15-251

Great Theoretical Ideas in Computer Science

Upcoming Events

Review Session on Saturday
(5 pm, Wean 5409)

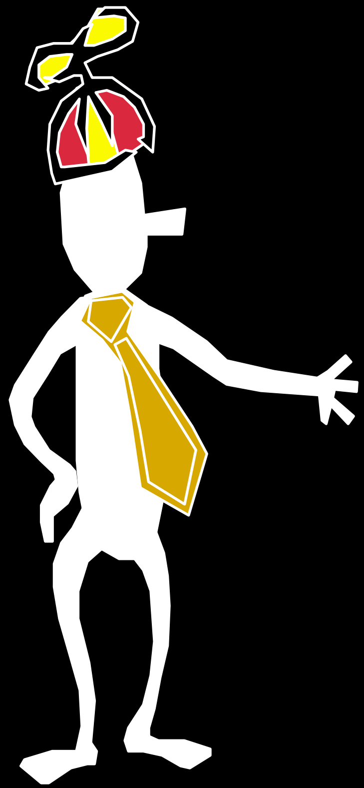
Test on Monday

Election Day

Graphs

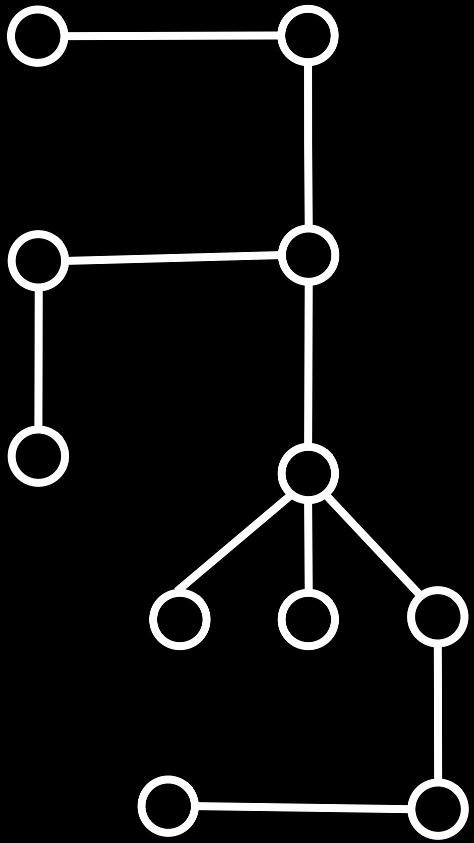
Lecture 18, October 23, 2008

What’s a tree?

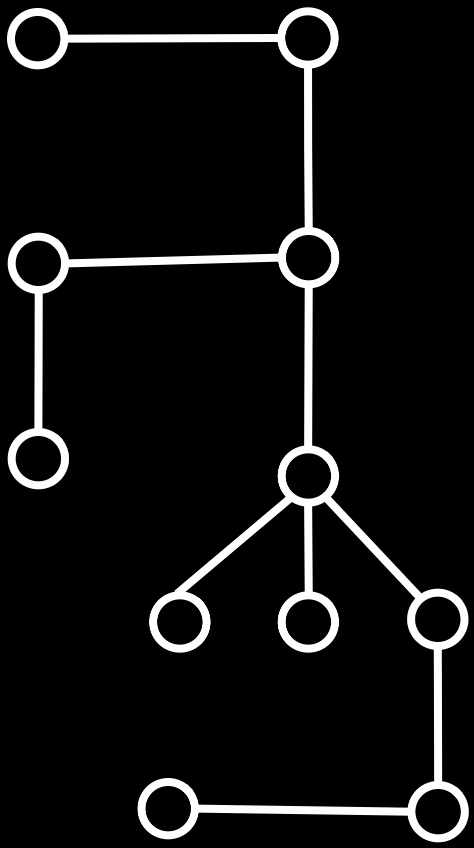


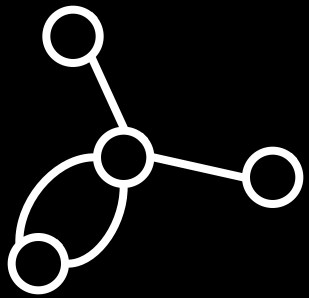
What's a tree?

