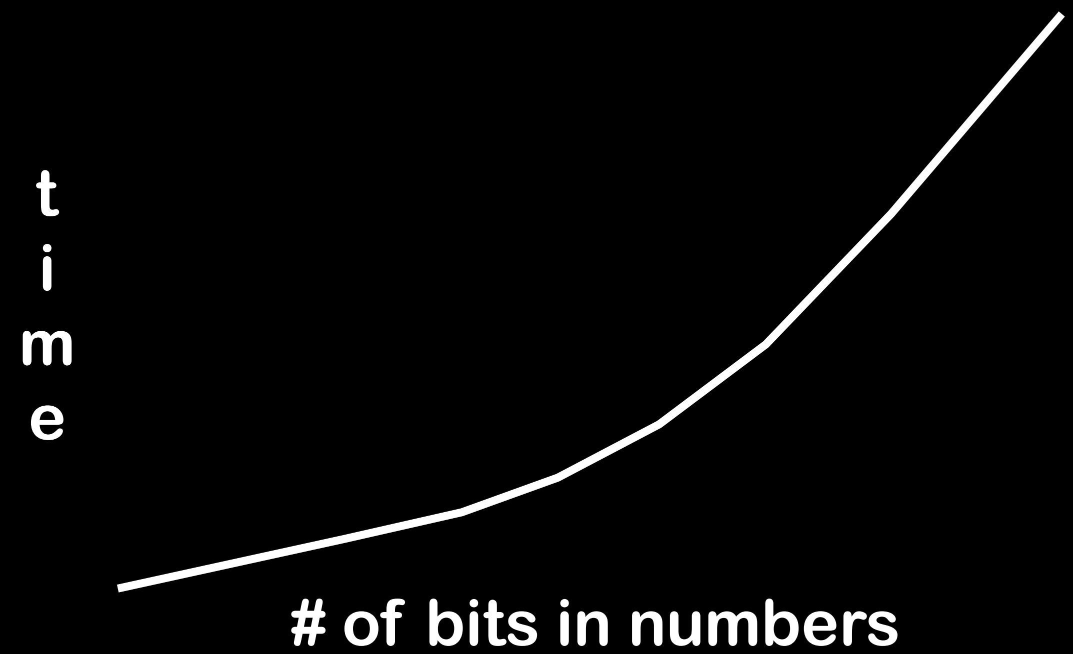
How much time does it take to square the number n using grade school multiplication?

Grade School Multiplication: Quadratic time



$c(\log n)^2$ time to square the number n

Grade School Multiplication: Quadratic time



$c(\log n)^2$ time to square the number n

Input size is measured in bits, unless we say otherwise.

How much time does it take?

Nursery School Addition

Input: Two n-bit numbers, a and b

Output: $a + b$

How much time does it take?

Nursery School Addition

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Start at a and increment (by 1) b times

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$T(n) = ?$

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If $b = 000\ldots0000$, then NSA takes almost no time

How much time does it take?

Nursery School Addition

Input: Two n-bit numbers, a and b

Output: $a + b$

Start at a and increment (by 1) b times

$T(n) = ?$

If $b = 000\ldots0000$, then NSA takes almost no time

If $b = 1111\ldots11111$, then NSA takes $cn2^n$ time

Worst Case Time

Worst Case Time T(n) for algorithm A:

$T(n) = \text{Max}_{[\text{all permissible inputs X of size n}]}$ (Running time of algorithm A on input X).

What is $T(n)$?

Kindergarten Multiplication

Input: Two n-bit numbers, a and b

Output: $a * b$

What is T(n)?

Kindergarten Multiplication

Input: Two n-bit numbers, a and b

Output: a * b

Start with a and add a, b-1 times

What is T(n)?

Kindergarten Multiplication

Input: Two n-bit numbers, a and b

Output: $a * b$

Start with a and add a, $b-1$ times

Remember, we always pick the WORST CASE input for the input size n.

Thus, $T(n) = cn2^n$

Thus, Nursery School adding and Kindergarten multiplication are exponential time.

They scale HORRIBLY as input size grows.

Grade school methods scale polynomially: just linear and quadratic. Thus, we can add and multiply fairly large numbers.

If $T(n)$ is not polynomial, the algorithm is not efficient: the run time scales too poorly with the input size.

