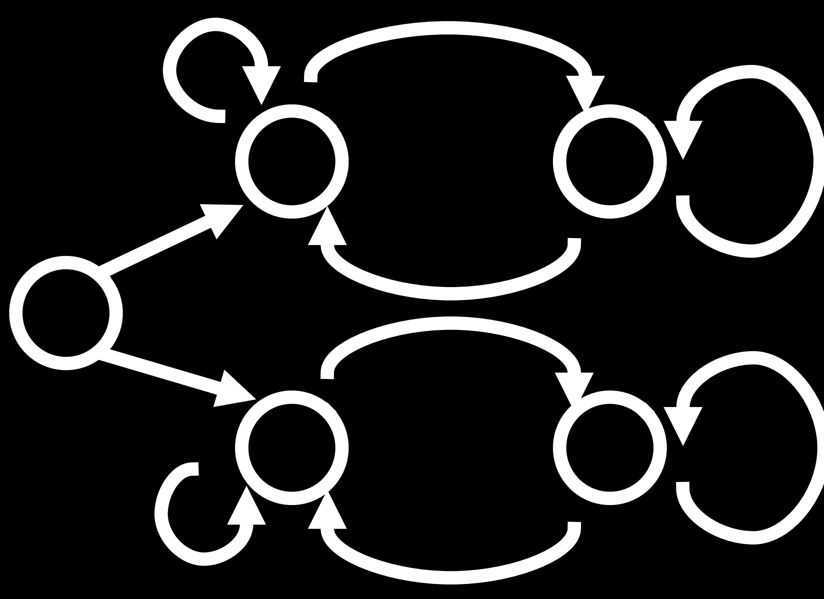
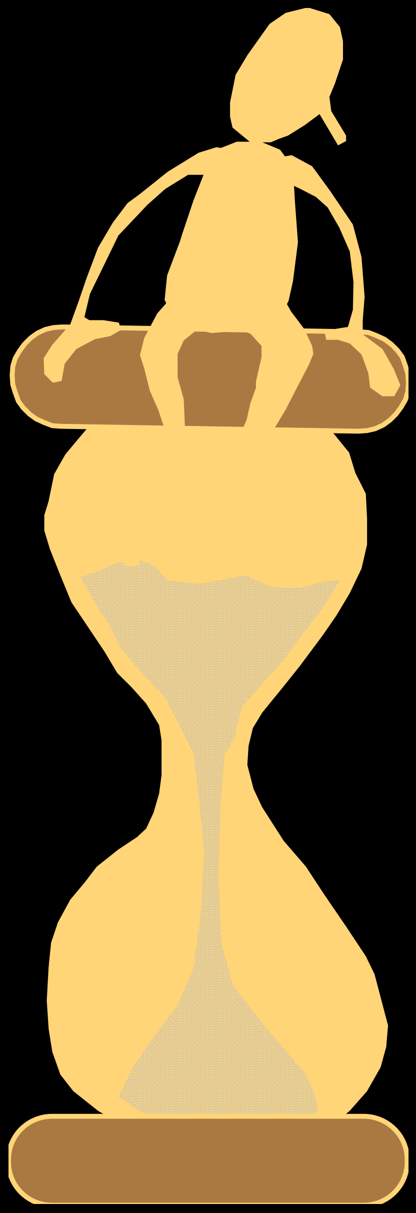
15-251

Great Theoretical Ideas in Computer Science

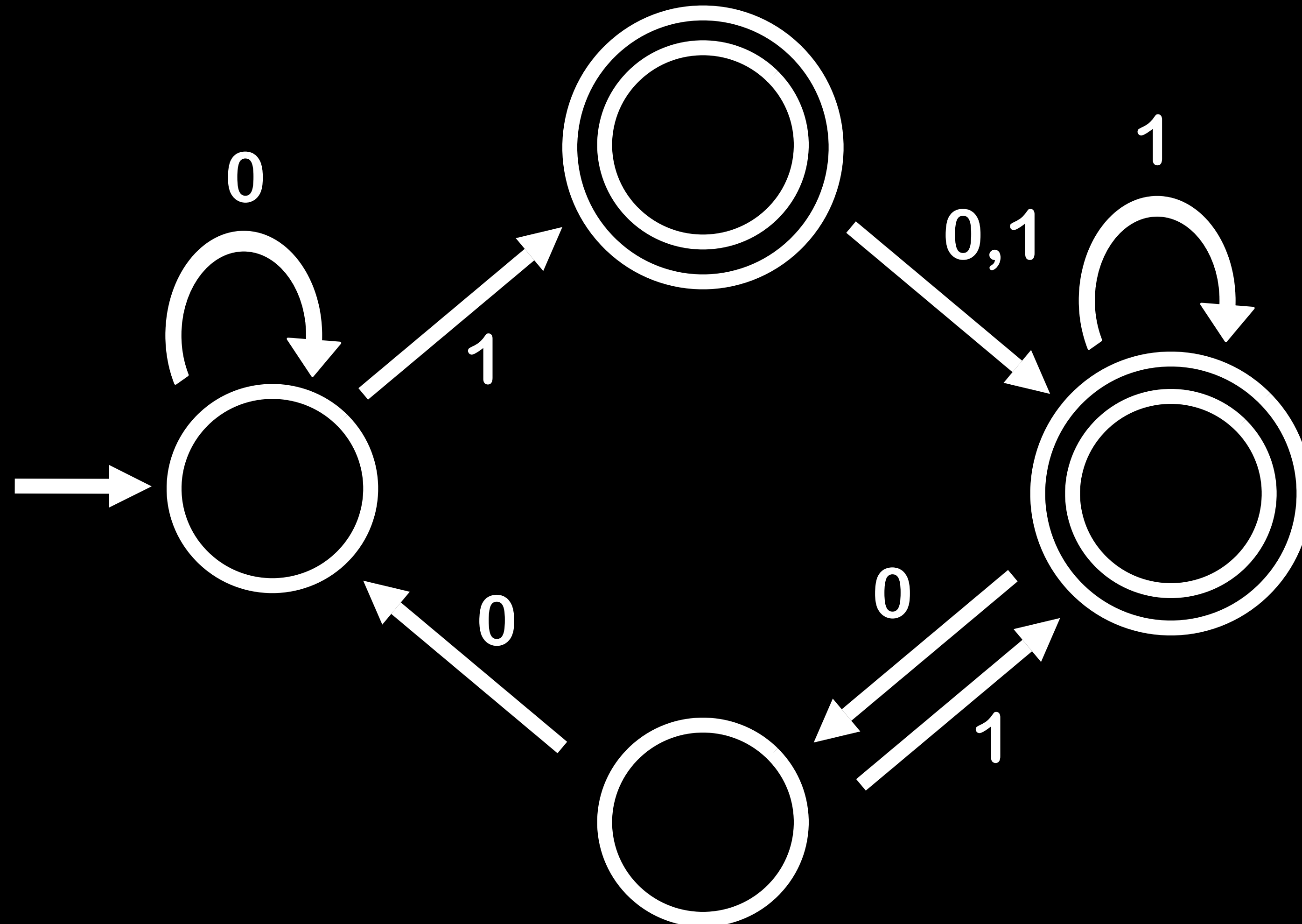
Deterministic Finite Automata

Lecture 20 (October 30, 2008)





Let me show you a machine so simple that you can understand it in less than two minutes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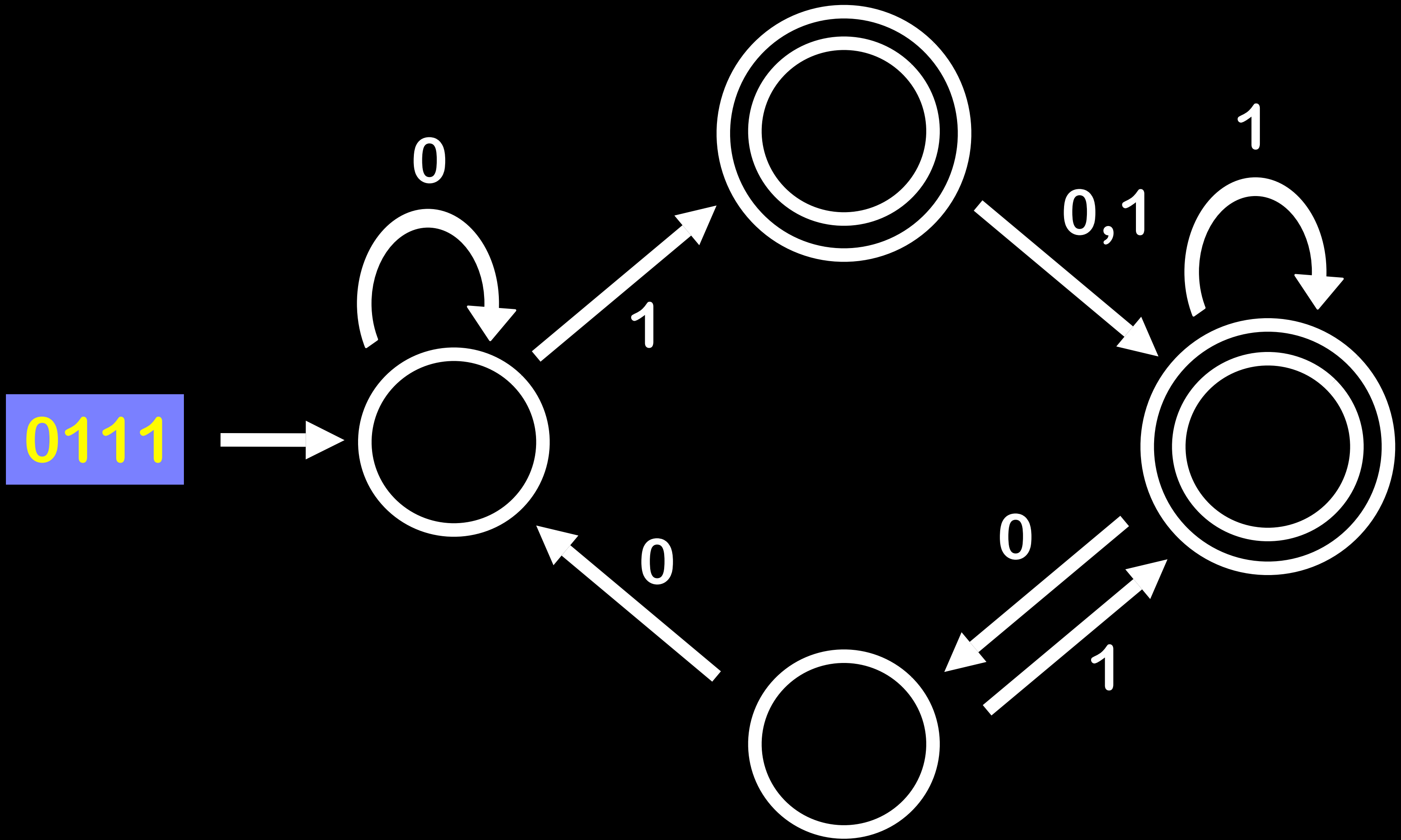
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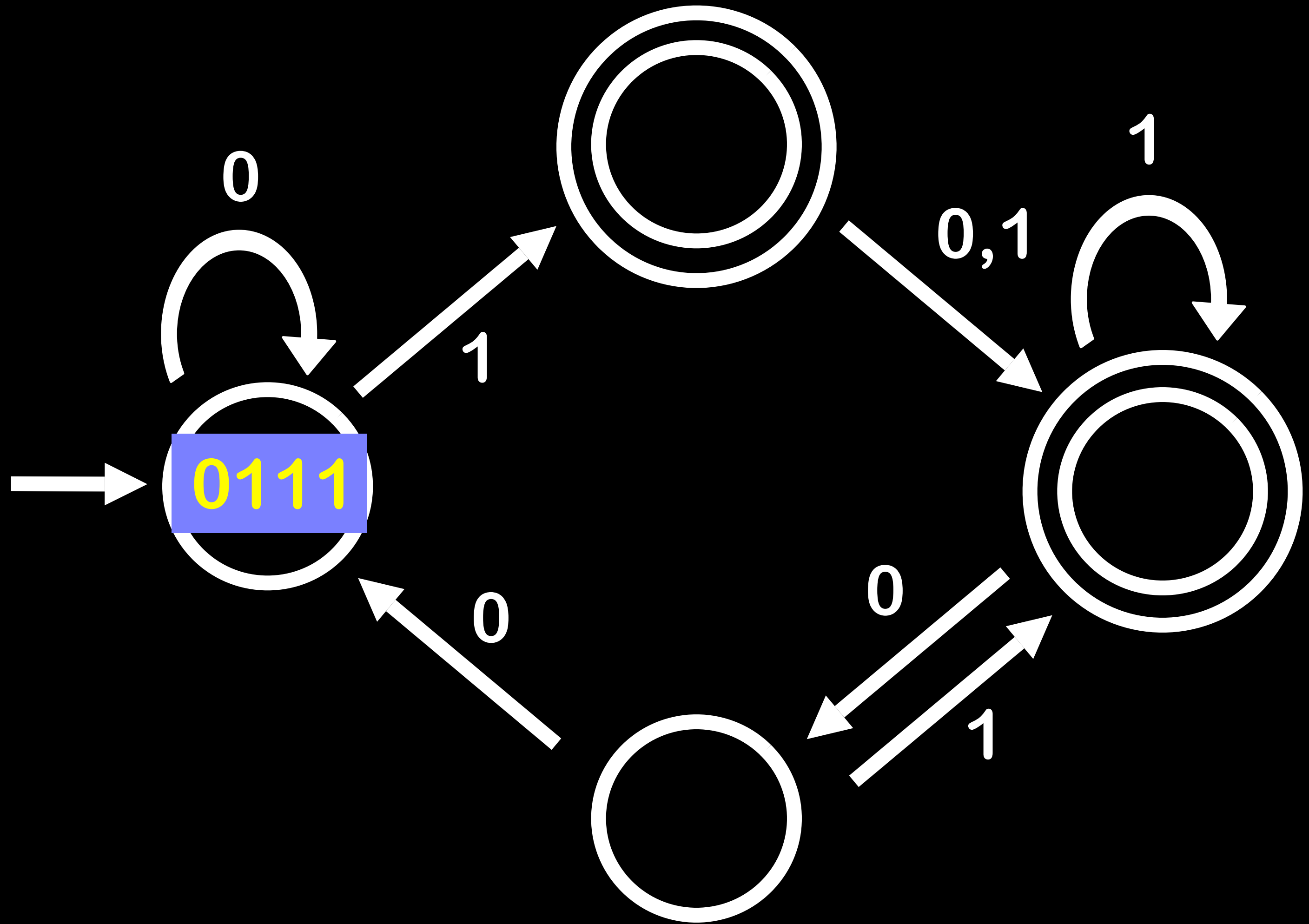