A **tree** is a connected graph with no cycles

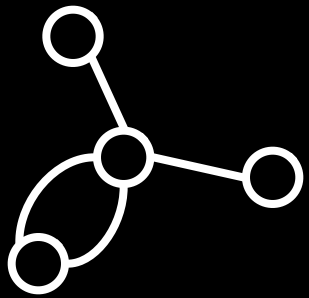


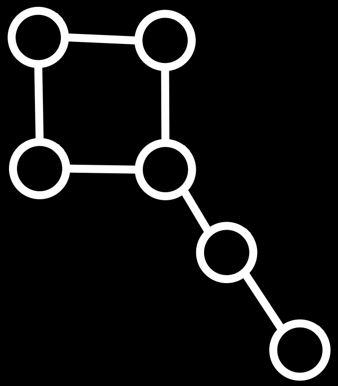
Tree





Not aTree





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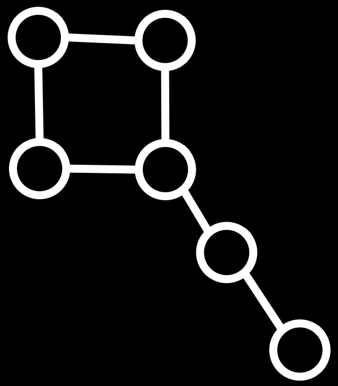

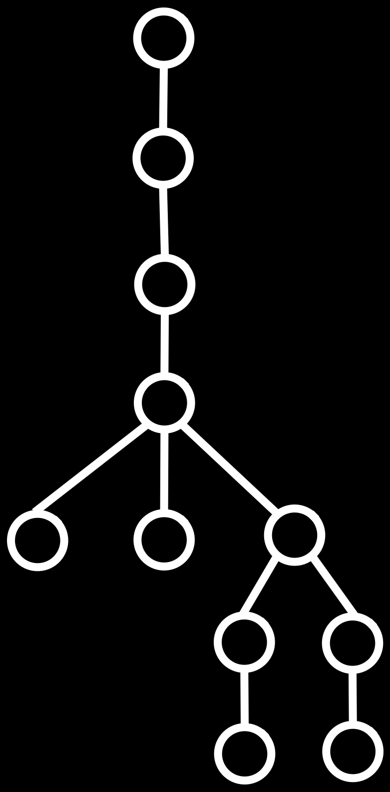

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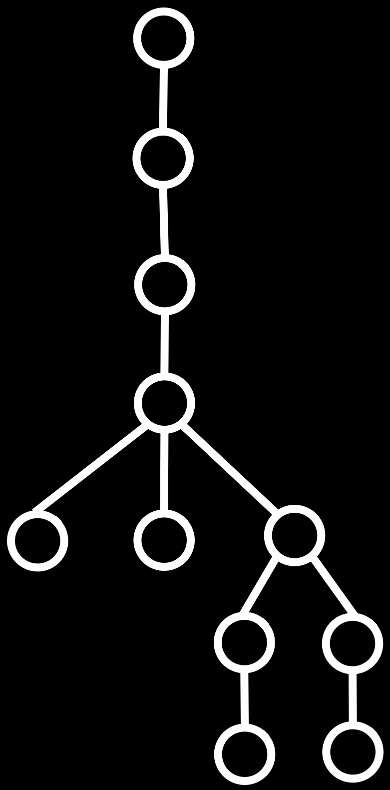

How Many n-Node Trees?

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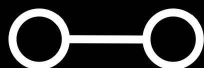
2: 

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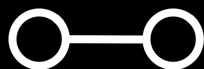
2: 

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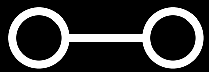
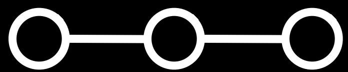
2: 

3: 

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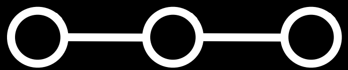
3: 