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This will be the yardstick with which we will measure "efficiency".

Multiplication is efficient, what about “reverse multiplication”?

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Let's define FACTORING(N) to be any method to produce a non-trivial factor of N, or to assert that N is prime.

Factoring The Number N
By Trial Division

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Trial division up to $\sqrt{N}$

Factoring The Number N By Trial Division

Trial division up to $\sqrt{N}$

```
for k = 2 to √ N do
        if k | N then
        return "N has a non-trivial factor k"
return "N is prime"
```

Factoring The Number N By Trial Division

Trial division up to $\sqrt{N}$

for k = 2 to $\sqrt{N}$ do

if $k \mid N$ then

return "N has a non-trivial factor k"

return "N is prime"

$c \sqrt{N} (\log N)^2$ time if division is $c (\log N)^2$ time

Factoring The Number N By Trial Division

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Is this efficient?

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$c \sqrt{N} (\log N)^2$ time if division is $c (\log N)^2$ time

Is this efficient?

No! The input length $n = \log N$. Hence we're using $c\, 2^{n/2} n^2$ time.

Can we do better?

Can we do better?

We know of methods for FACTORING that are sub-exponential (about $2^{n^{1/3}}$ time) but nothing efficient.

Notation to Discuss Growth Rates

For any monotonic function **f** from the positive integers to the positive integers, we say

"$f = O(n)$" or "f is $O(n)$"

Notation to Discuss Growth Rates

For any monotonic function f from the positive integers to the positive integers, we say

“f = O(n)” or “f is O(n)”

If some constant times n eventually dominates f

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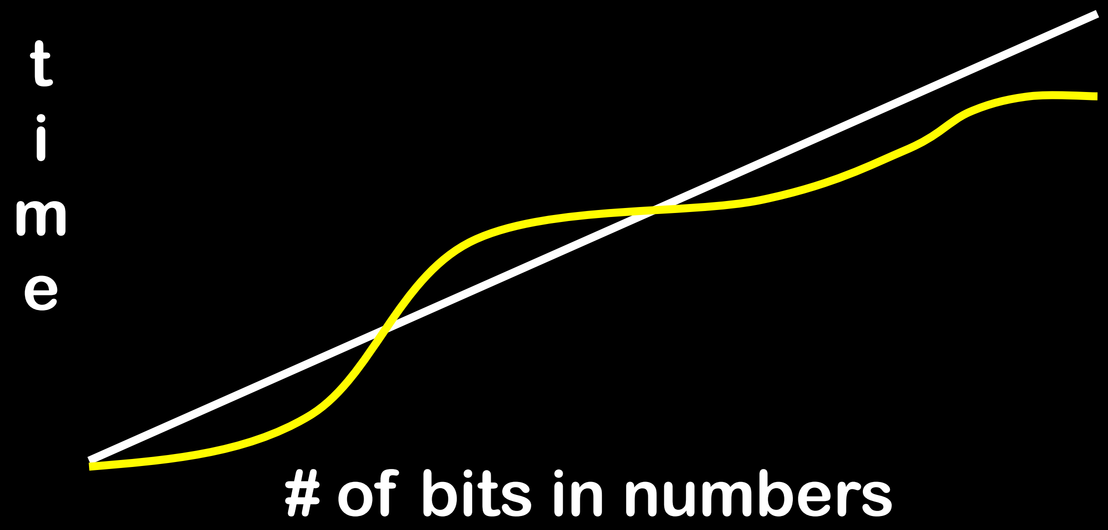
For any monotonic function f from the positive integers to the positive integers, we say

"f = O(n)" or "f is O(n)"

If some constant times n eventually dominates f

[Formally: there exists a constant c such that for all sufficiently large n: $f(n) \leq cn$]

f = O(n) means that there is a line that can be drawn that stays **above** **f** from some point on



Other Useful Notation: Ω

For any monotonic function f from the positive integers to the positive integers, we say

"$f = \Omega(n)$" or "f is $\Omega(n)$"