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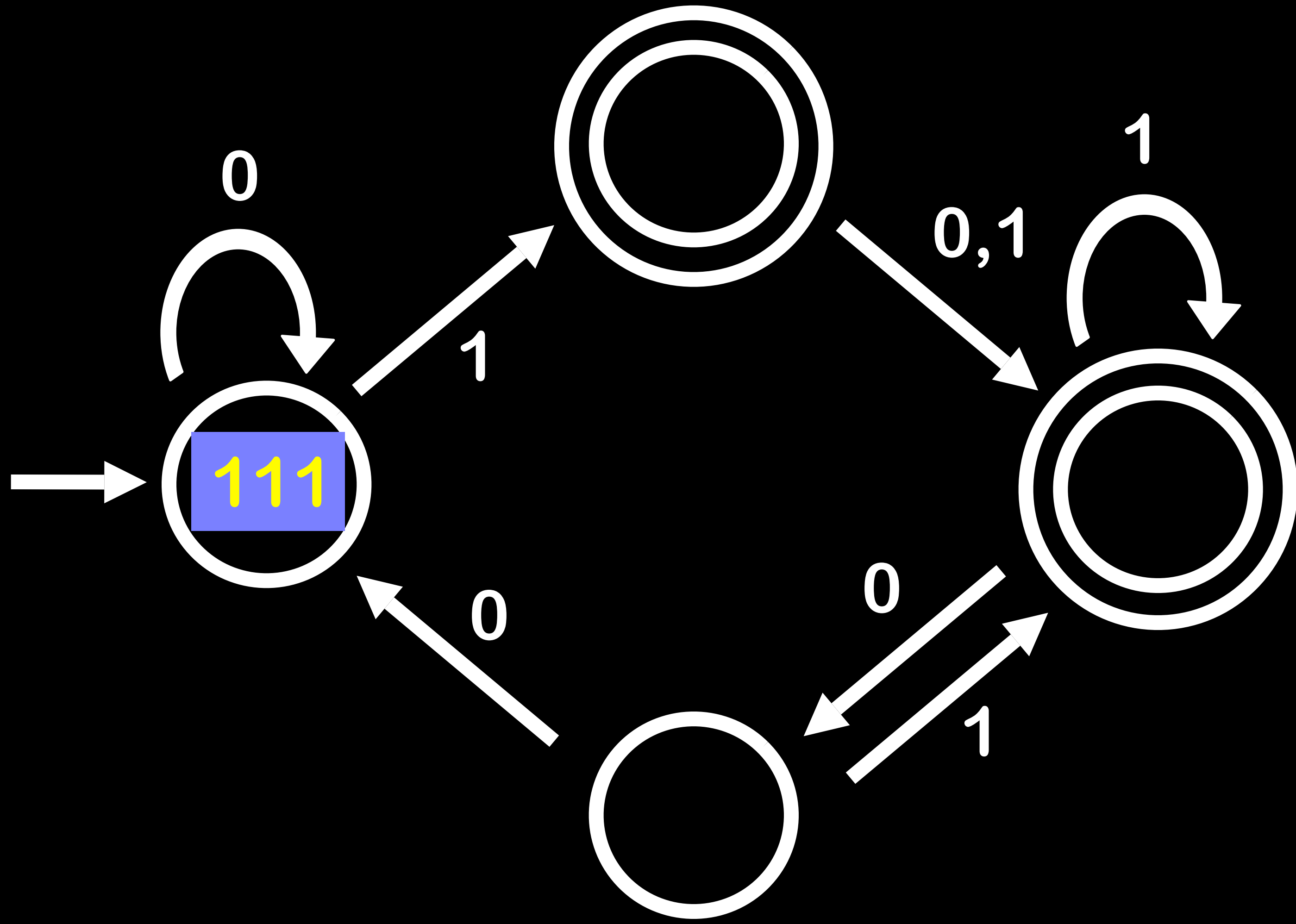
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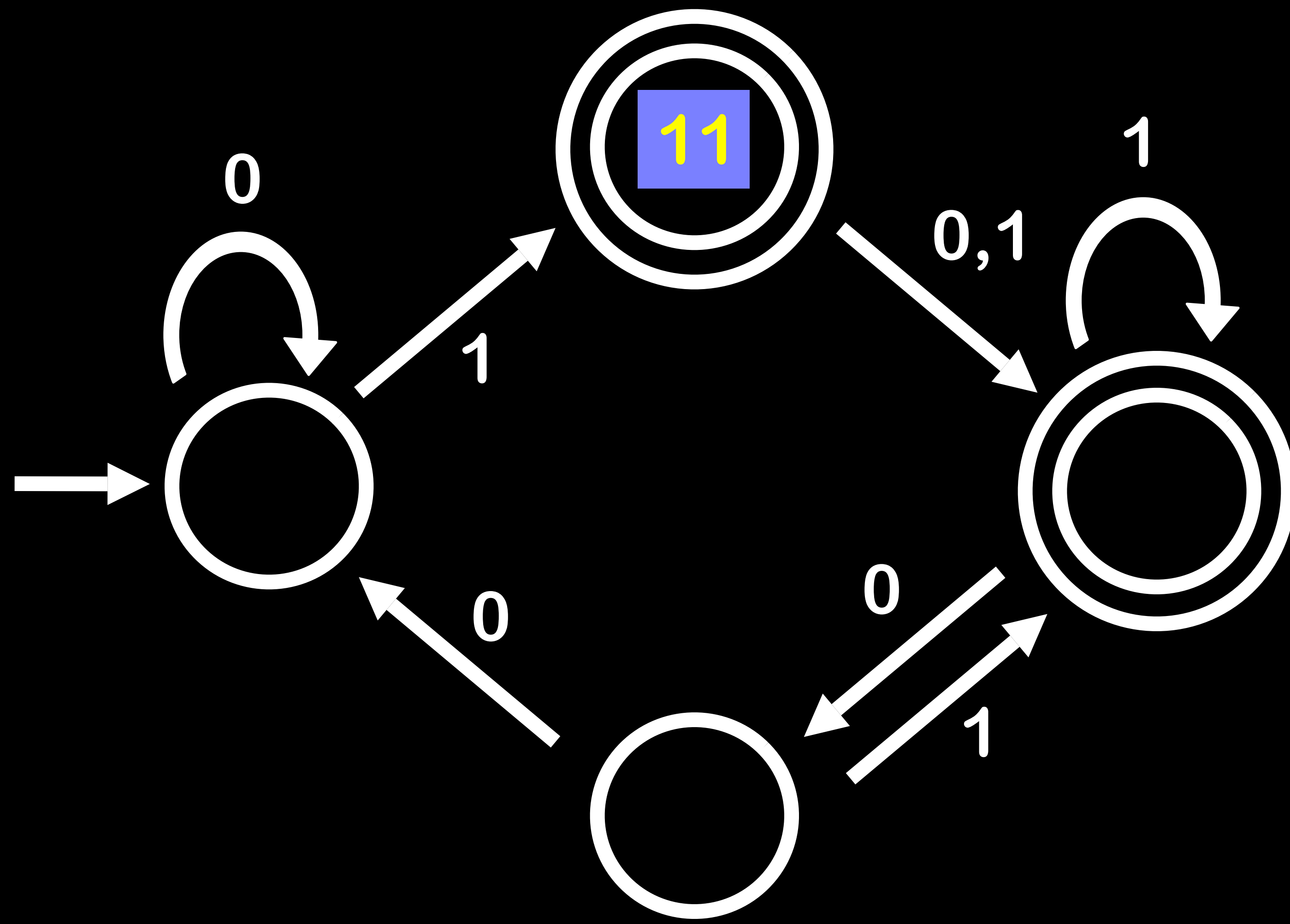

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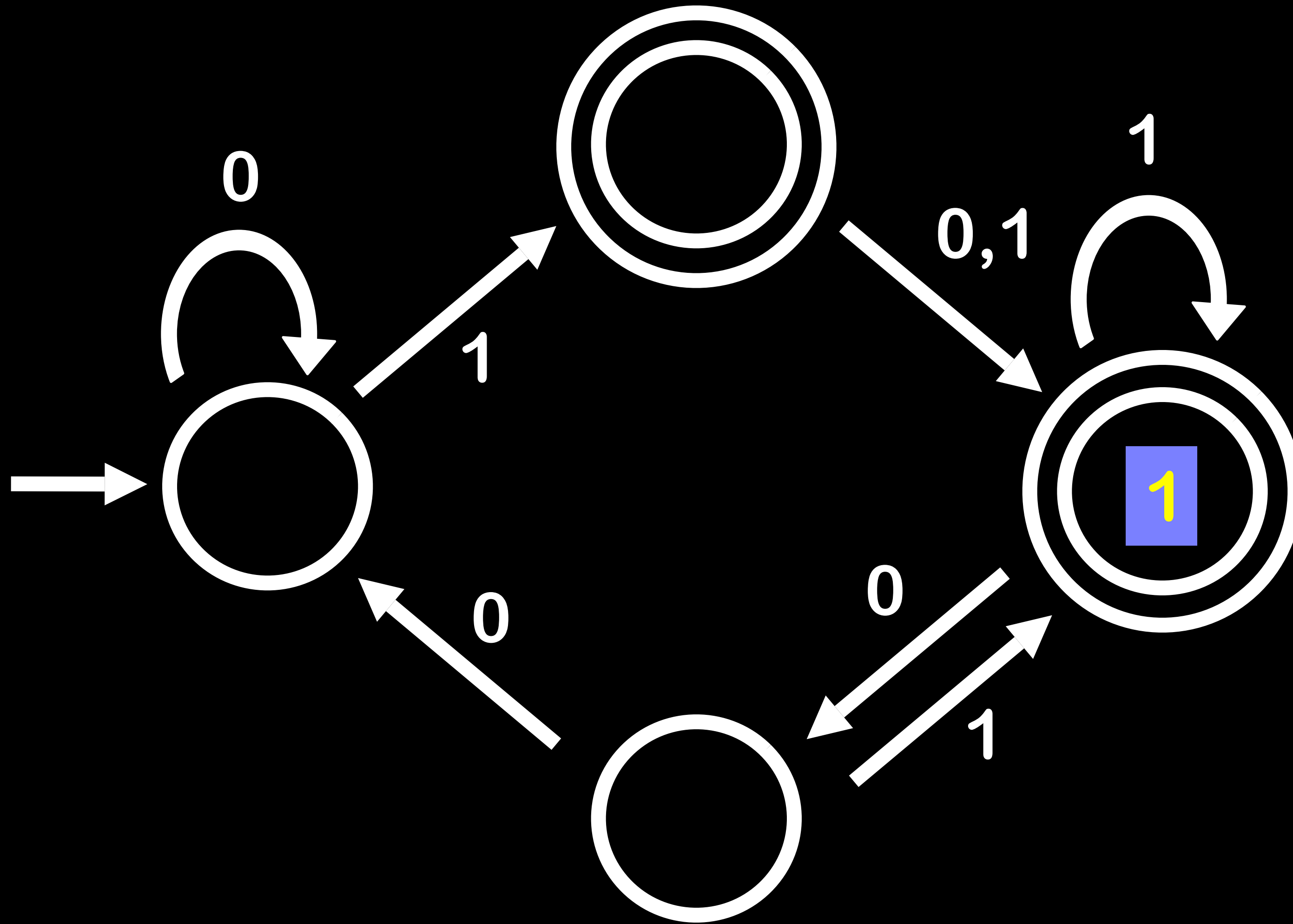
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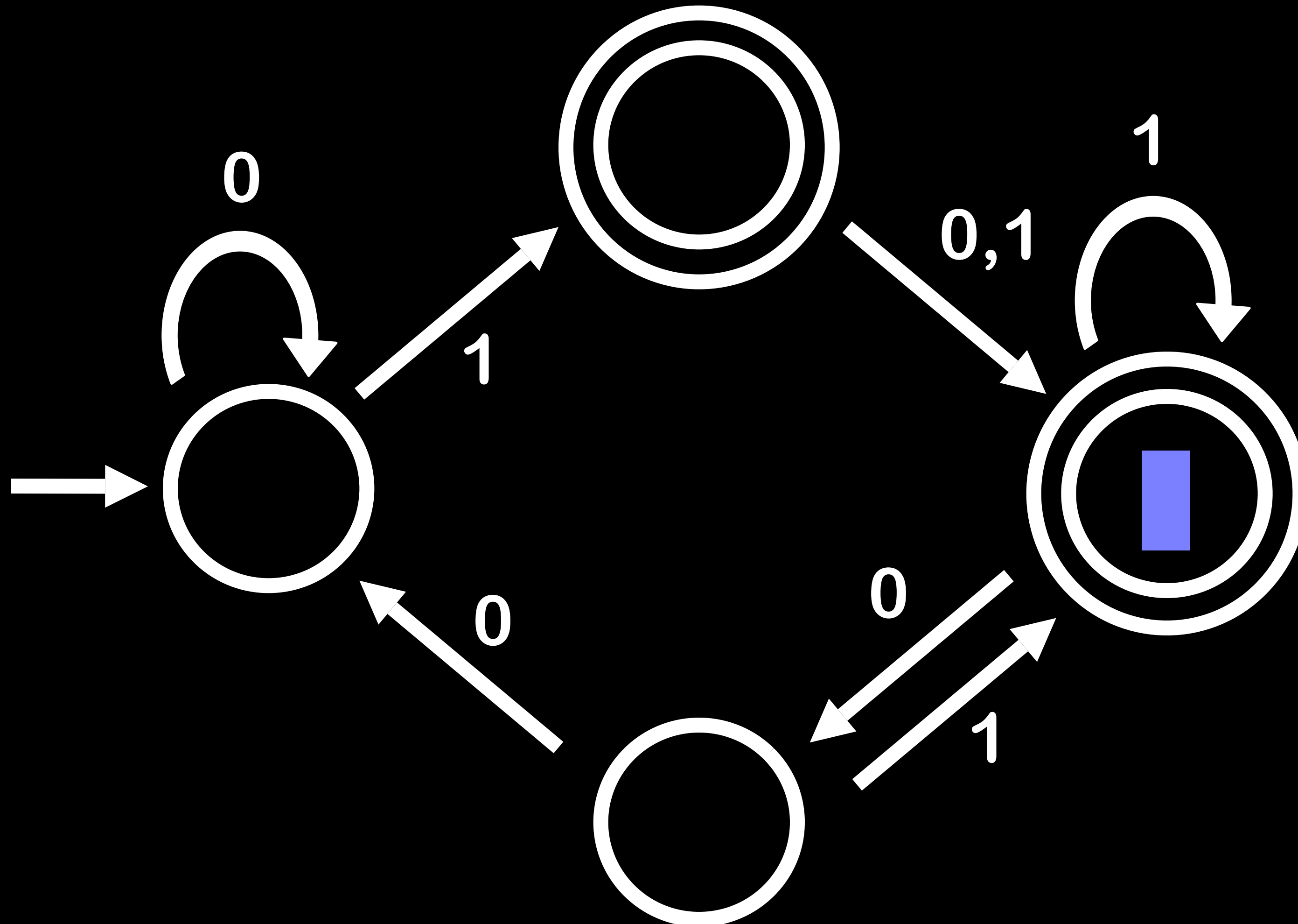
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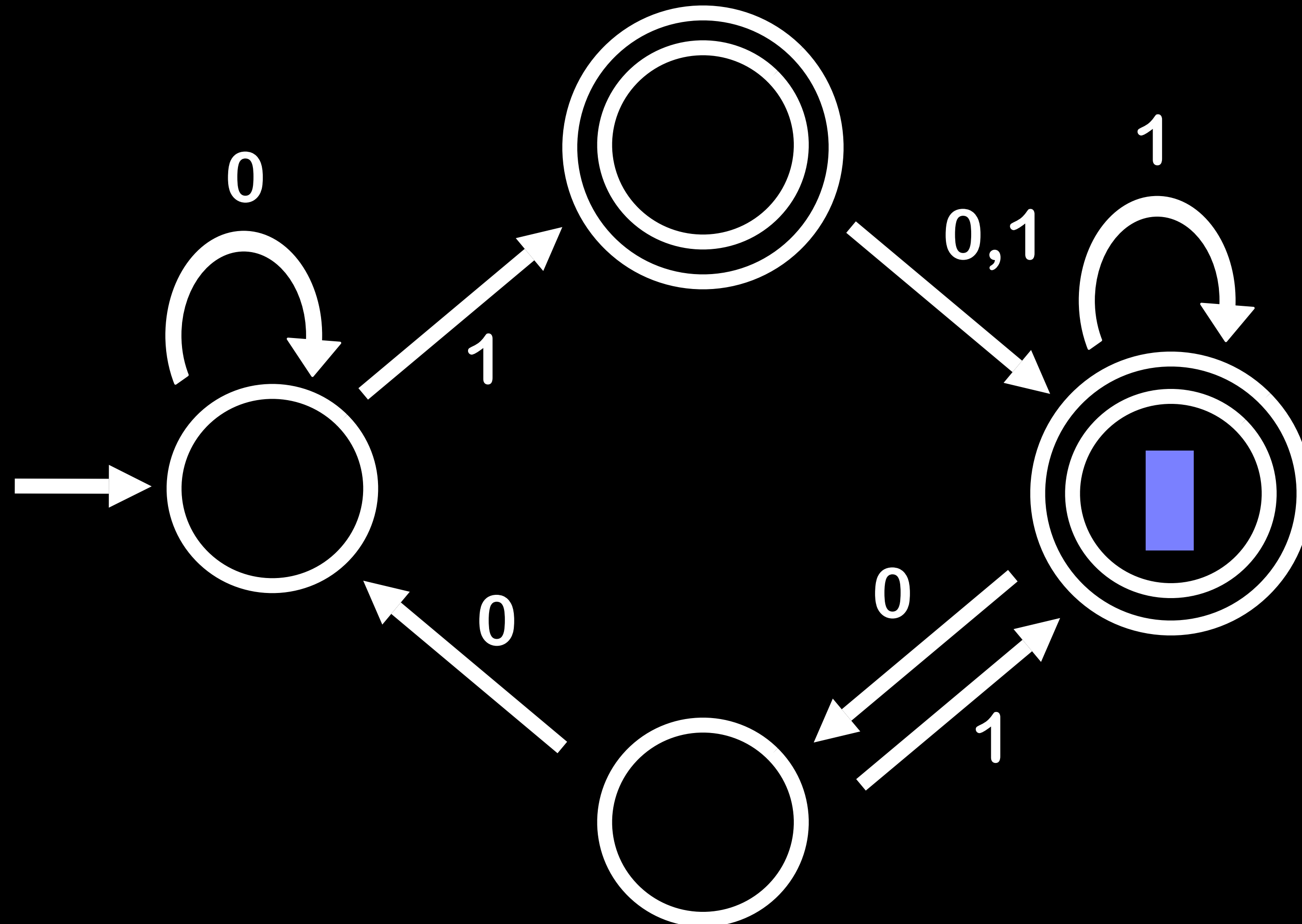
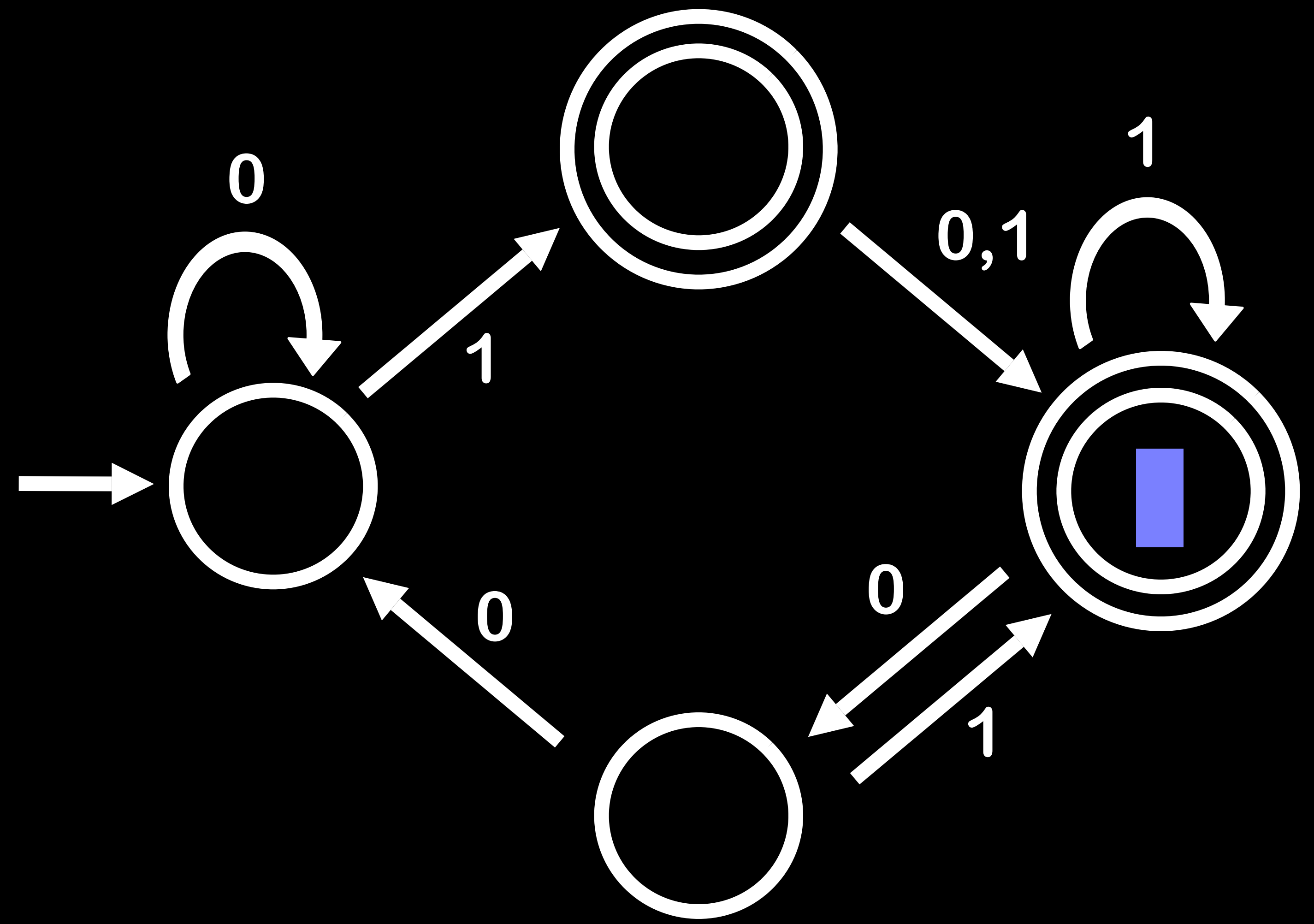