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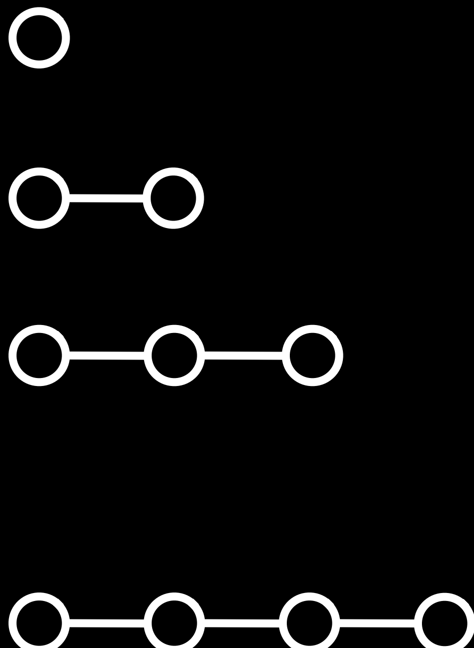
3:

4:



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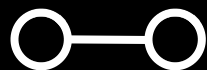
2: 

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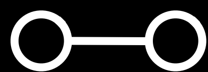
2: 

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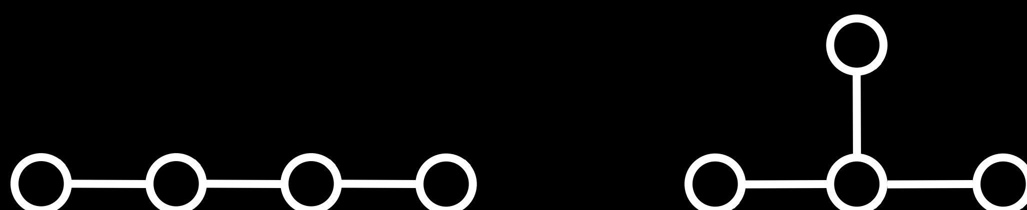
4: 

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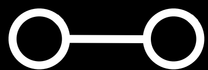
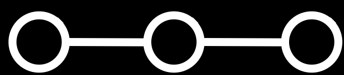
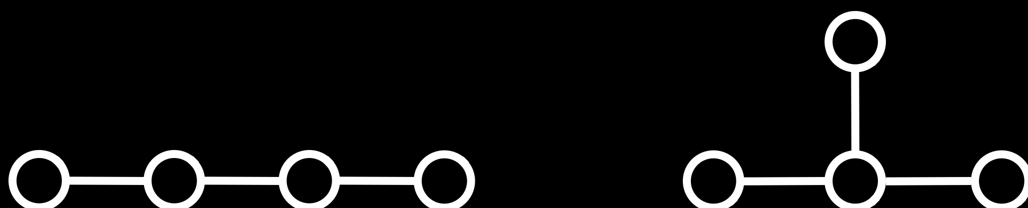
2: 

3: 

4: 