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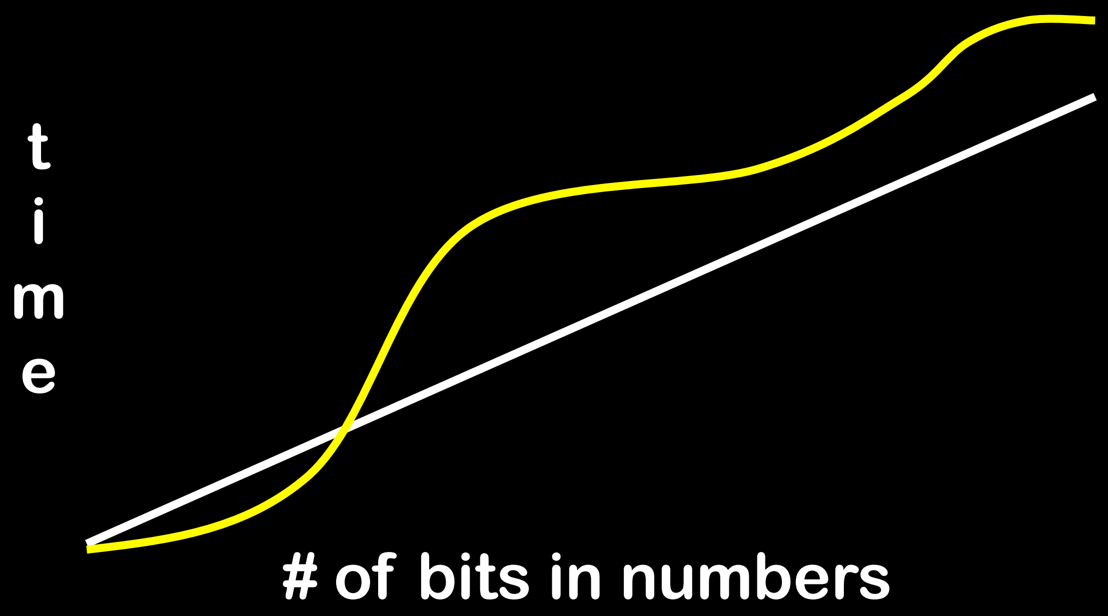
f = Ω(n) means that there is a line that can be drawn that stays below f from some point on



Yet More Useful Notation: Θ

For any monotonic function f from the positive integers to the positive integers, we say

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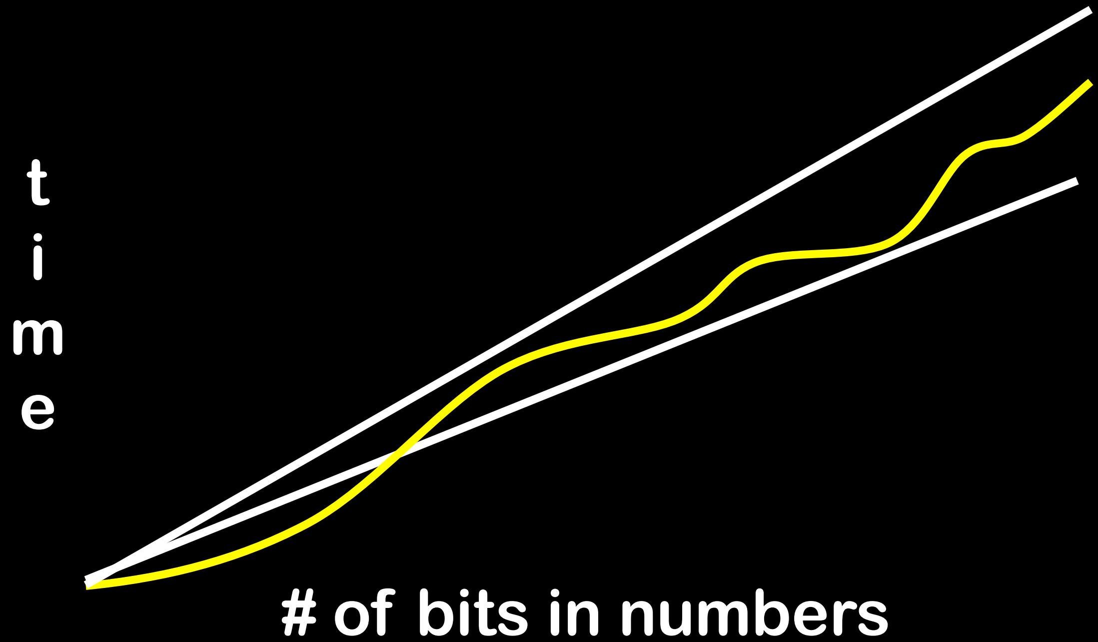
Yet More Useful Notation: Θ

For any monotonic function f from the positive integers to the positive integers, we say

"$f = \Theta(n)$" or "f is $\Theta(n)$"

if: $f = O(n)$ and $f = \Omega(n)$

$f = \Theta(n)$ means that f can be sandwiched between two lines from some point on.