The machine **accepts** a string if the process ends in a double circle



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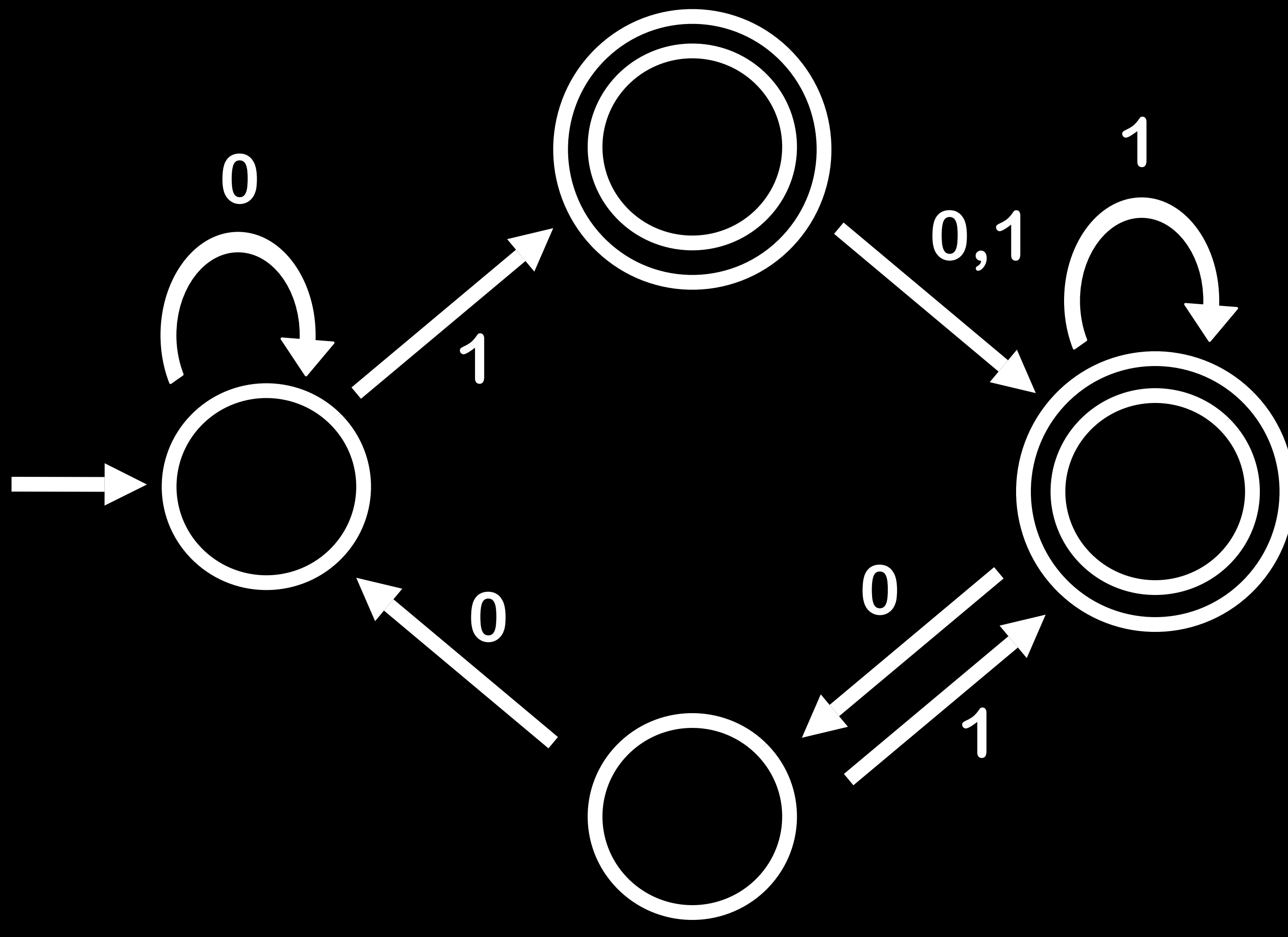
Anatomy of a Deterministic Finite Automaton



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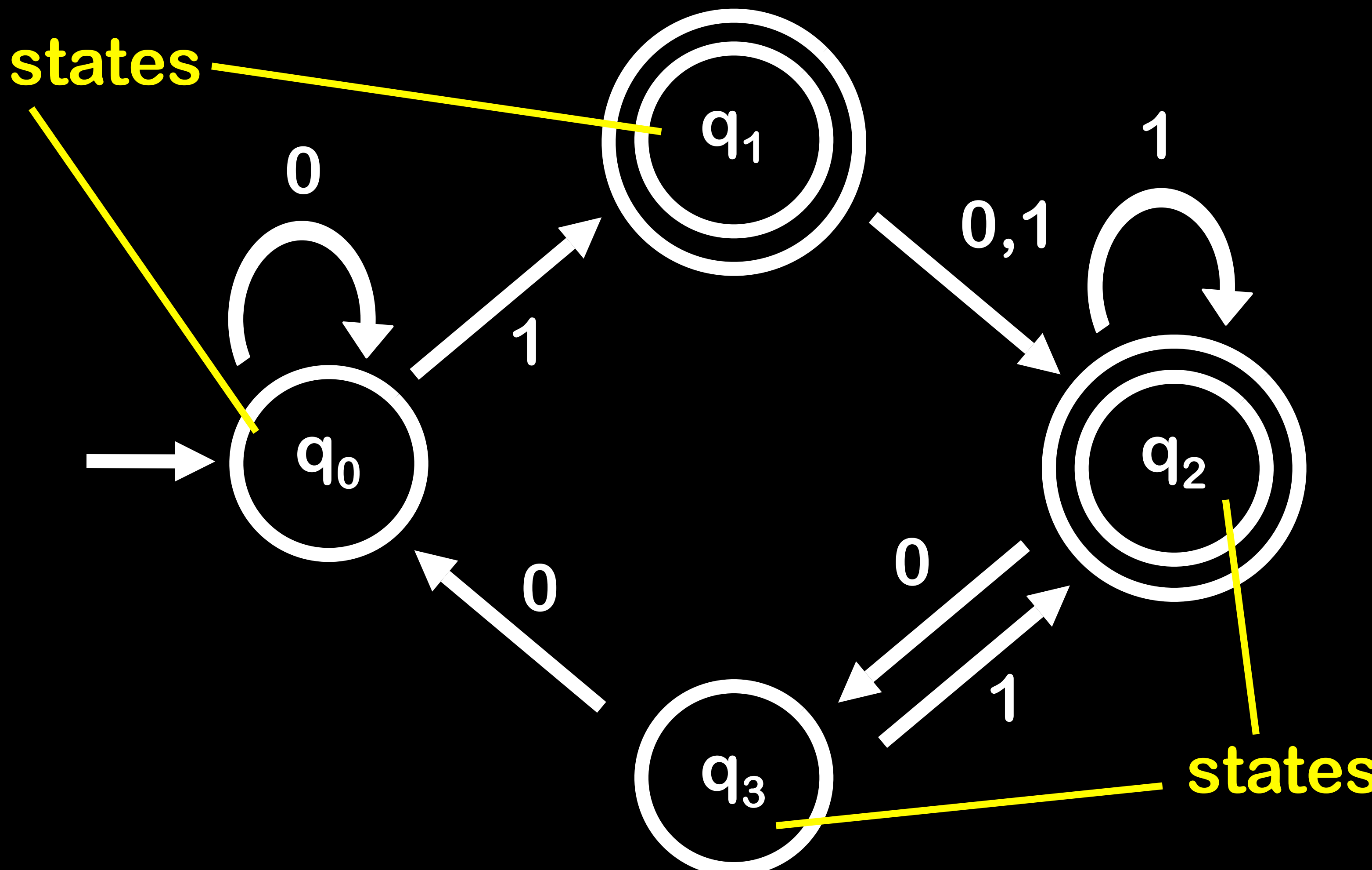

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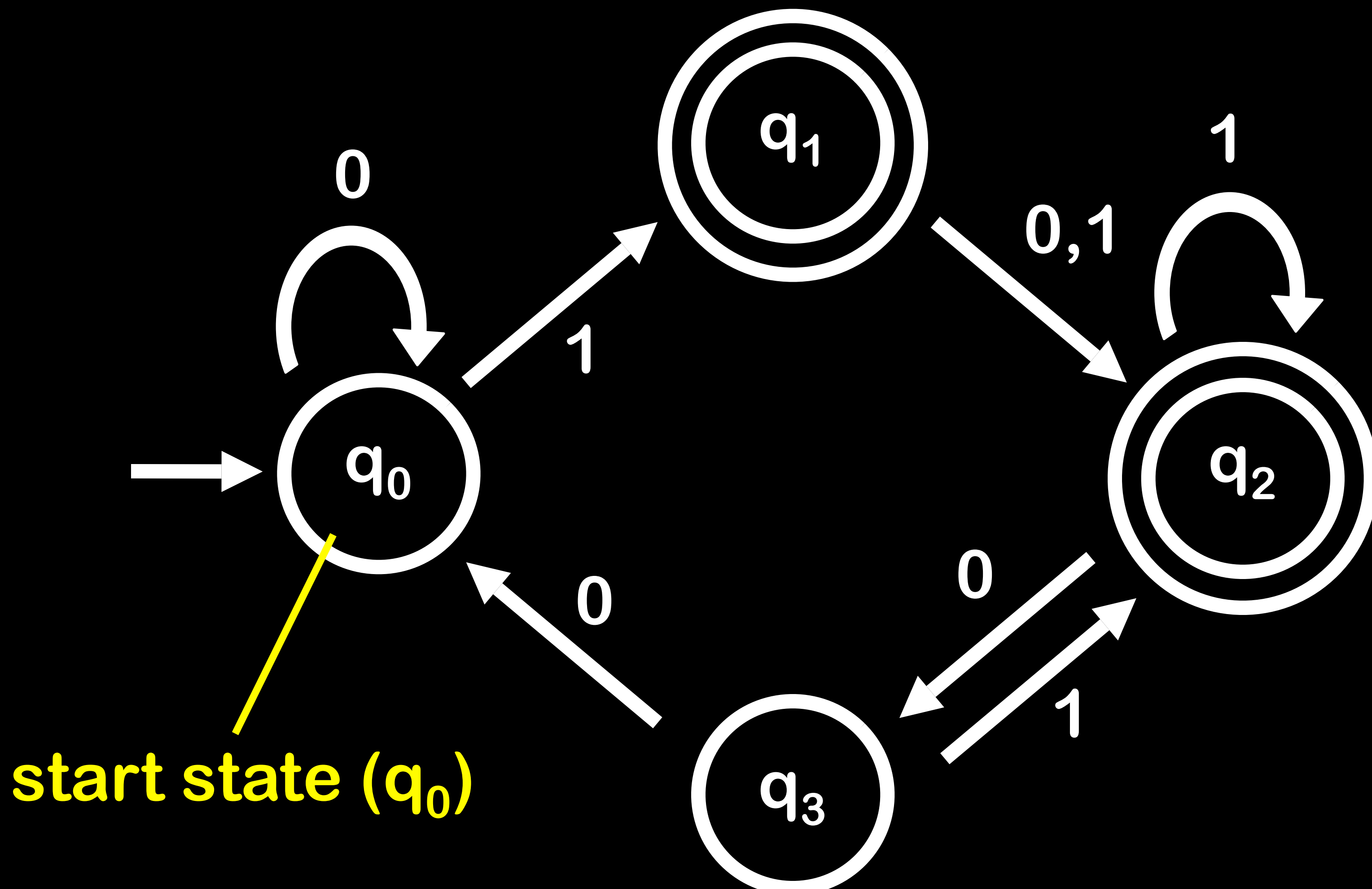

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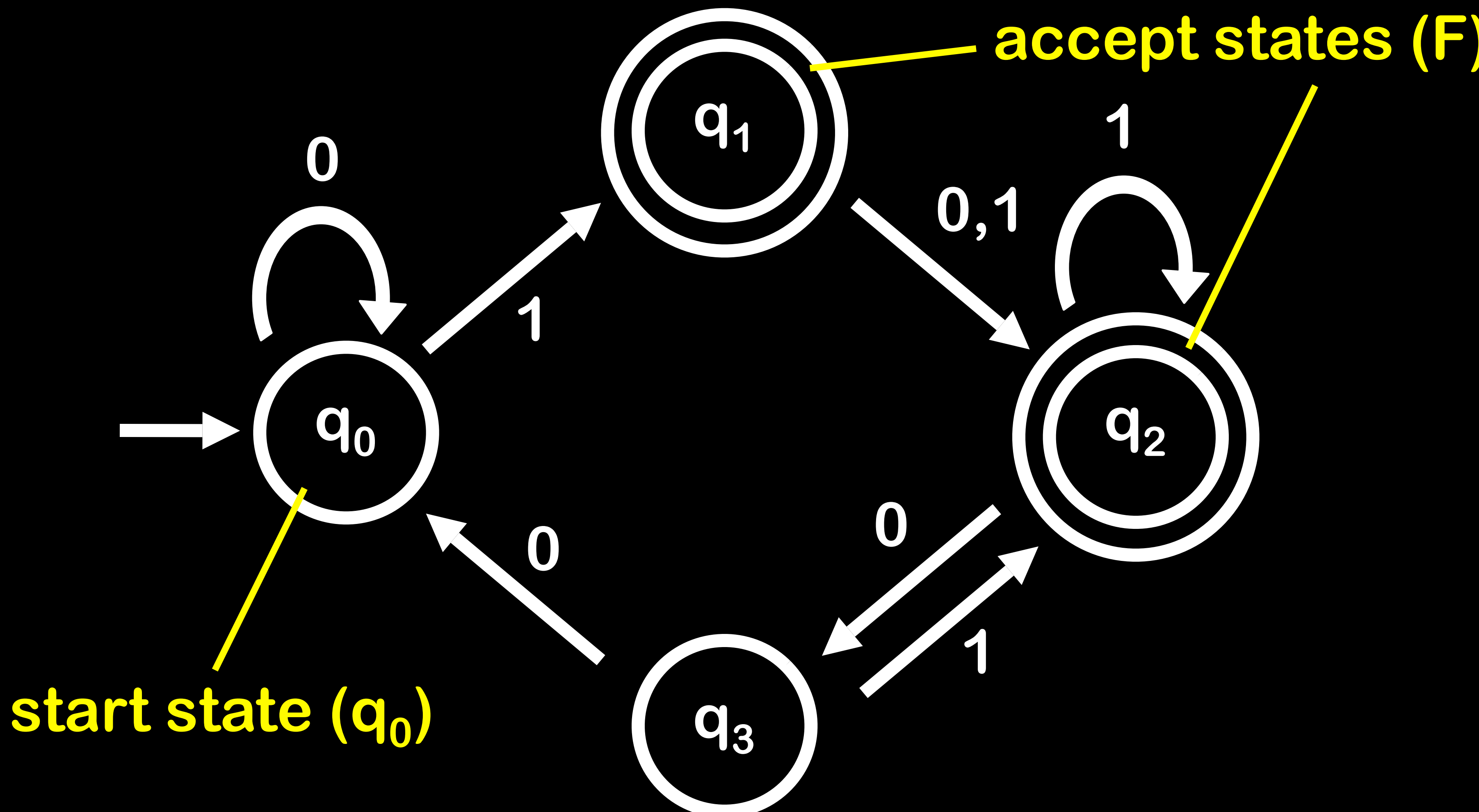

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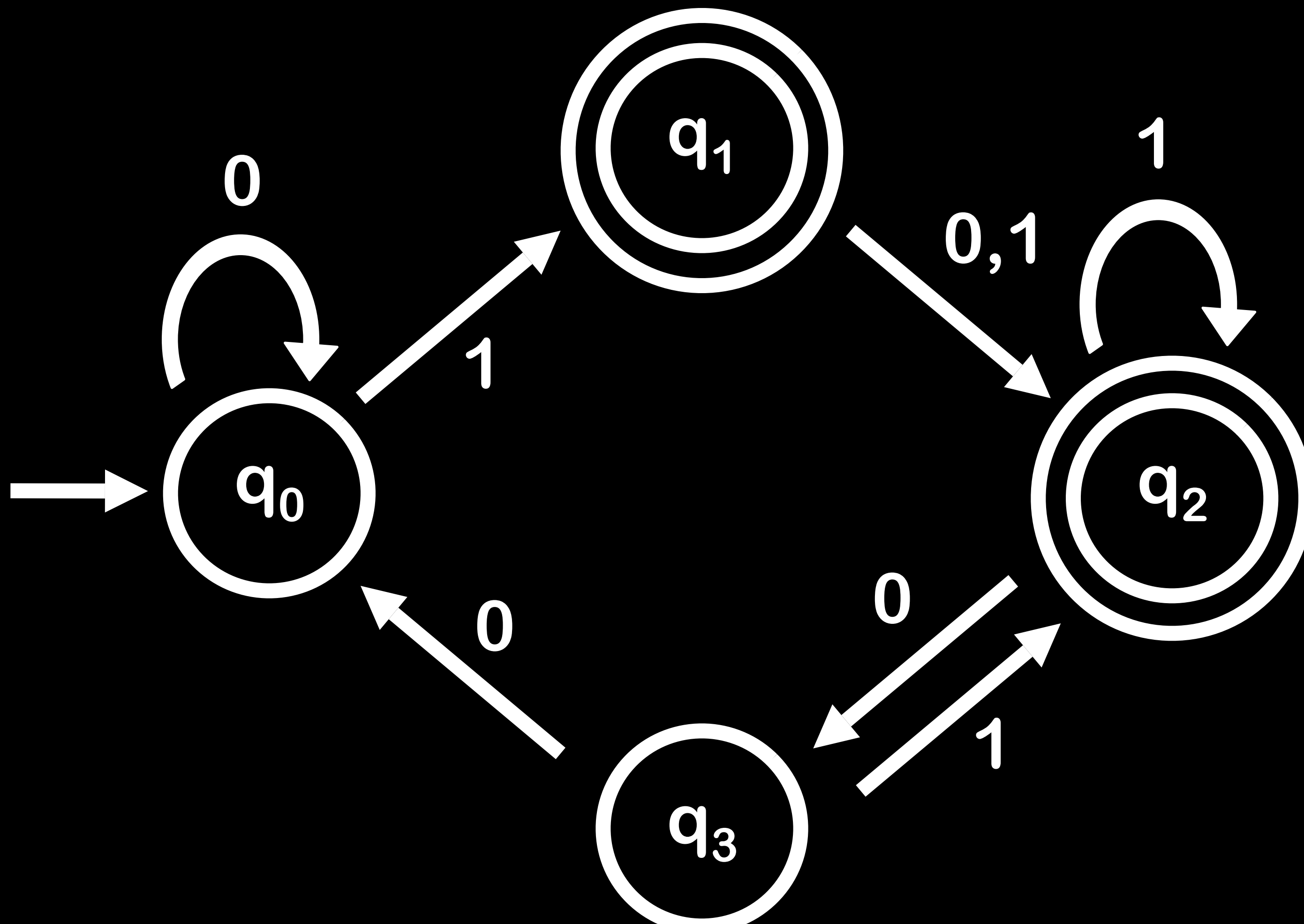