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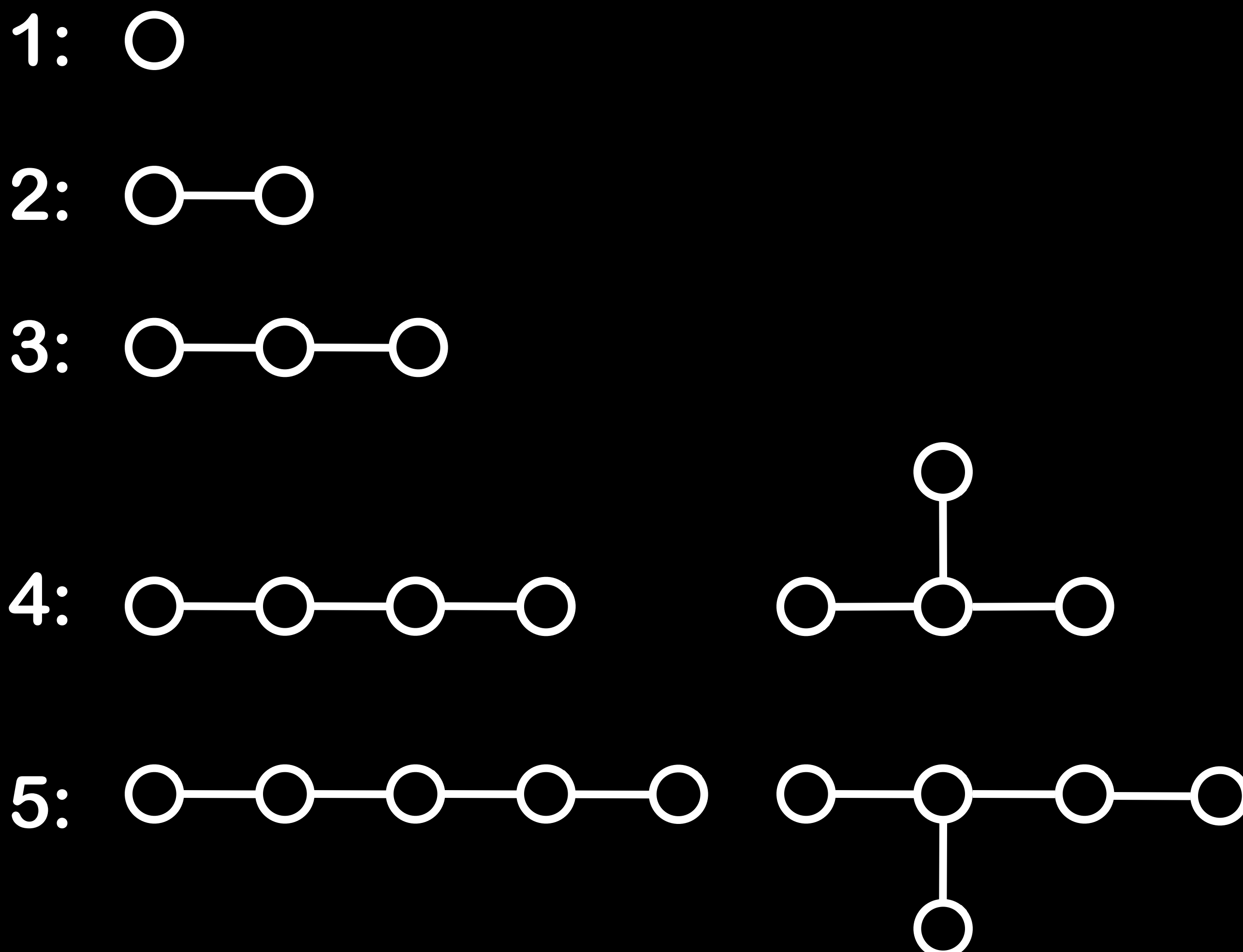

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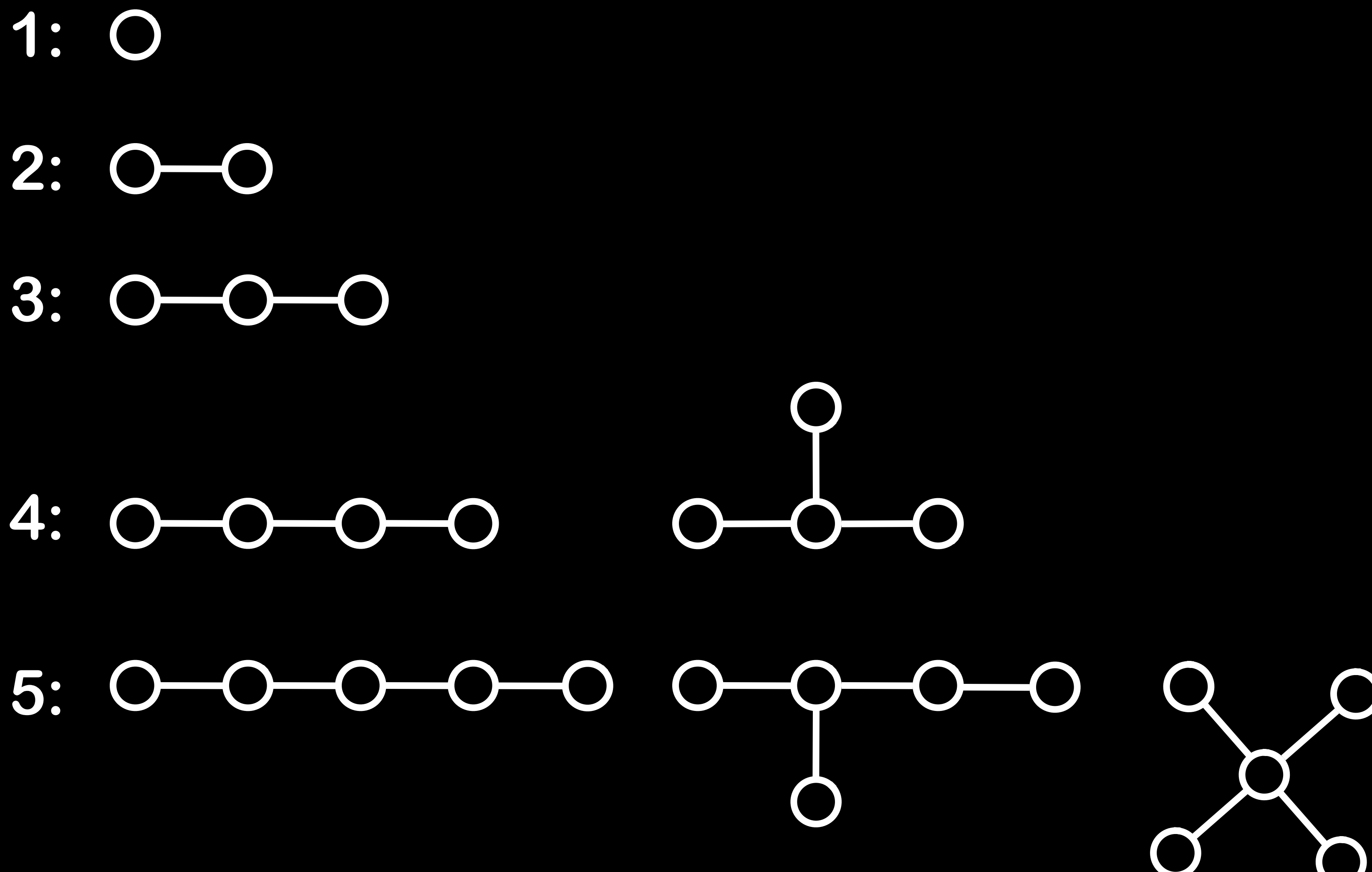