Notation to Discuss Growth Rates

For any two monotonic functions **f and g** from the positive integers to the positive integers, we say

“**f = O(g)**” or “**f is O(g)**”

Notation to Discuss Growth Rates

For any two monotonic functions f and g from the positive integers to the positive integers, we say

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Notation to Discuss Growth Rates

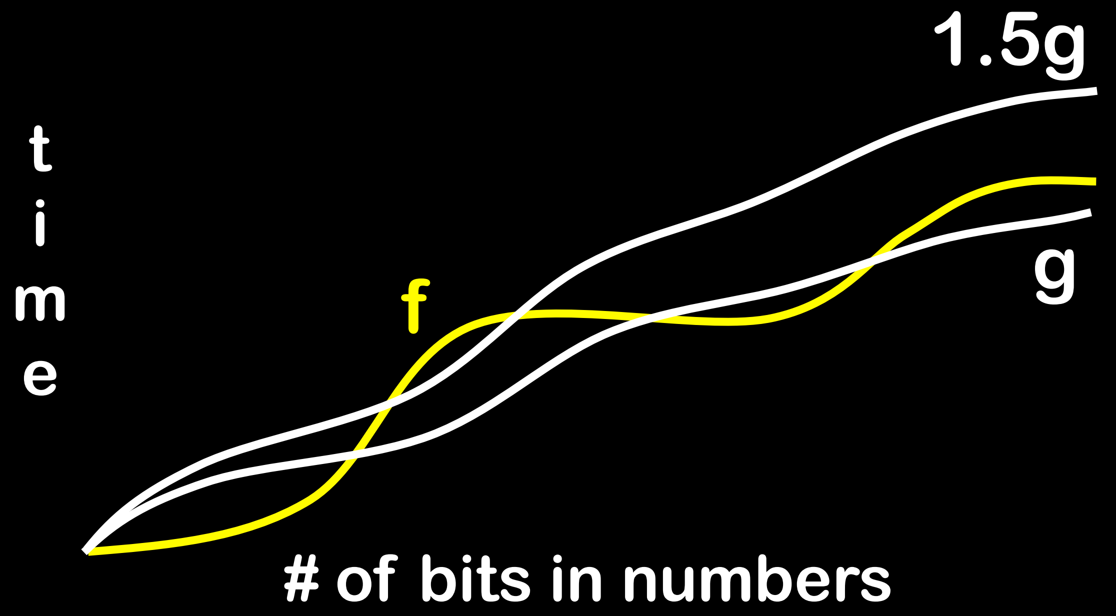
For any two monotonic functions f and g from the positive integers to the positive integers, we say

"f = O(g)" or "f is O(g)"

If some constant times g eventually dominates f

[Formally: there exists a constant c such that for all sufficiently large n: $f(n) \leq c\, g(n)$]

f = O(g) means that there is some constant c such that c g(n) stays above f(n) from some point on.



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For any two monotonic functions **f and g** from the positive integers to the positive integers, we say

“**$f = \Omega(g)$**” or “**f is $\Omega(g)$**”

Other Useful Notation: Ω

For any two monotonic functions f and g from the positive integers to the positive integers, we say

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If f eventually dominates some constant times g

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If f eventually dominates some constant times g

[Formally: there exists a constant c such that for all sufficiently large n: $f(n) \geq c\, g(n)$]

Yet More Useful Notation: Θ

For any two monotonic functions f and g from the positive integers to the positive integers, we say

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Yet More Useful Notation: Θ

For any two monotonic functions f and g from the positive integers to the positive integers, we say

"$f = \Theta(g)$" or "f is $\Theta(g)$"

If: $f = O(g)$ and $f = \Omega(g)$

- $n = O(n^2)$?