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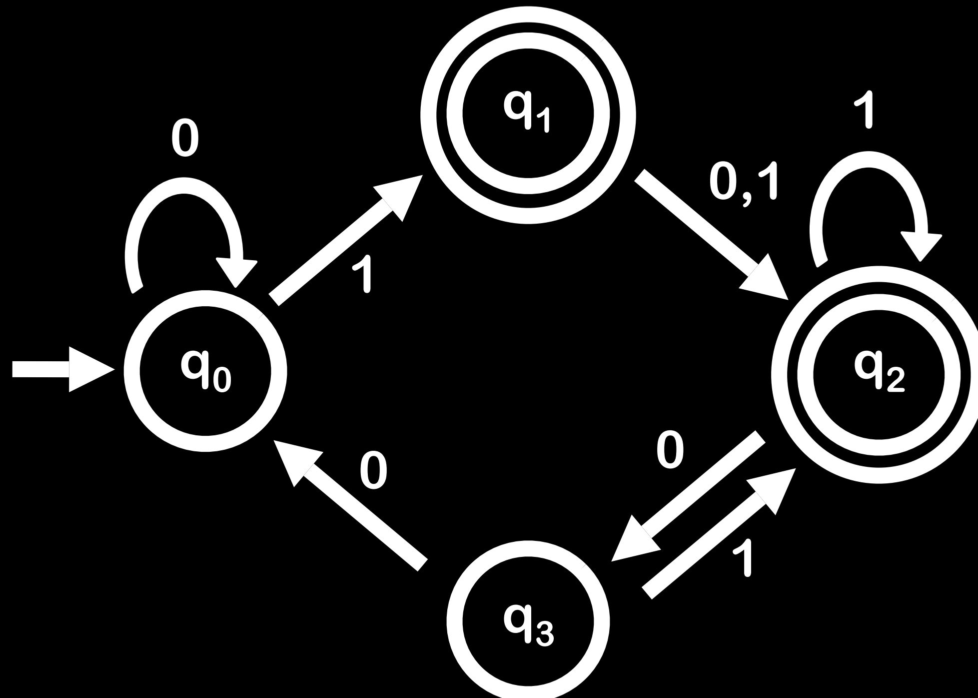

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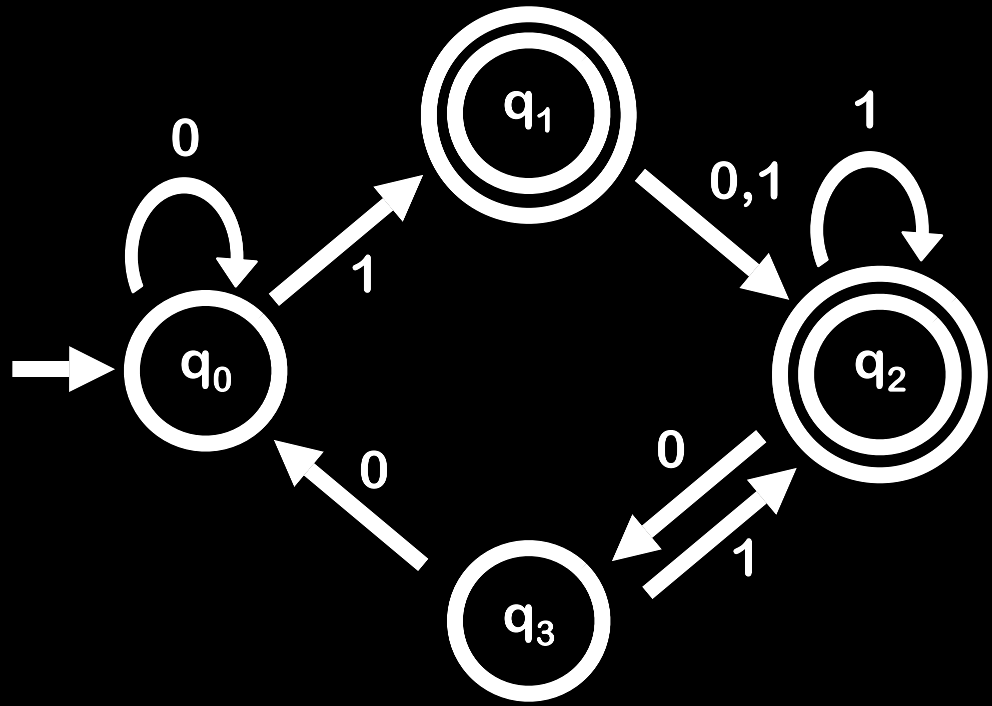

The **alphabet** of a finite automaton is the set where the symbols come from:

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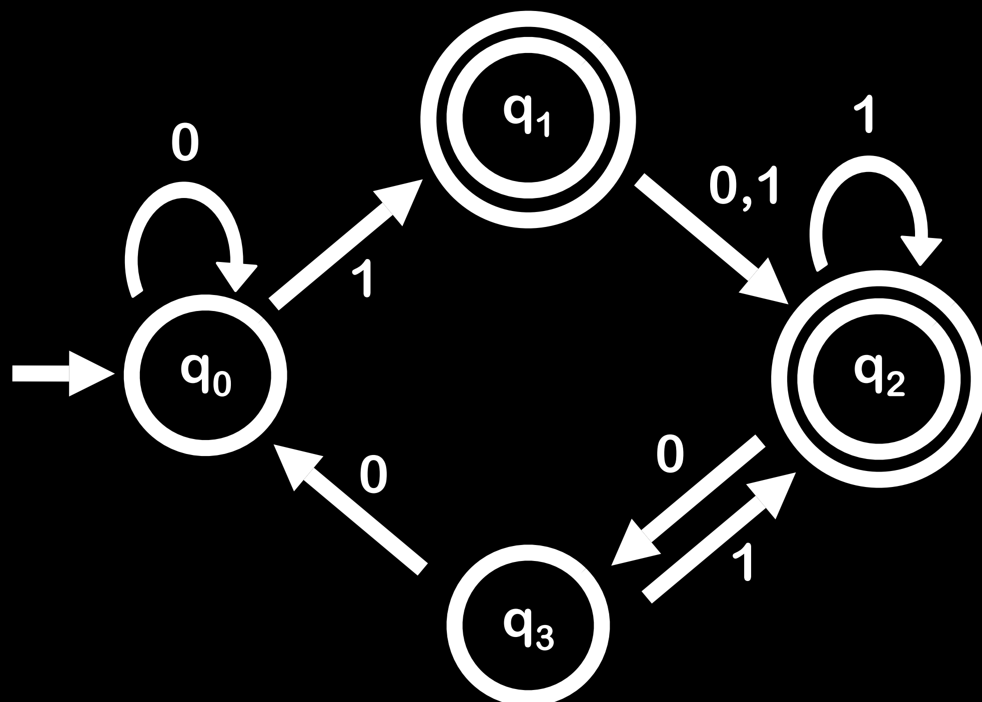

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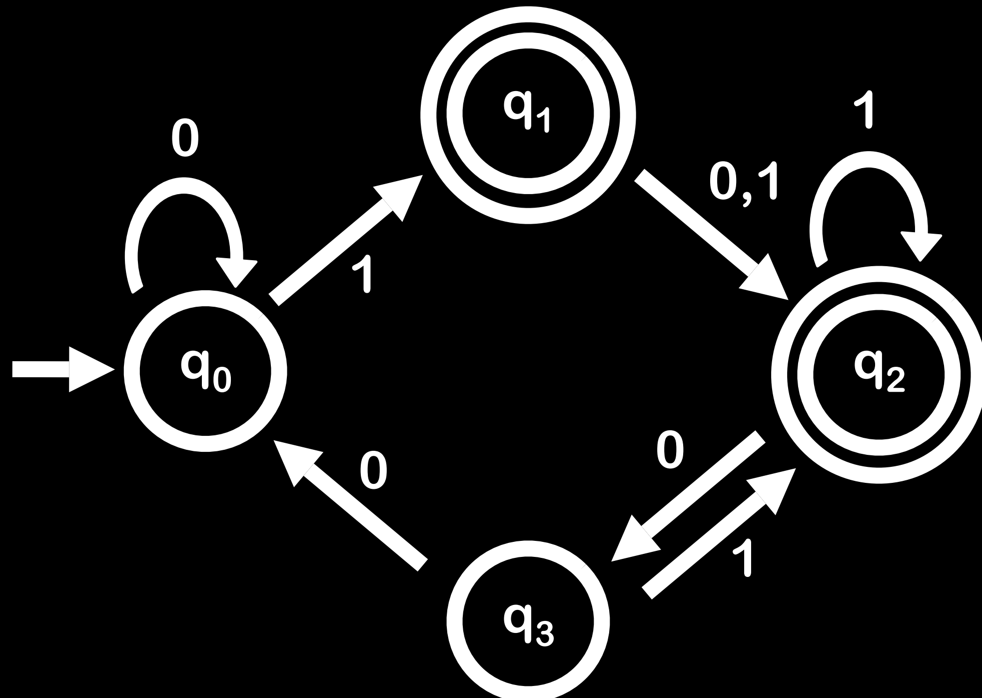
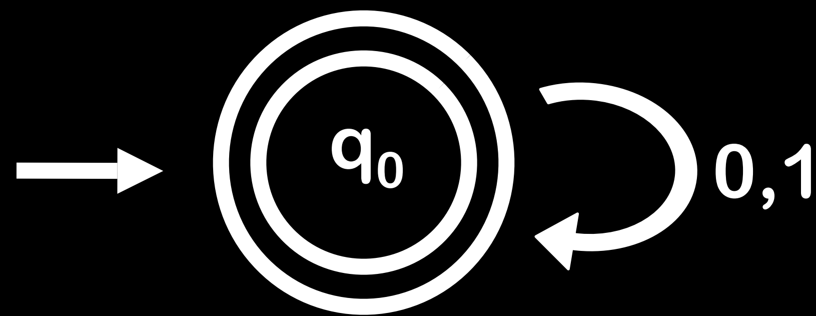


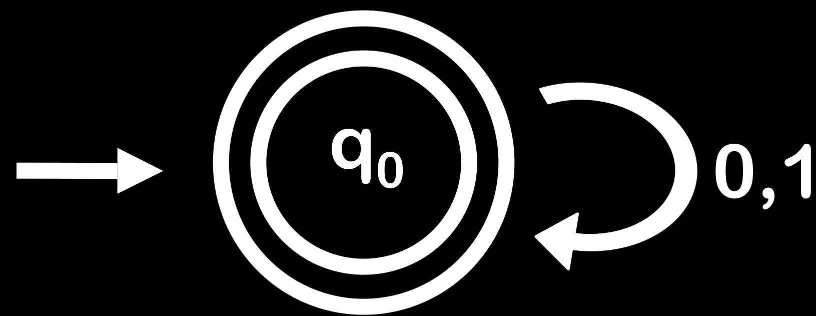
The **alphabet** of a finite automaton is the set where the symbols come from: {0,1}

The **language** of a finite automaton is the set of strings that it accepts



L(M) =

The Language of Machine M



L(M) = All strings of 0s and 1s

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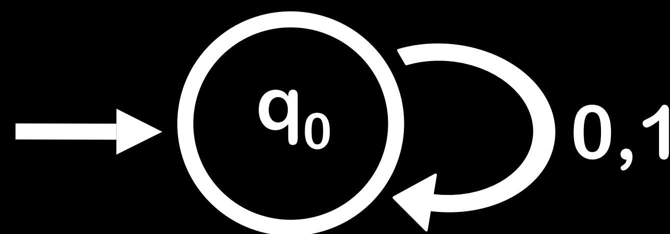

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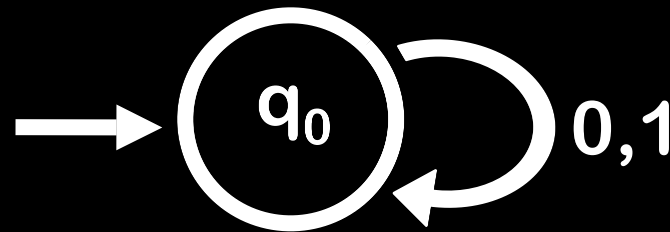
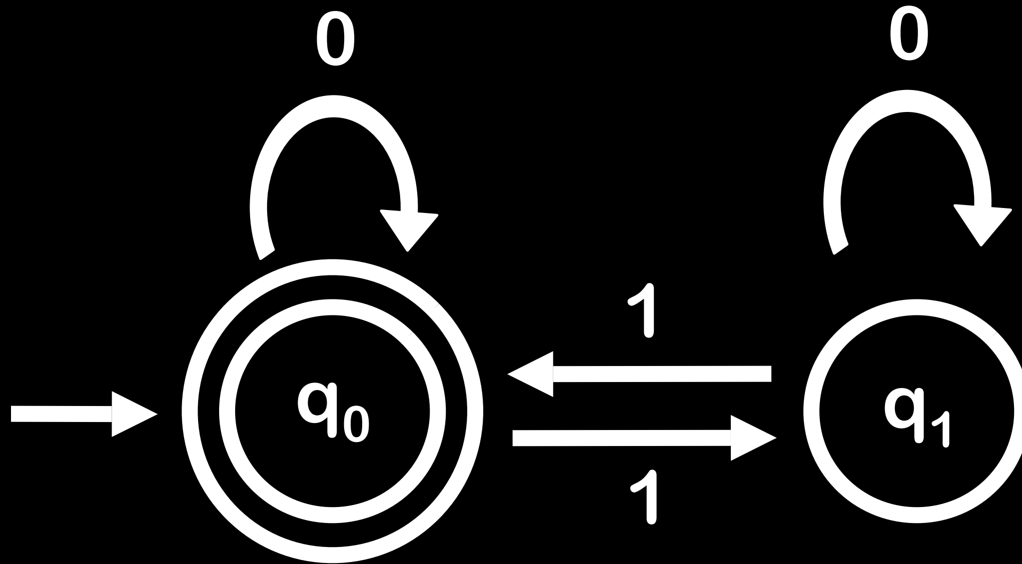
The Language of Machine M



$L(M) = \varnothing$

The Language of Machine M



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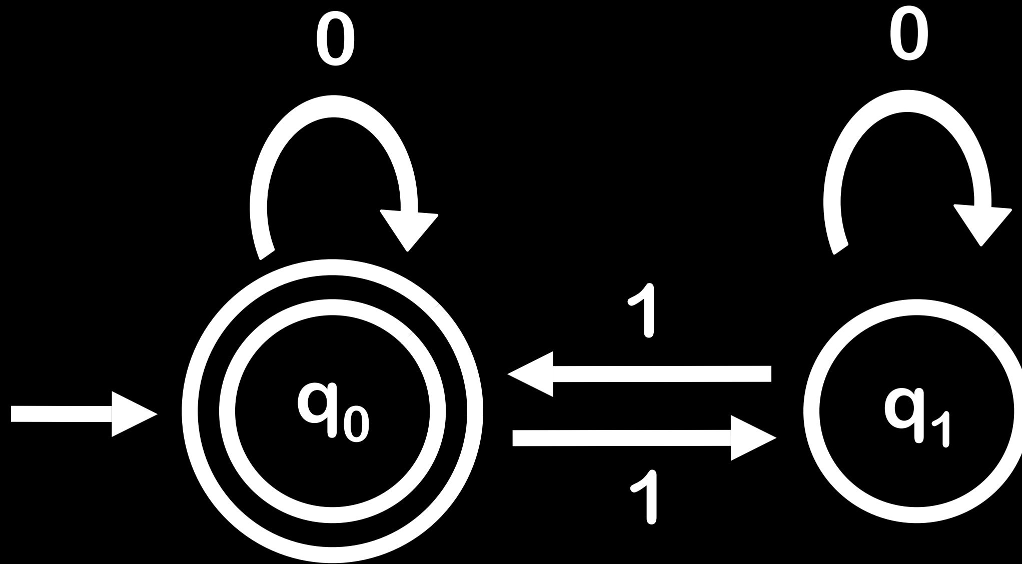

L(M) = { w | w has an even number of 1s}

Notation

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An alphabet Σ is a finite set (e.g., $\Sigma = \{0,1\}$)

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A **language over Σ** is a set of strings over Σ

A finite automaton is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

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$F \subseteq Q$ is the set of accept states