We'll pass around a piece of paper. Draw a new 8-node tree, and put your name next to it. (There are 23 of them…)

The Shy People Party

At the shy people party, people enter one-by-one, and as a person comes in, (s)he shakes hand with only one person already at the party.

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At the shy people party, people enter one-by-one, and as a person comes in, (s)he shakes hand with only one person already at the party.

Prove that at a shy party with n people (n >= 2), at least two people have shaken hands with only one other person.

The Shy People Party

Notation

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n will denote the number of nodes in a graph

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In this lecture:

- n will denote the number of nodes in a graph
- e will denote the number of edges in a graph

Theorem: Let G be a graph with n nodes and e edges

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3. G is connected and $n = e + 1$
4. G is acyclic and $n = e + 1$
5. G is acyclic and if any two non-adjacent points are joined by a line, the resulting graph has exactly one cycle

To prove this, it suffices to show

$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

1 => 2

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Proof: (by contradiction)

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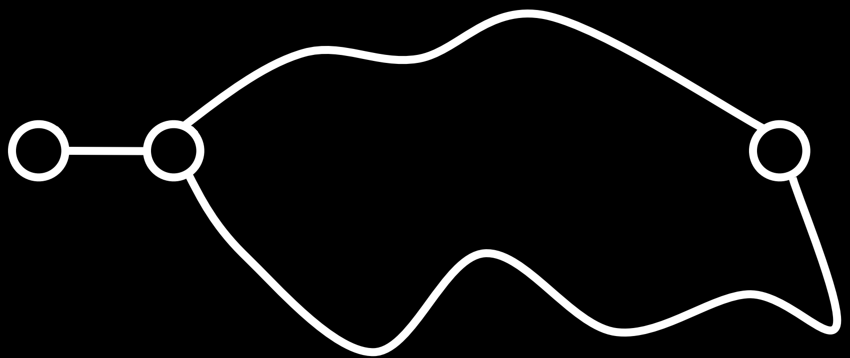

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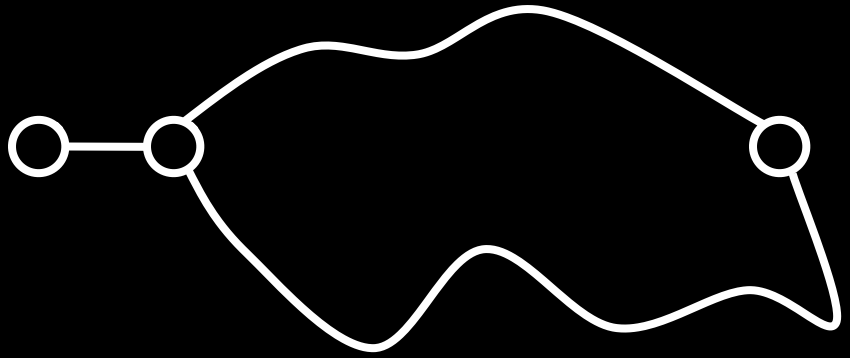

Then there exists a cycle!