- $n = O(n^2)$?   Yes!

- $n = O(n^2)$? **Yes!**

Take $c = 1$

For all $n \geq 1$, it holds that $n \leq cn^2$

- $n = O(n^2)$? Yes!

- $n = O(\sqrt{n})$?

- $n = O(n^2)$? **Yes!**

- $n = O(\sqrt{n})$? **No**

- $n = O(n^2)$? Yes!

- $n = O(\sqrt{n})$? No

Suppose it were true that $n \le c \sqrt{n}$
for some constant c and large enough n

- $n = O(n^2)$? **Yes!**
- $n = O(\sqrt{n})$? **No**

Suppose it were true that $n \leq c \sqrt{n}$
for some constant c and large enough n

Cancelling, we would get $\sqrt{n} \leq c$.

Which is false for $n > c^2$

- $n = O(n^2)$? Yes!
- $n = O(\sqrt{n})$? No
- $3n^2 + 4n + 4 = O(n^2)$?
- $3n^2 + 4n + 4 = \Omega(n^2)$?
- $n^2 = \Omega(n \log n)$?
- $n^2 \log n = \Theta(n^2)$?

- $n = O(n^2)$? Yes!
- $n = O(\sqrt{n})$? No
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- $n^2 = \Omega(n \log n)$? Yes!
- $n^2 \log n = \Theta(n^2)$? No

- $f = O(g)$ and $g = O(h)$

  then $f = O(h)$?

- $f = O(g)$

  then $g = \Omega(f)$

- $f = O(g)$ and $g = O(h)$

  then $f = O(h)$?

Yes!

- $f = O(g)$

  then $g = \Omega(f)$

- $f = O(g)$ and $g = O(h)$

  then $f = O(h)$?

Yes!

$f(n) \leq c\, g(n)$ for all $n \geq n_0$.
and $g(n) \leq c'\, h(n)$ for all $n \geq n_0'$.
So $f(n) \leq (cc')\, h(n)$ for all $n \geq \max(n_0, n_0')$

- $f = O(g)$

  then $g = \Omega(f)$

- $f = O(g)$ and $g = O(h)$

  then $f = O(h)$?

Yes!

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and $g(n) \leq c'\, h(n)$ for all $n \geq n_0'$.

So $f(n) \leq (cc')\, h(n)$ for all $n \geq \max(n_0, n_0')$

- $f = O(g)$

  then $g = \Omega(f)$

Yes!

Names For Some Growth Rates

Linear Time: $T(n) = O(n)$

Quadratic Time: $T(n) = O(n^2)$

Cubic Time: $T(n) = O(n^3)$

Names For Some Growth Rates

Linear Time: $T(n) = O(n)$

Quadratic Time: $T(n) = O(n^2)$

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Polynomial Time:

Names For Some Growth Rates

Linear Time: $T(n) = O(n)$

Quadratic Time: $T(n) = O(n^2)$

Cubic Time: $T(n) = O(n^3)$

Polynomial Time:

for some constant k, $T(n) = O(n^k)$.

Example: $T(n) = 13n^5$

Large Growth Rates

Exponential Time:

Large Growth Rates

Exponential Time:

for some constant k, $T(n) = O(k^n)$

Example: $T(n) = n2^n = O(3^n)$

Small Growth Rates

Logarithmic Time: $T(n) = O(\log n)$

Example: $T(n) = 15\log_2(n)$

Small Growth Rates

Logarithmic Time: $T(n) = O(\log n)$

Example: $T(n) = 15\log_2(n)$

Polylogarithmic Time:

for some constant k, $T(n) = O(\log^k(n))$

Small Growth Rates

Logarithmic Time: $T(n) = O(\log n)$
Example: $T(n) = 15\log_2(n)$

Polylogarithmic Time:
for some constant k, $T(n) = O(\log^k(n))$

Note: These kind of algorithms can't possibly read all of their inputs.

Small Growth Rates

Logarithmic Time: $T(n) = O(\log n)$
Example: $T(n) = 15\log_2(n)$

Polylogarithmic Time:
for some constant k, $T(n) = O(\log^k(n))$

Note: These kind of algorithms can’t possibly read all of their inputs.