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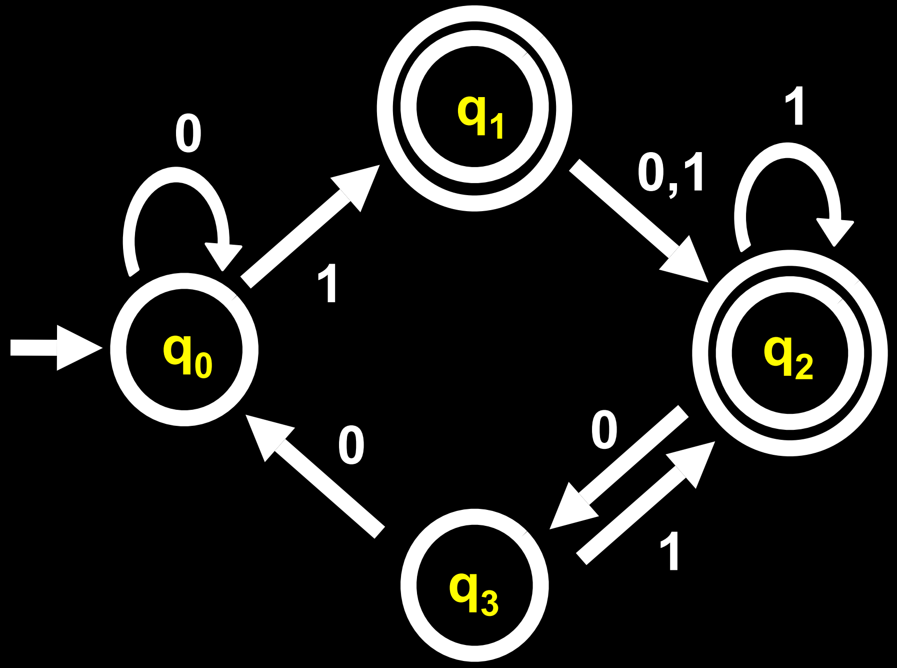
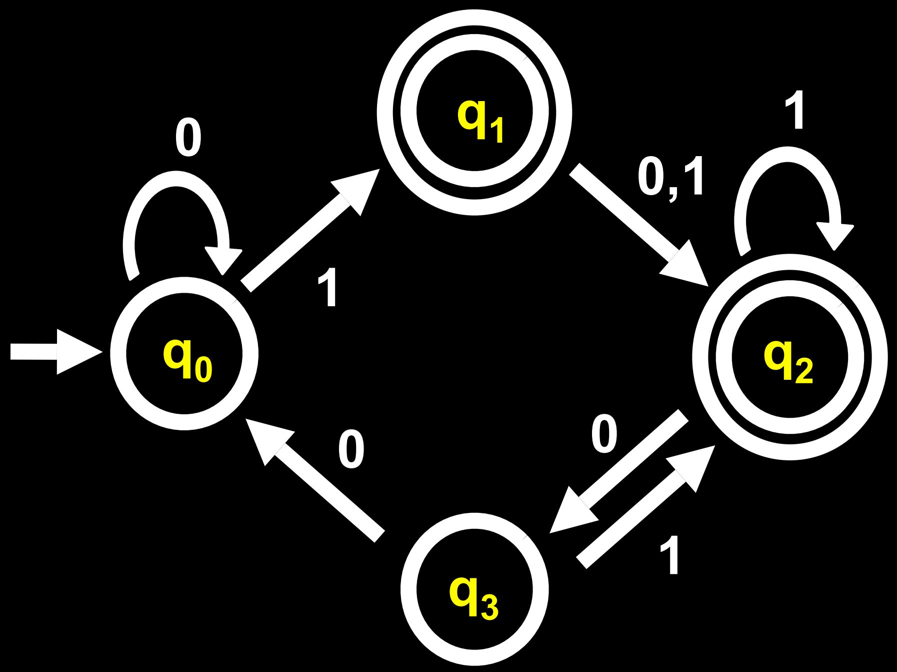
L(M) = the language of machine M
= set of all strings machine M accepts


q0
q1
q2
q3
0
1
0,1
1
0
1
0

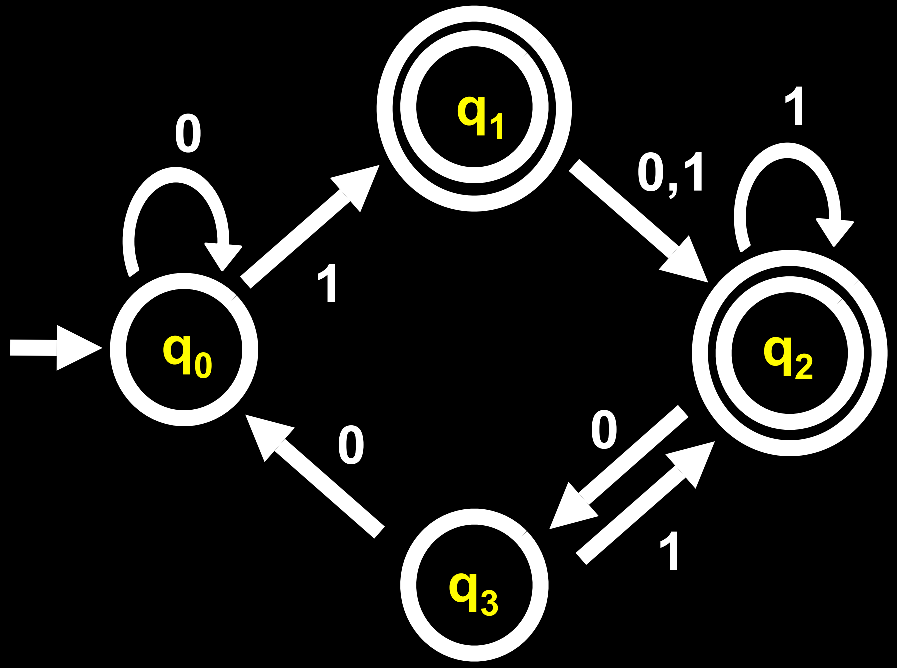
$M = (Q, \Sigma, \delta, q_0, F)$

where



$M = (Q, \Sigma, \delta, q_0, F)$ where

$Q = \{q_0, q_1, q_2, q_3\}$



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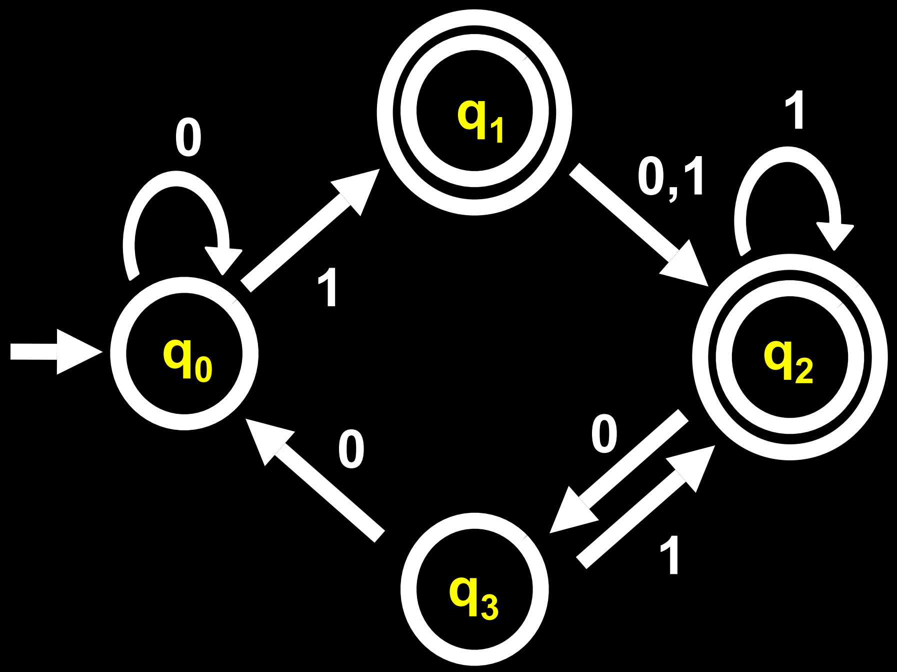

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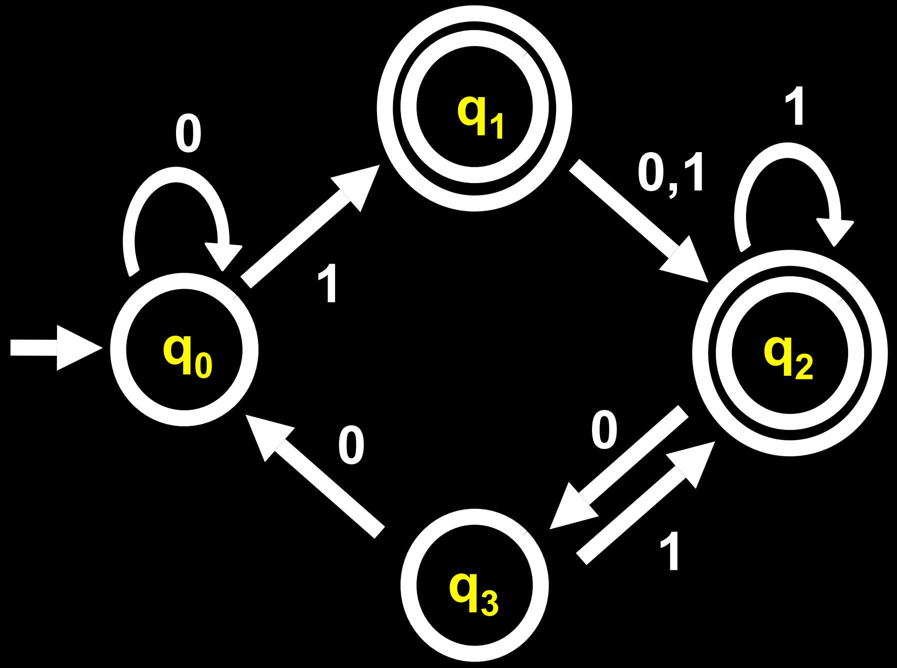
$q_0 \in Q$ is start state



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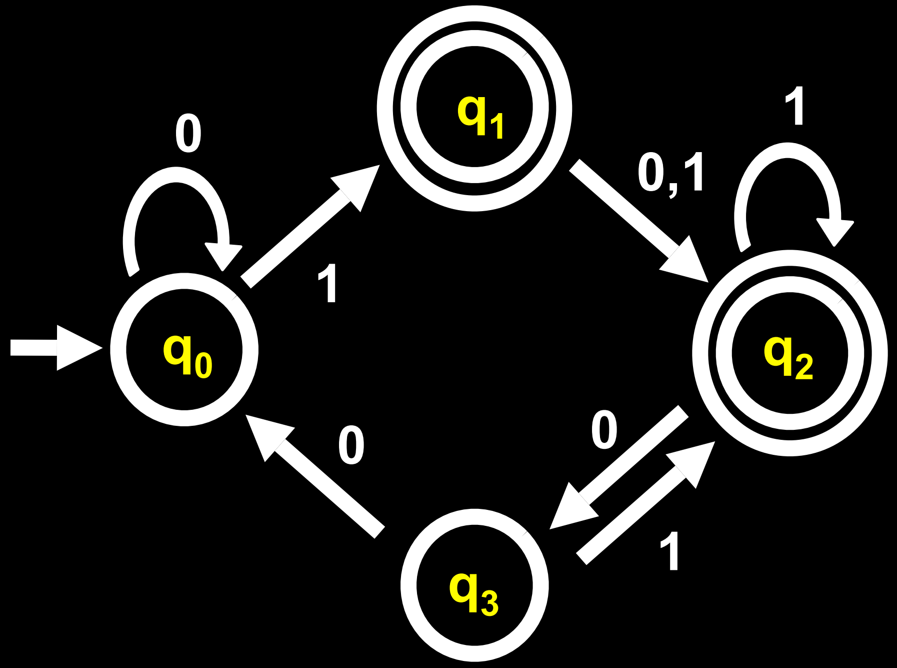

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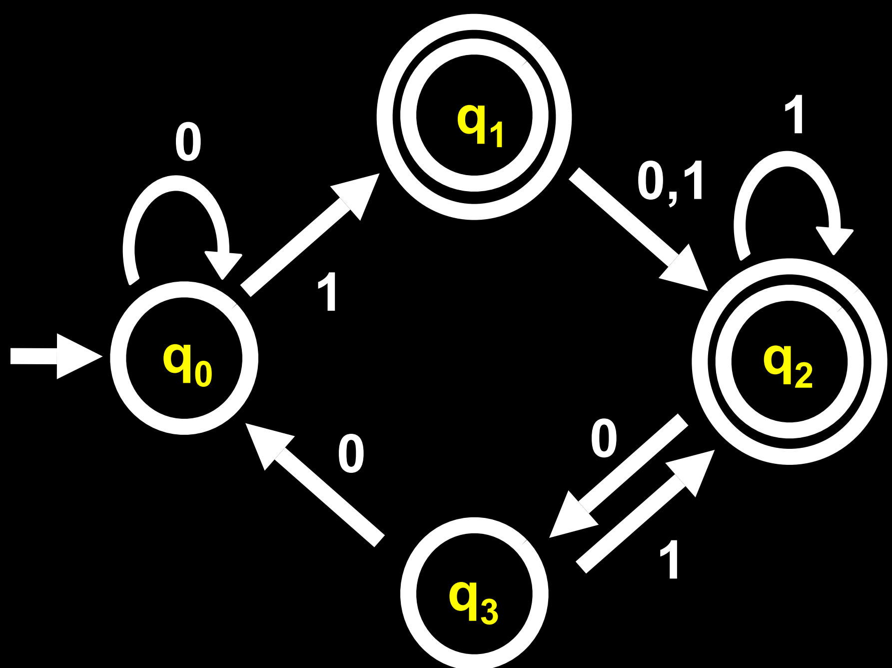
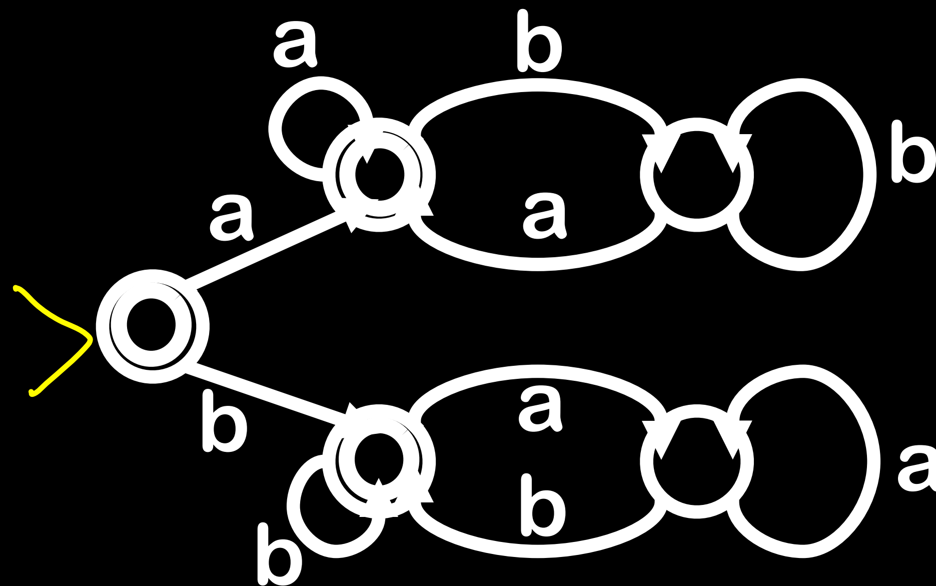
$\delta : Q \times \Sigma \rightarrow Q$ transition function

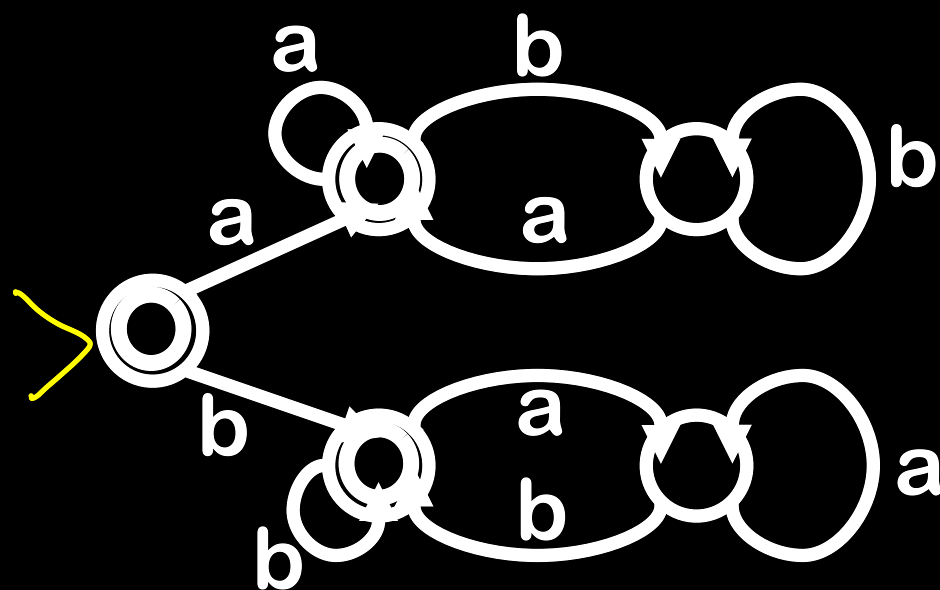


"ABA" The Automaton



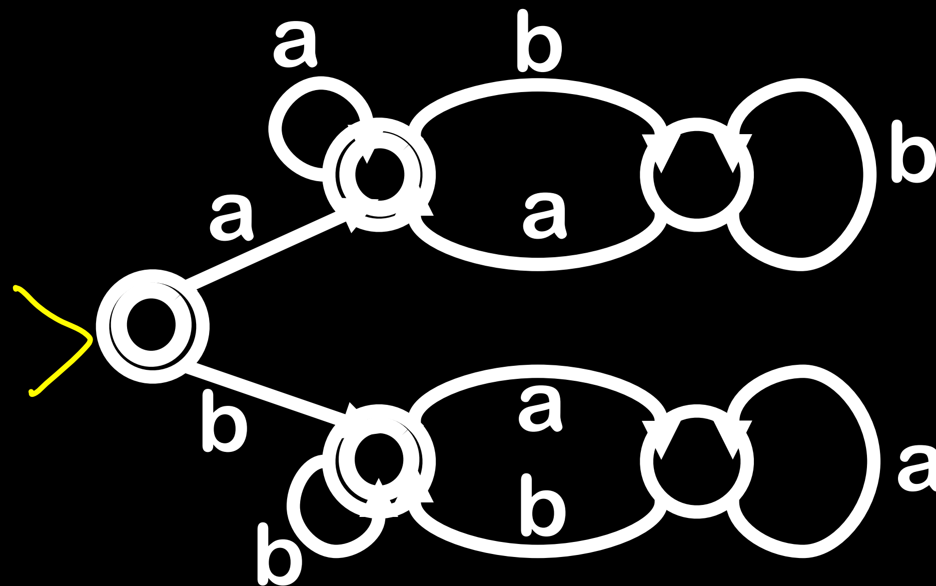
Input String	Result
aba	
aabb	
aabba	
ε	

“ABA” The Automaton



Input String	Result
aba	Accept
aabb	
aabba	
ε	

“ABA” The Automaton



Input String	Result
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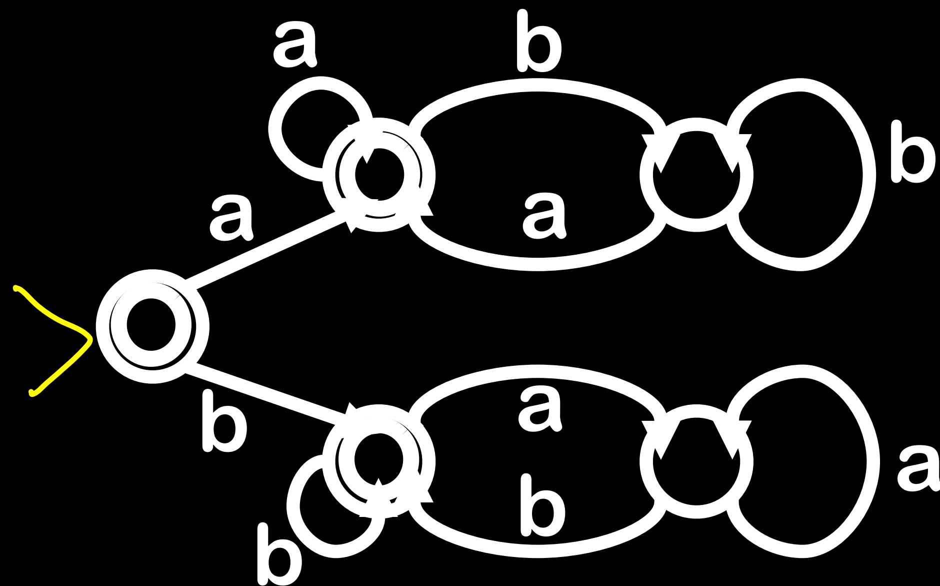