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Let G have n nodes and let x and y be adjacent

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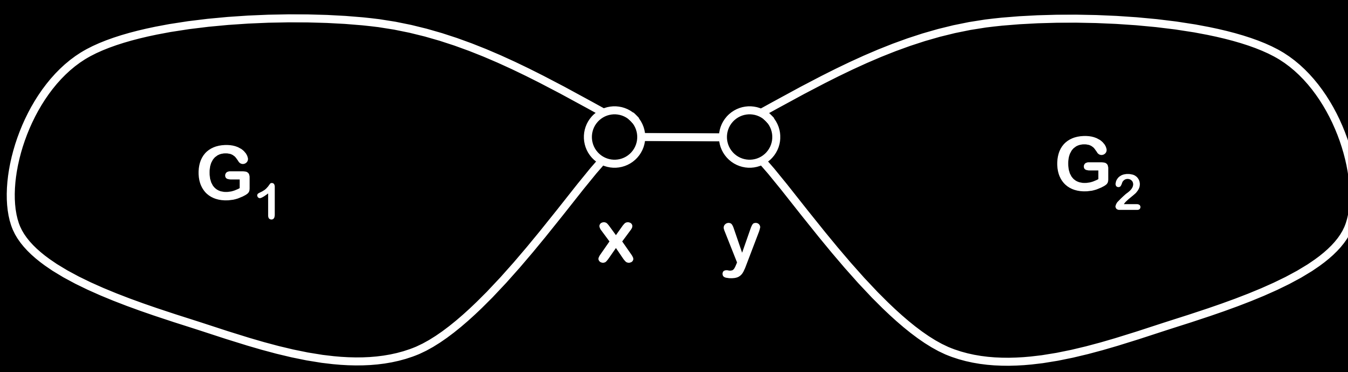

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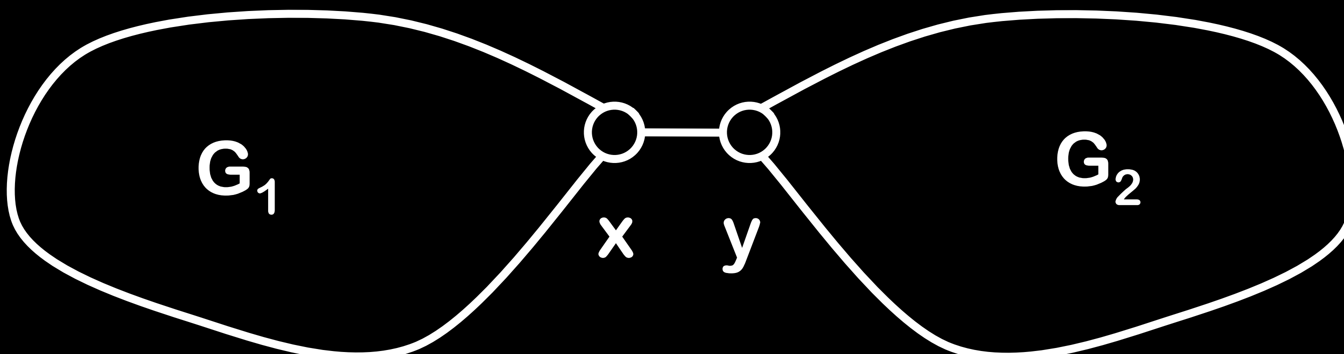