A very common example of logarithmic time is looking up a word in a sorted dictionary (binary search)

Some Big Ones

Doubly Exponential Time means that for some constant k

Some Big Ones

Doubly Exponential Time means that for some constant k

$$T(n) = 2^{2^{kn}}$$

Some Big Ones

Doubly Exponential Time means that for some constant k

$$T(n) = 2^{2^{k\,n}}$$

Triply Exponential

Some Big Ones

Doubly Exponential Time means that for some constant k

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Triply Exponential

$$T(n) = 2^{2^{2^{kn}}}$$

Faster and Faster: 2STACK

$2STACK(0) = 1$

$2STACK(n) = 2^{2STACK(n-1)}$

Faster and Faster: 2STACK

$\text{2STACK}(0) = 1$

$\text{2STACK}(n) = 2^{\text{2STACK}(n-1)}$

$\text{2STACK}(1) = 2$

Faster and Faster: 2STACK

$2STACK(0) = 1$
$2STACK(n) = 2^{2STACK(n-1)}$

$2STACK(1) = 2$

$2STACK(2) = 4$

Faster and Faster: 2STACK

$2STACK(0) = 1$
$2STACK(n) = 2^{2STACK(n-1)}$

$2STACK(1) = 2$

$2STACK(2) = 4$

$2STACK(3) = 16$

Faster and Faster: 2STACK

2STACK(0) = 1
2STACK(n) = $2^{\text{2STACK(n-1)}}$

2STACK(1) = 2

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2STACK(3) = 16

2STACK(4) = 65536

Faster and Faster: 2STACK

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$2STACK(5) \geq 10^{80}$

Faster and Faster: 2STACK

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2STACK(5) $\geq 10^{80}$
= atoms in universe

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2STACK(1) = 2

2STACK(2) = 4

2STACK(3) = 16

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2STACK(5) $\geq 10^{80}$
= atoms in universe

$$\text{2STACK(n)} = 2^{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}$$

"tower of n 2's"

And the inverse of 2STACK: log*

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2STACK(1) = 2

2STACK(2) = 4

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2STACK(4) = 65536

2STACK(5) $\geq 10^{80}$

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So an algorithm that can be shown to run in $O(n \log^* n)$ Time is Linear Time for all practical purposes!!

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	n=0	1	2	3	4	5	
m=0							
1							
2							
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5							

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$A(5,2)$ can't be written out as decimal in this universe

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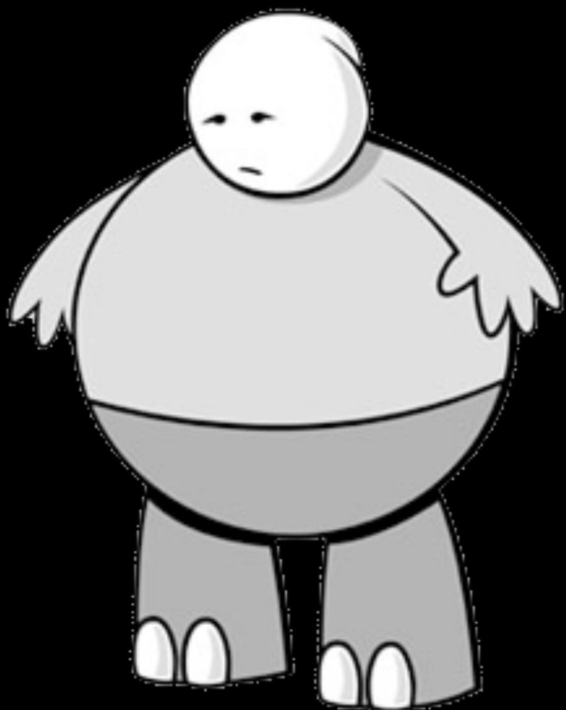
The inverse Ackermann function – in fact, $\Theta(n\,\alpha(n))$ arises in the seminal paper of:

D. D. Sleator and R. E. Tarjan. A data structure for dynamic trees. Journal of Computer and System Sciences, 26(3): 362-391, 1983.



Here's What You Need to Know…

- How is "time" measured
- Definitions of:
  - O, Ω, Θ
  - linear, quadratic time, etc
  - log*(n)
  - Ackerman Function