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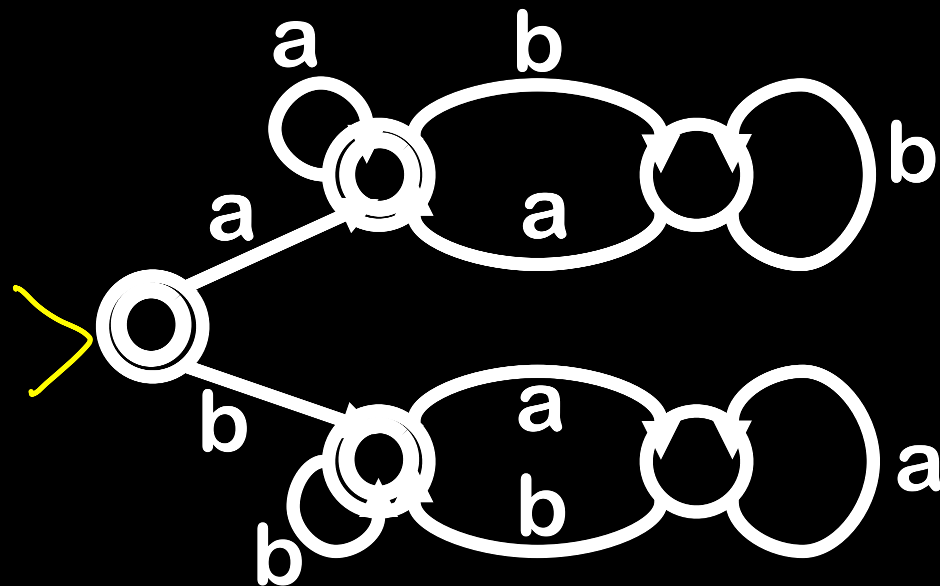

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What machine accepts this language?

$L =$ all strings in $\{a,b\}^*$ that contain at least one a

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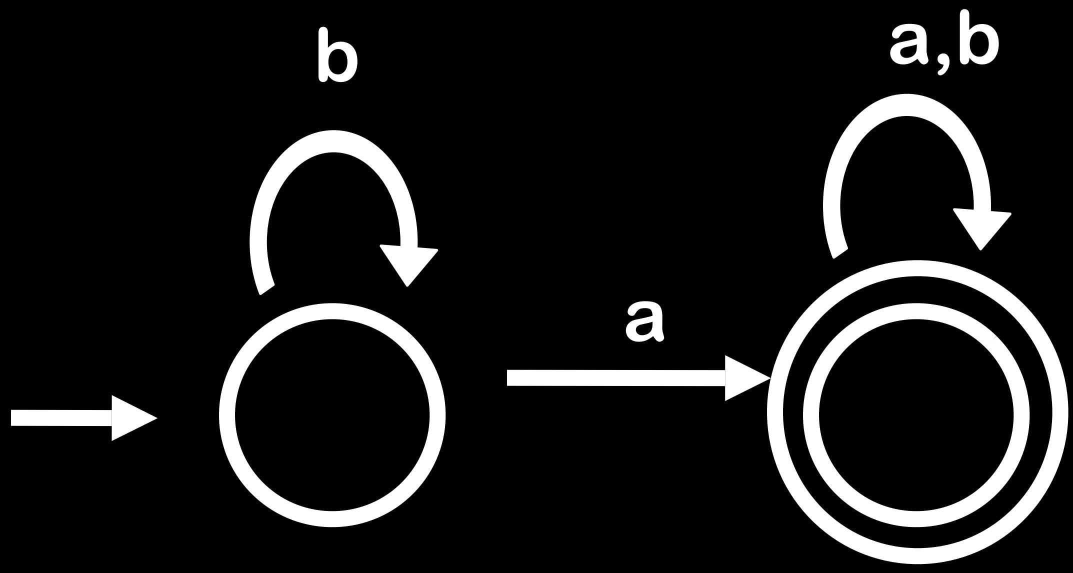
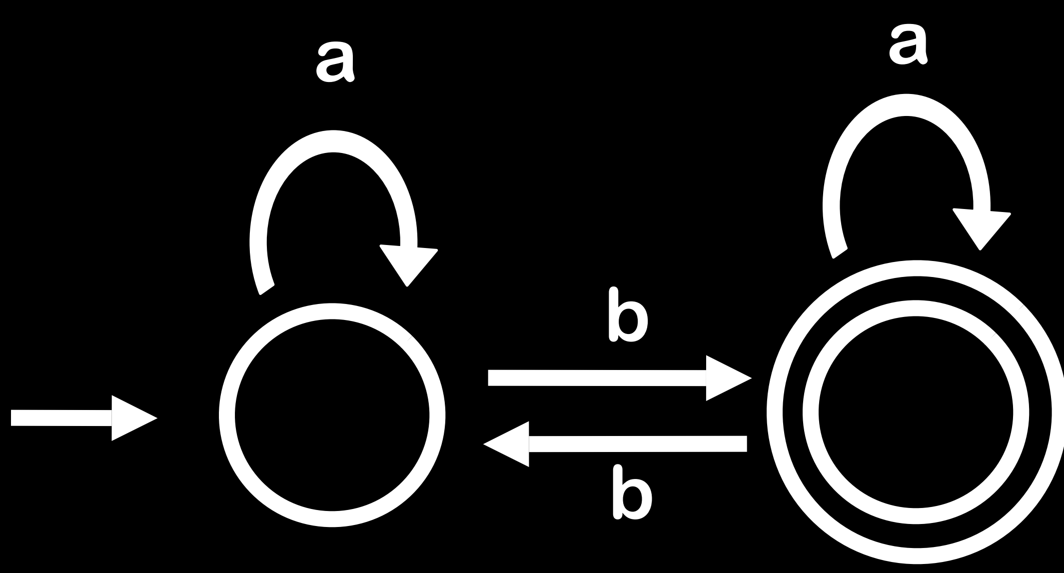


What machine accepts this language?

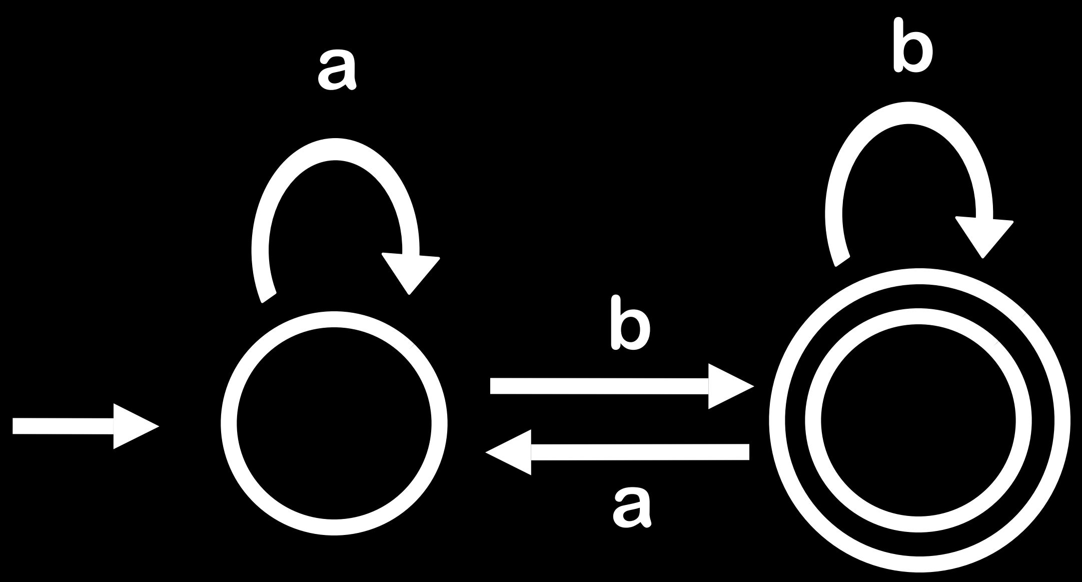
L = strings with an odd number of b's and any number of a's

What machine accepts this language?

L = strings with an odd number of **b**'s and any number of **a**'s



What is the language accepted by this machine?



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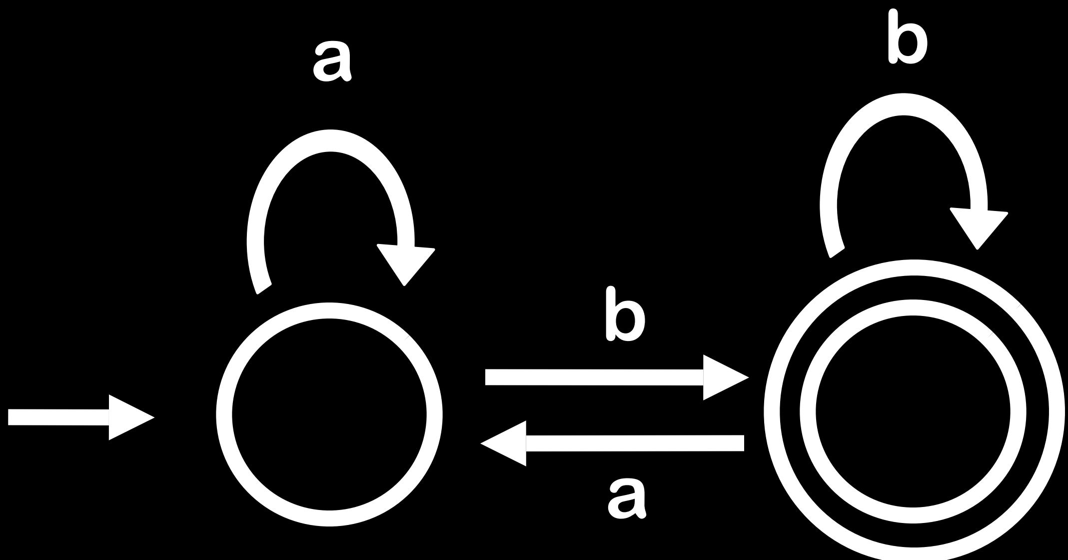

L = any string ending with a b

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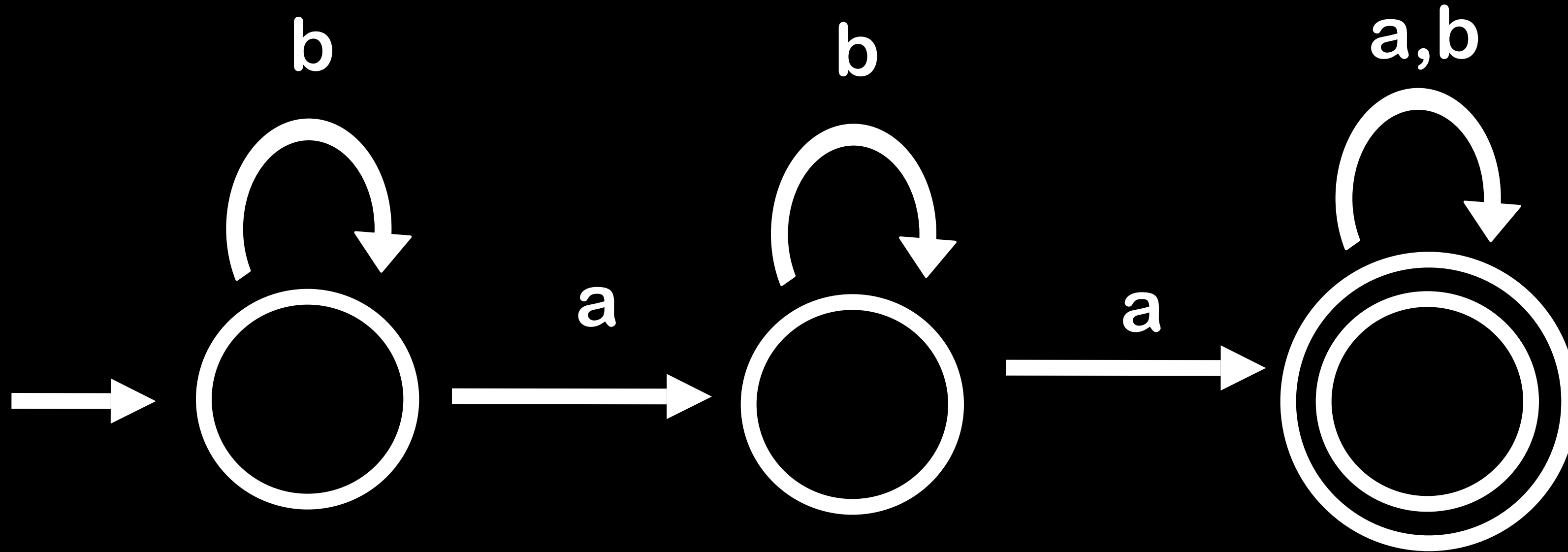

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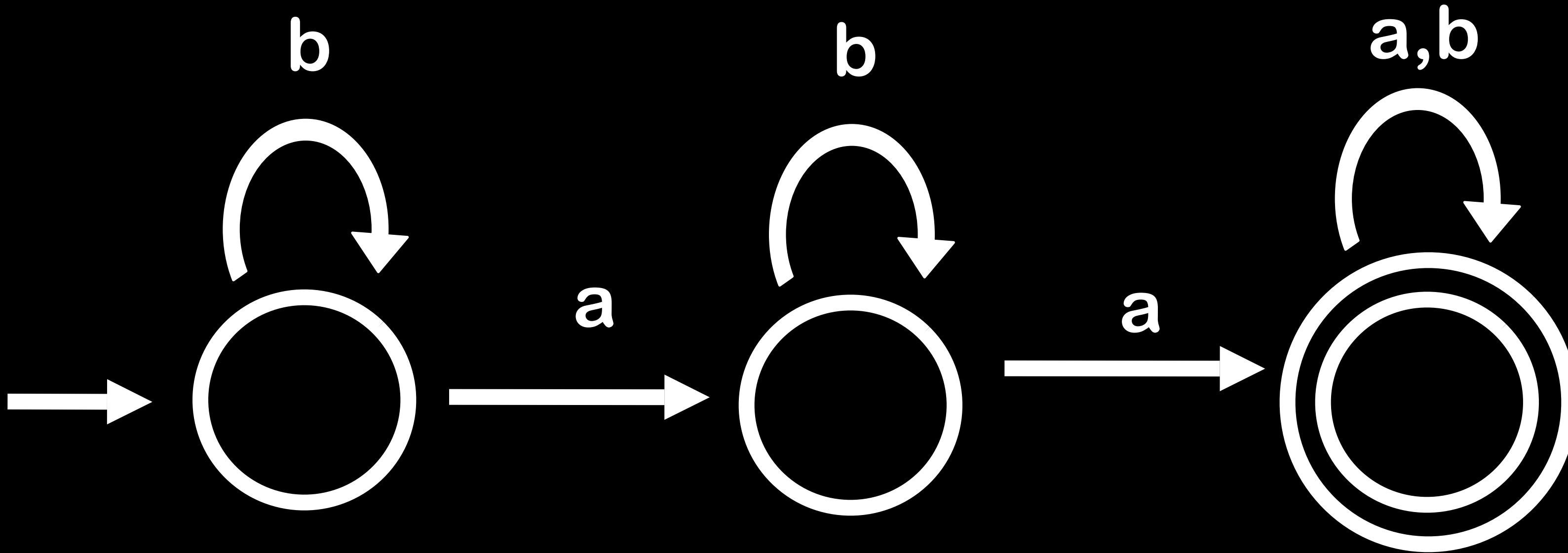


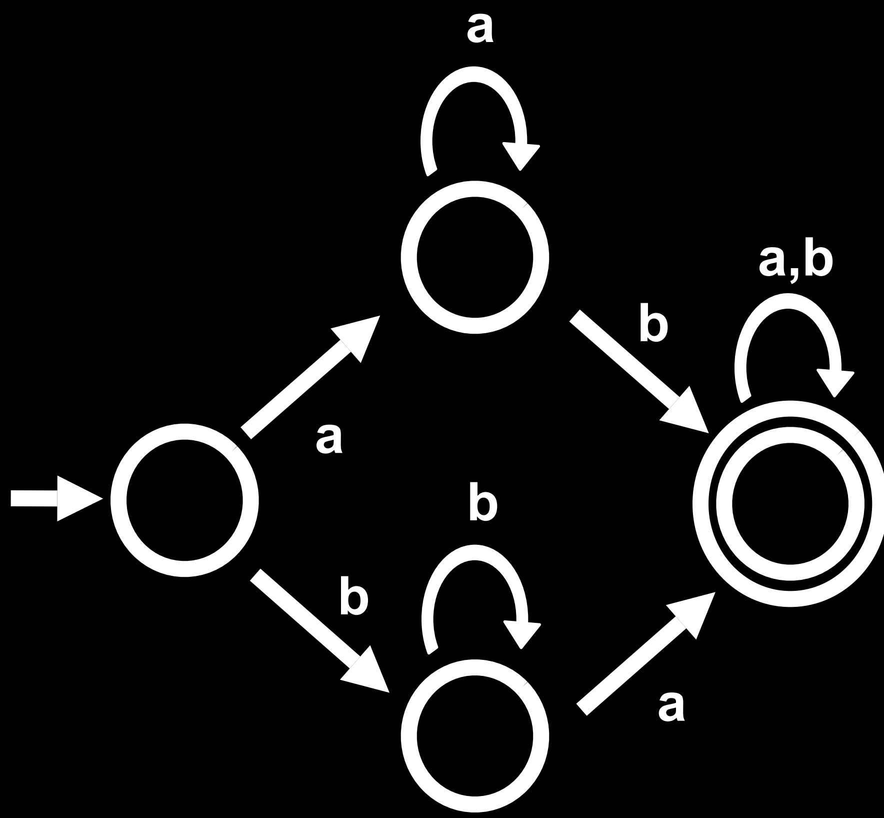
L(M) = any string with at least two **a**’s

What machine accepts this language?