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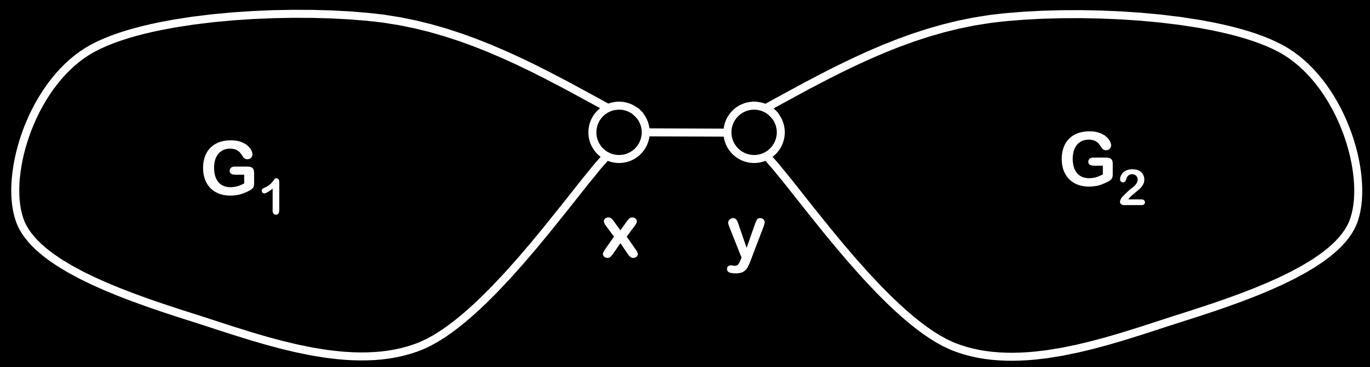

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Then $n = n_1 + n_2 = e_1 + e_2 + 2 = e + 1$

3 => 4

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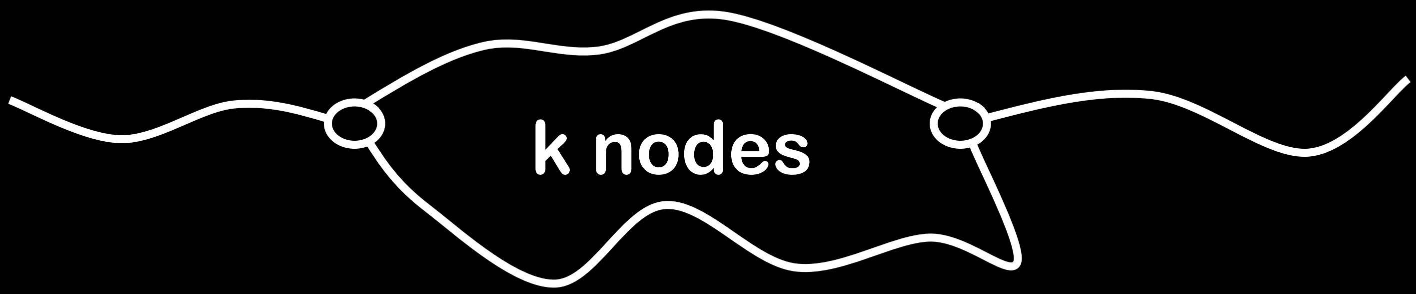

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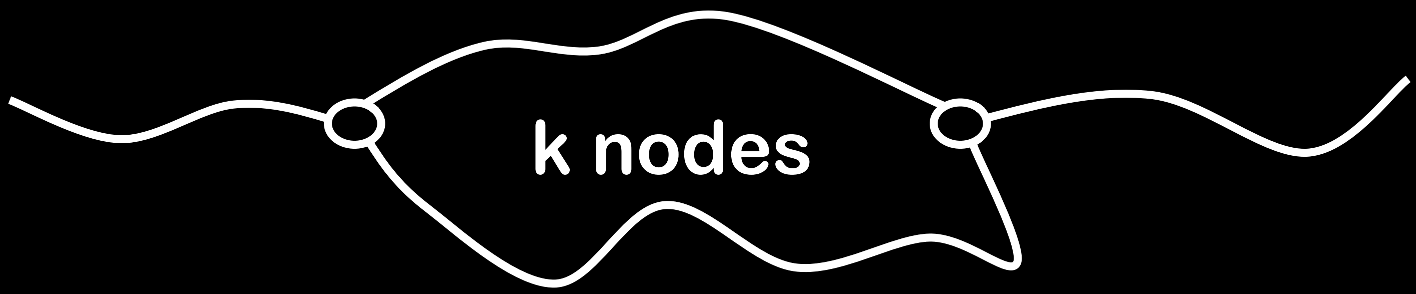