L = any string with an a and a b

What machine accepts this language?

L = any string with an **a** and a **b**



What machine accepts this language?

L = strings with an even number of **ab** pairs

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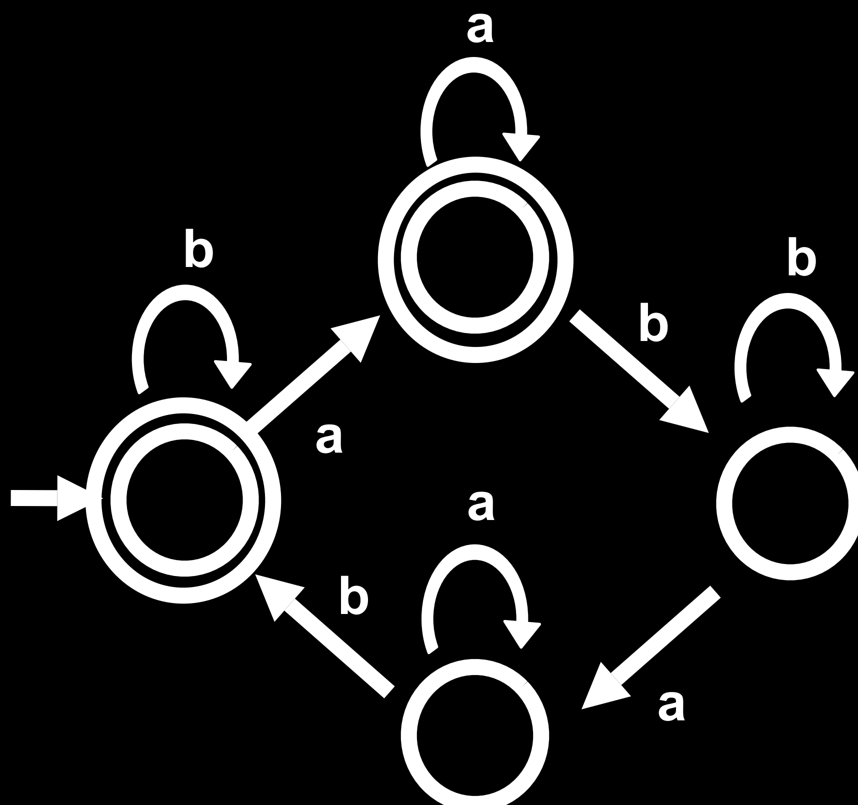
L = strings with an even number of **ab** pairs



Build an automaton that accepts all and only those strings that contain 001

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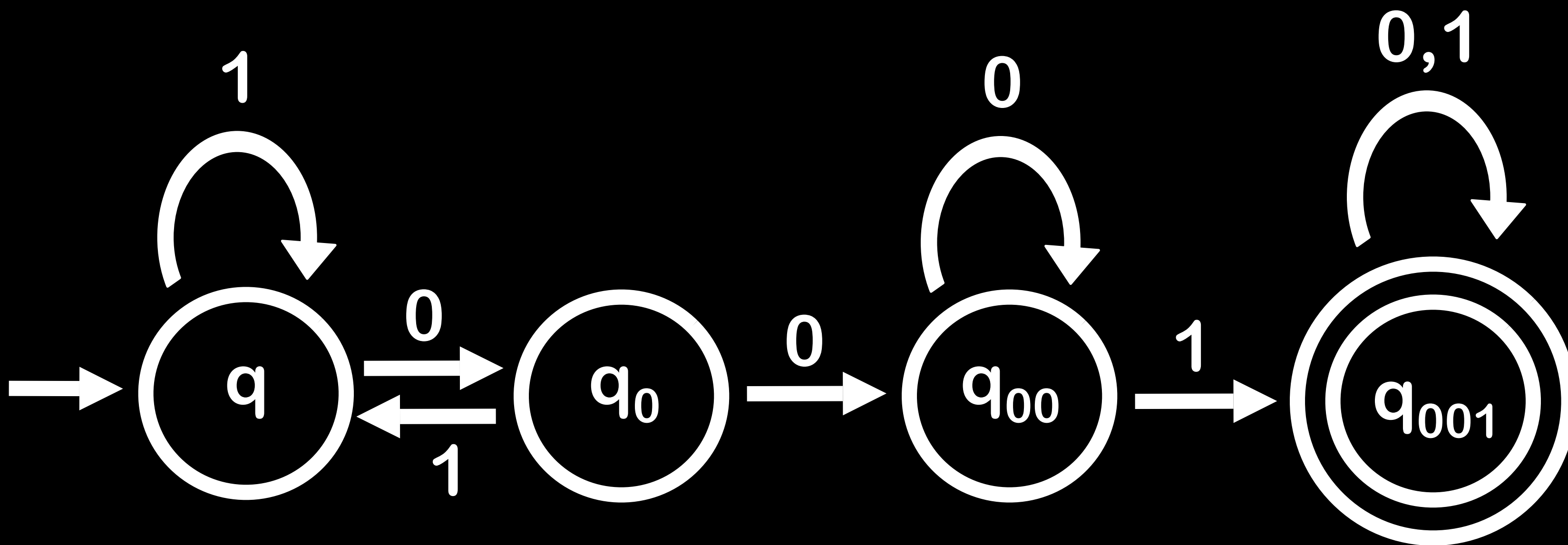

$L =$ **all strings containing** $ababb$ **as a consecutive substring**

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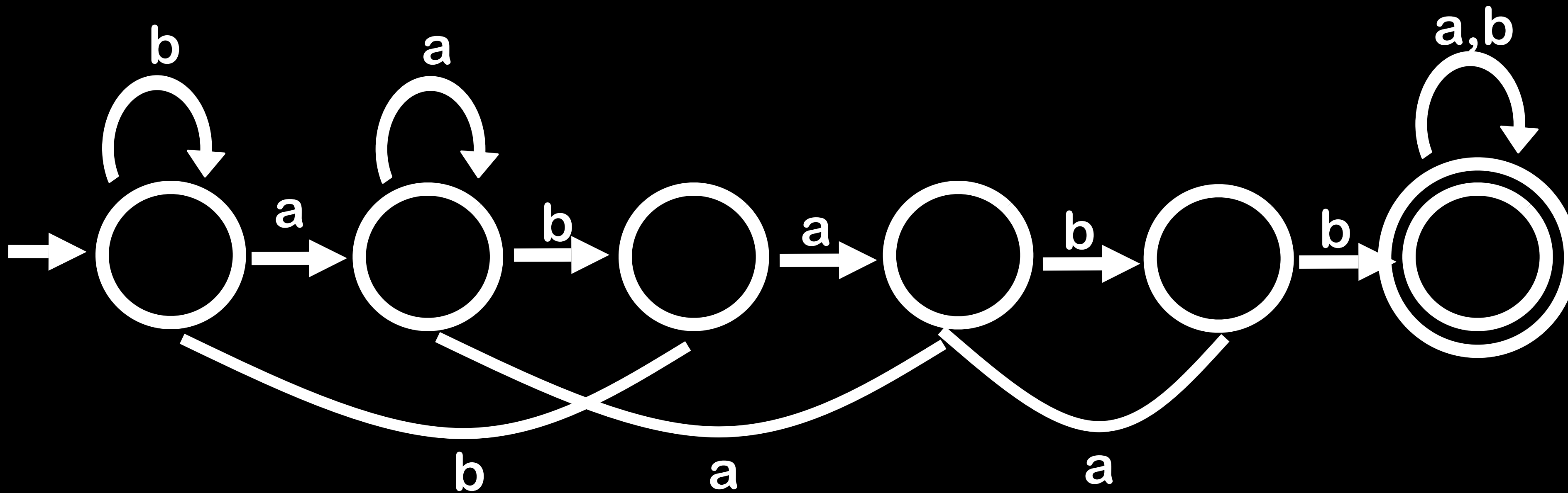

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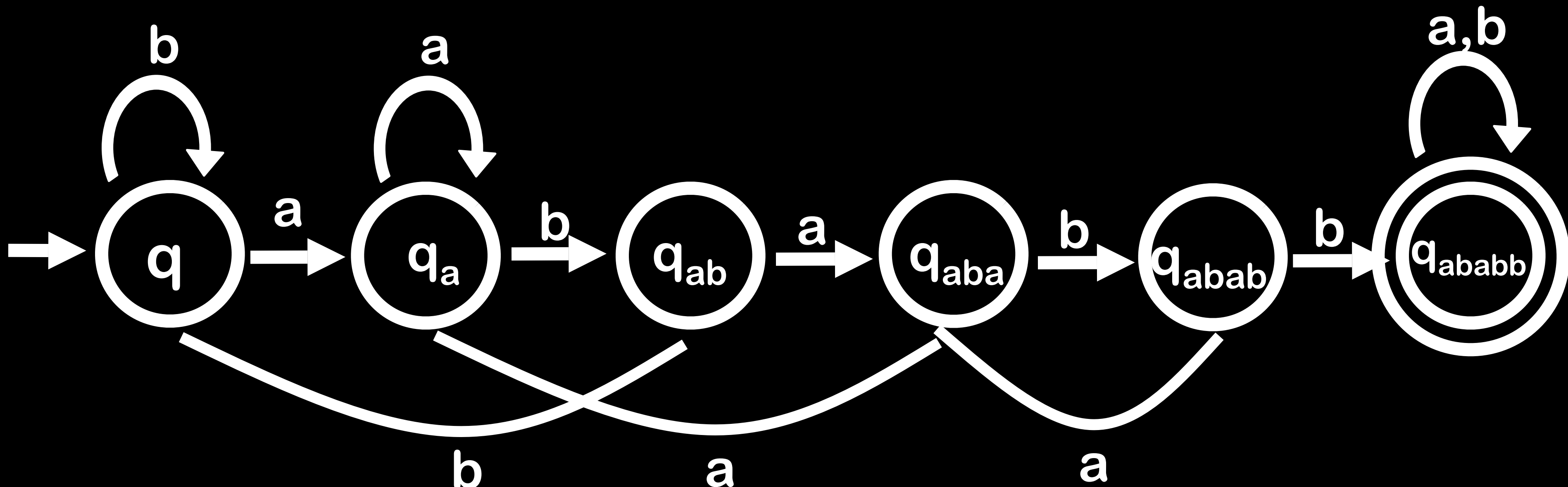

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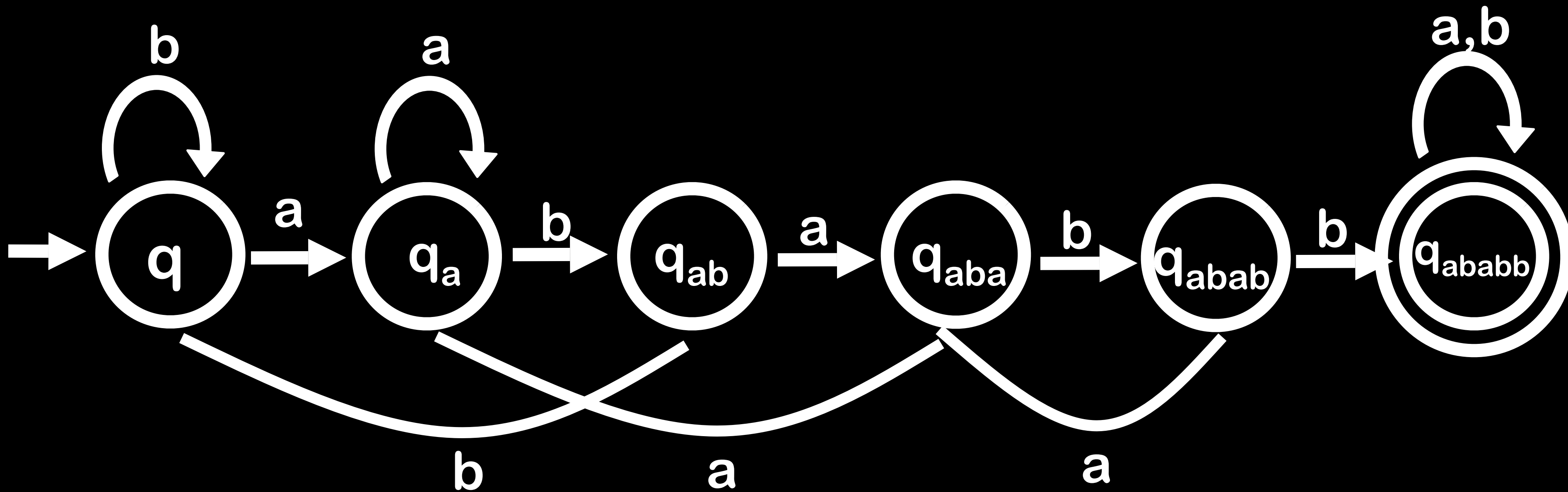

Invariant:
I am state s exactly when s is the longest suffix of the input (so far) forming a prefix of ababb.

The “Grep” Problem

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Input: Text T of length t, string S of length n

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Input: Text **T** of length **t**, string **S** of length **n**

Problem: Does string **S** appear inside text **T**?

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Naïve method:

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Naïve method:

$$a_1, a_2, a_3, a_4, a_5, \ldots, a_t$$

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Naïve method:

$$a_1,\ a_2,\ \boxed{a_3,\ a_4,\ a_5,}\ \ldots,\ a_t$$

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Problem: Does string S appear inside text T?

Naïve method:

$$a_1, a_2, \boxed{a_3, a_4, a_5,} \ldots, a_t$$

Cost:

The “Grep” Problem

Input: Text **T** of length **t,** string **S** of length **n**

Problem: Does string **S** appear inside text **T**?

Naïve method:

$$a_1, a_2, \boxed{a_3, a_4, a_5,} \ldots, a_t$$

Cost: Roughly nt comparisons

Automata Solution

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Build a machine M that accepts any string with S as a consecutive substring

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Cost: t comparisons + time to build M

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As luck would have it, the Knuth, Morris, Pratt algorithm builds M quickly

Real-life Uses of DFAs

Grep

Coke Machines

Thermostats (fridge)

Elevators

Train Track Switches

Lexical Analyzers for Parsers

A language is regular if it is recognized by a deterministic finite automaton

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L = { w | w contains 001} is regular

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L = { w | w contains 001} is regular

L = { w | w has an even number of 1s} is regular

Union Theorem

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Given two languages, L_1 and L_2, define the union of L_1 and L_2 as

$$L_1 \cup L_2 = \{ w \mid w \in L_1 \text{ or } w \in L_2 \}$$

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Theorem: The union of two regular languages is also a regular language

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Proof Sketch: Let

$M_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$ be finite automaton for L_1

and

$M_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$ be finite automaton for L_2

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We want to construct a finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $L = L_1 \cup L_2$

Idea: Run both M_1 and M_2 at the same time!

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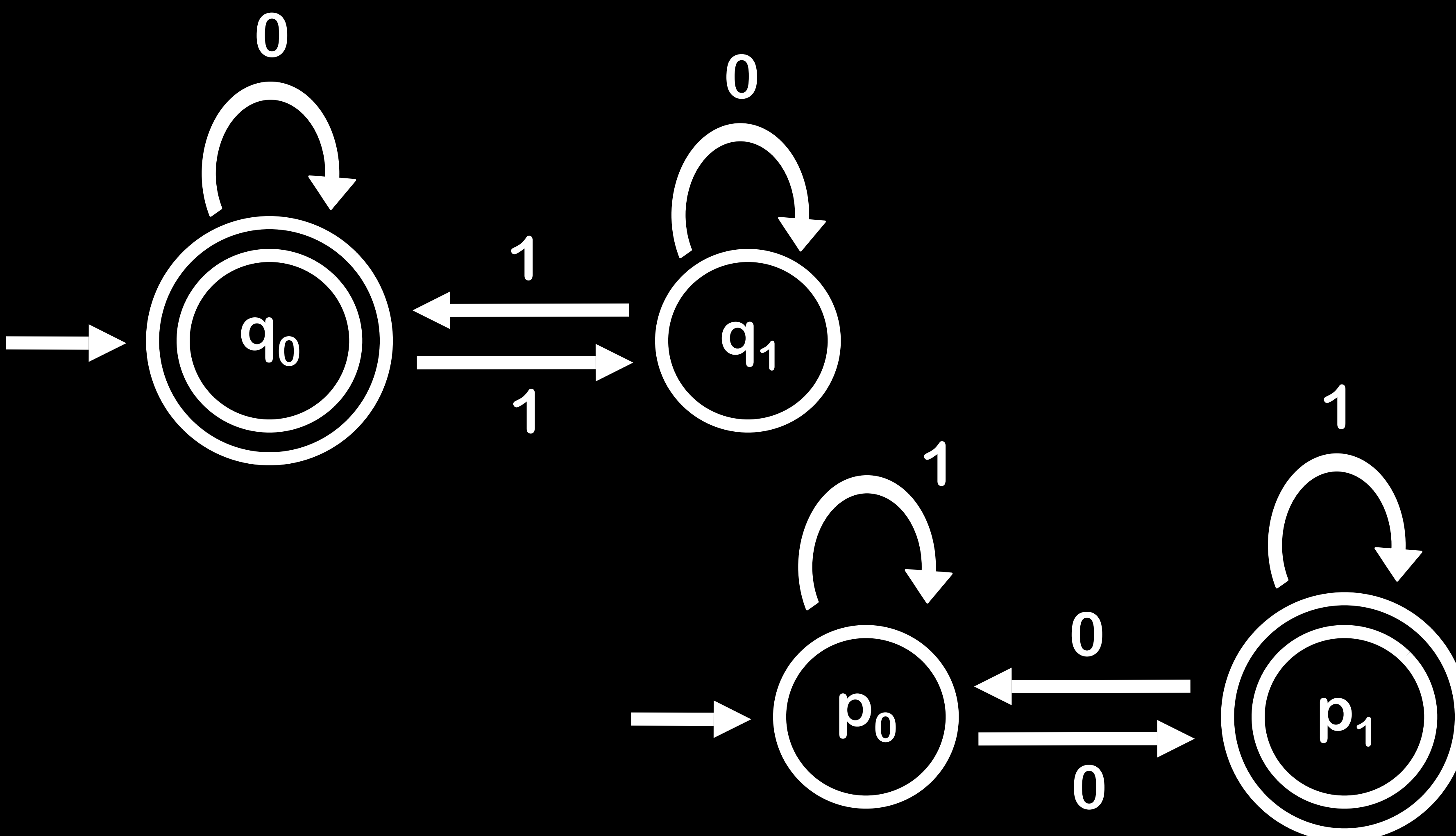
Q = pairs of states, one from M_1 and one from M_2

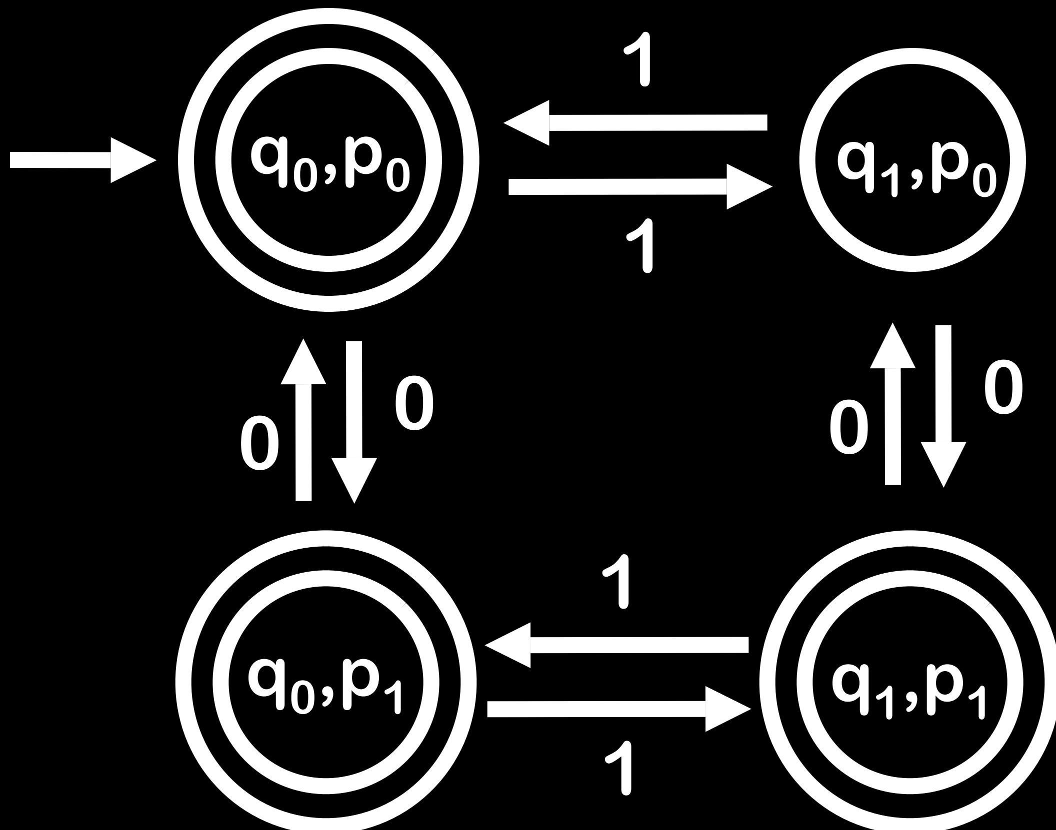
$= \{ (q_1, q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \}$

$= Q_1 \times Q_2$

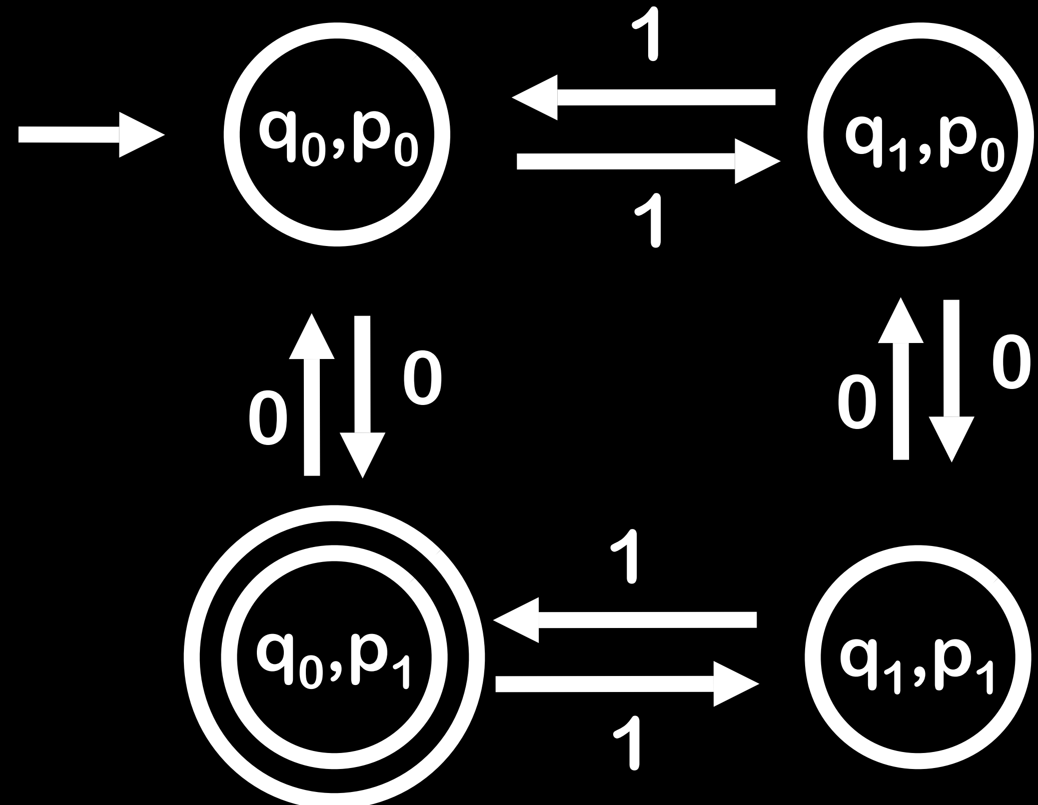
Theorem: The union of two regular languages is also a regular language



Automaton for Union



Automaton for Intersection



Theorem: The union of two regular languages is also a regular language

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Corollary: Any finite language is regular

The Regular Operations

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Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$

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Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

Star: $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

Regular Languages Are Closed Under The Regular Operations

We have seen part of the proof for Union. The proof for intersection is very similar. The proof for negation is easy.

Are all languages regular?

Consider the language $L = \{ a^n b^n \mid n > 0 \}$

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i.e., a bunch of a's followed by an equal number of b's

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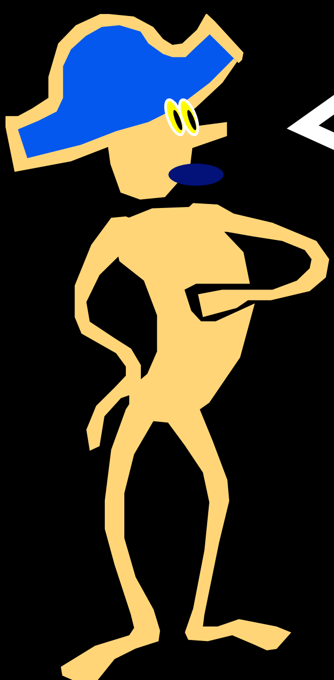
Can you prove this?



$a^n b^n$ **is not regular. No machine has enough states to keep track of the number of a’s it might encounter**

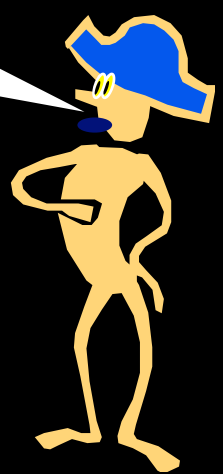
That is a fairly weak argument

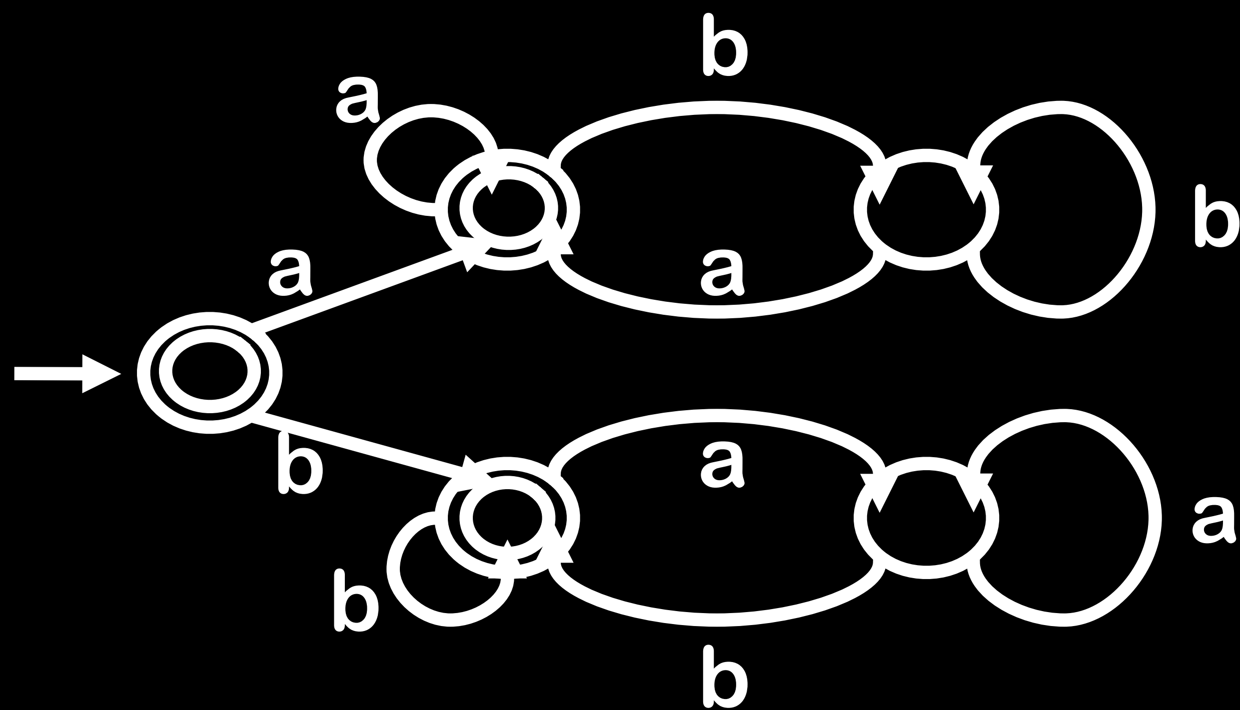
Consider the following example…

L = strings where the # of occurrences of the pattern **ab** is equal to the number of occurrences of the pattern **ba**

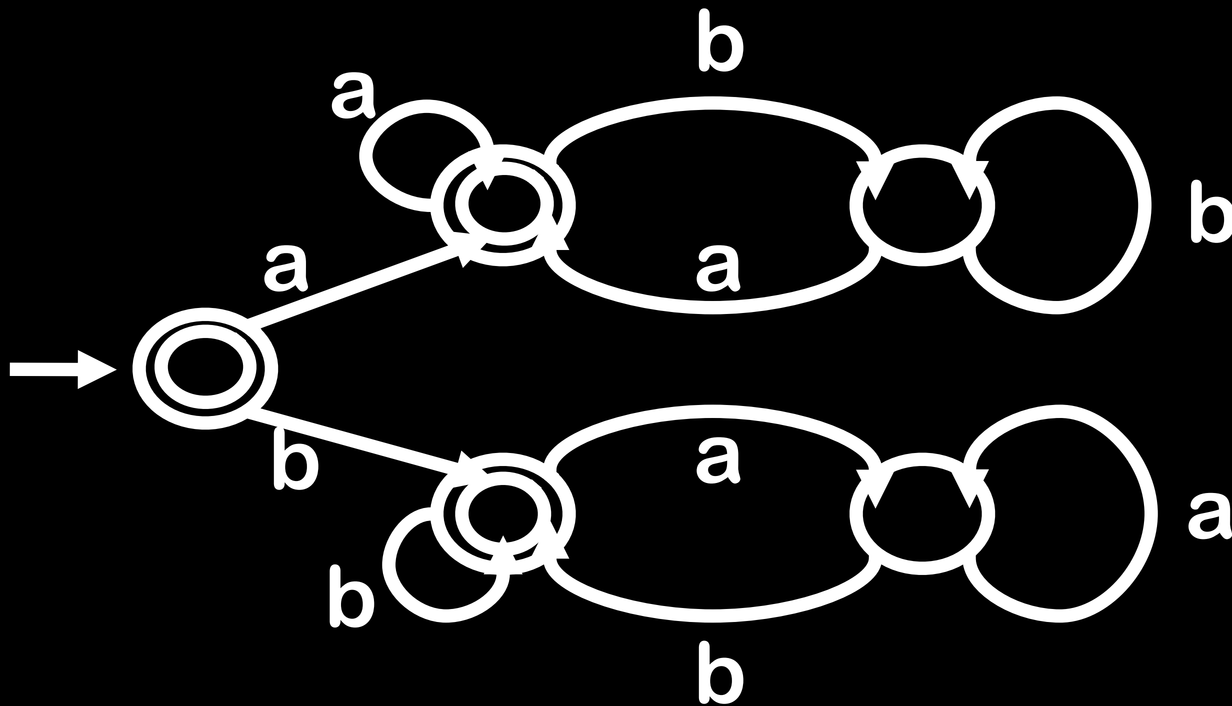
L = strings where the # of occurrences of the pattern ab is equal to the number of occurrences of the pattern ba

Can't be regular. No machine has enough states to keep track of the number of occurrences of ab





a
b
b
a
a
b
b
a
b
a



M accepts only the strings with an equal number of ab's and ba's!

Let me show you a professional strength proof that $a^n b^n$ is not regular...

Pigeonhole principle:

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Given n boxes and m > n objects, at least one box must contain more than one object

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Letterbox principle:

If the average number of letters per box is x, then some box will have at least x letters (similarly, some box has at most x)

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$\exists i,j \le k$ such that $S_i = S_j$, but $i \neq j$

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Proof (by contradiction):

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For each $0 \leq i \leq k$, let S_i be the state M is in after reading a^i

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M will do the same thing on $a^i b^i$ and $a^j b^i$

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Proof (by contradiction):

Assume that L is regular

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For each $0 \le i \le k$, let S_i be the state M is in after reading a^i

$\exists i,j \le k$ such that $S_i = S_j$, but $i \neq j$

M will do the same thing on $a^i b^i$ and $a^j b^i$

But a valid M must reject $a^j b^i$ and accept $a^i b^i$

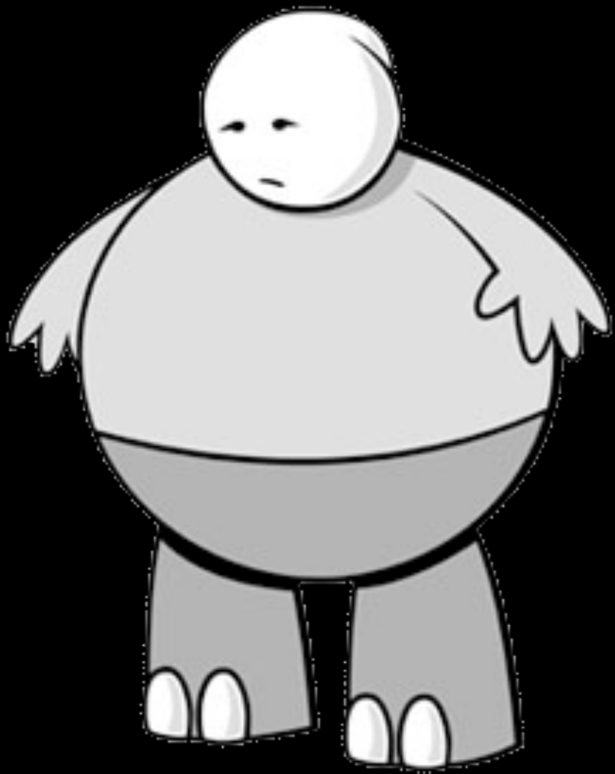
Advertisement

You can learn much more about these creatures in the FLAC course.

Formal Languages, Automata, and Computation

- There is a unique smallest automaton for any regular language

- It can be found by a fast algorithm.



Here's What You Need to Know…

Deterministic Finite Automata

- Definition
- Testing if they accept a string
- Building automata

Regular Languages

- Definition
- Closed Under Union, Intersection, Negation
- Using Pigeonhole Principle to show language not regular