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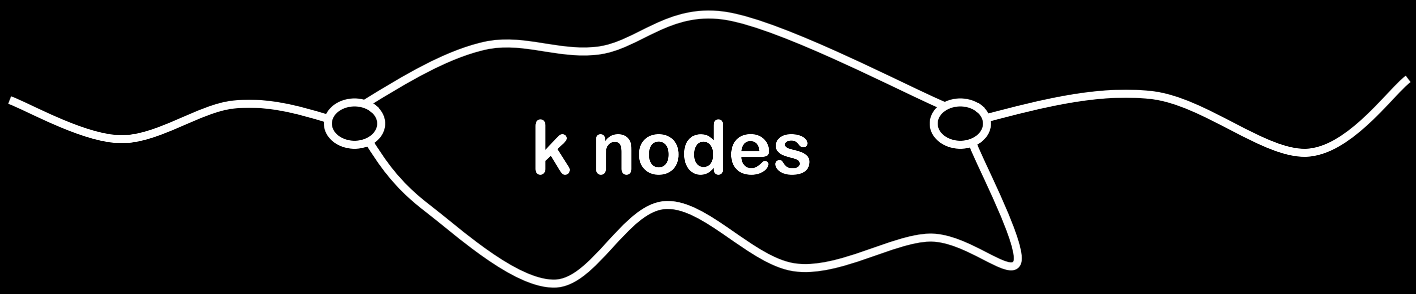

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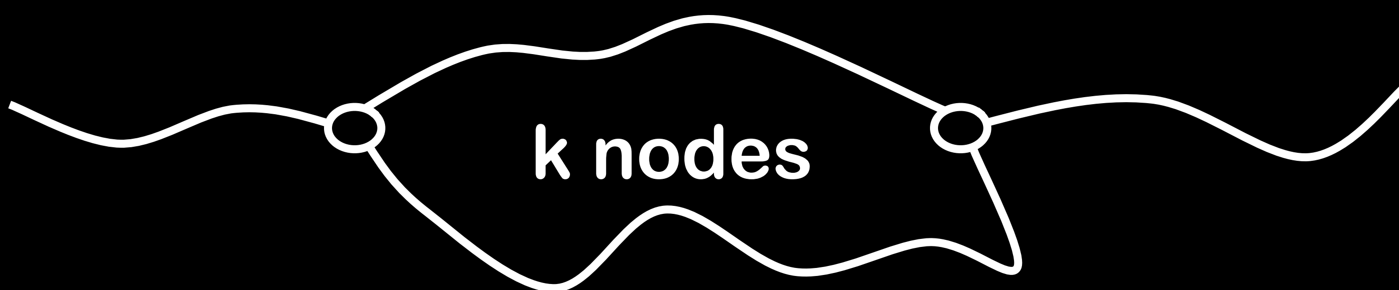

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Number of edges in the graph will be at least n

Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

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In any graph, sum of the degrees = 2e

Then the total number of edges in the tree is at least (2n-1)/2 = n - 1/2 > n - 1

How many labeled trees are there with three nodes?

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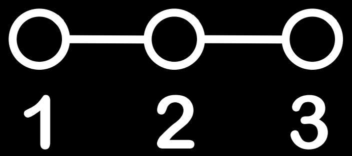

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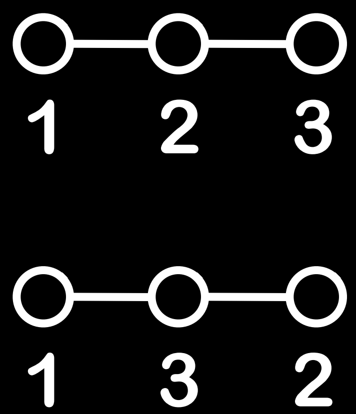

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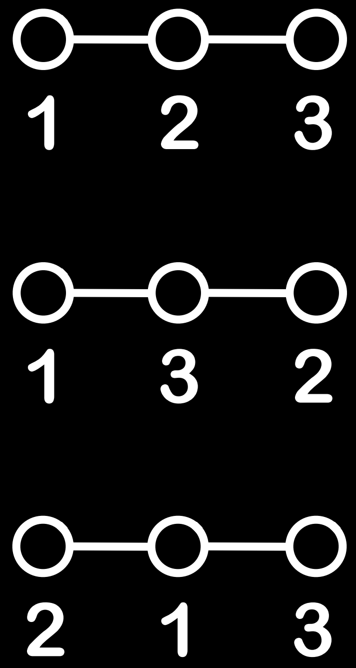
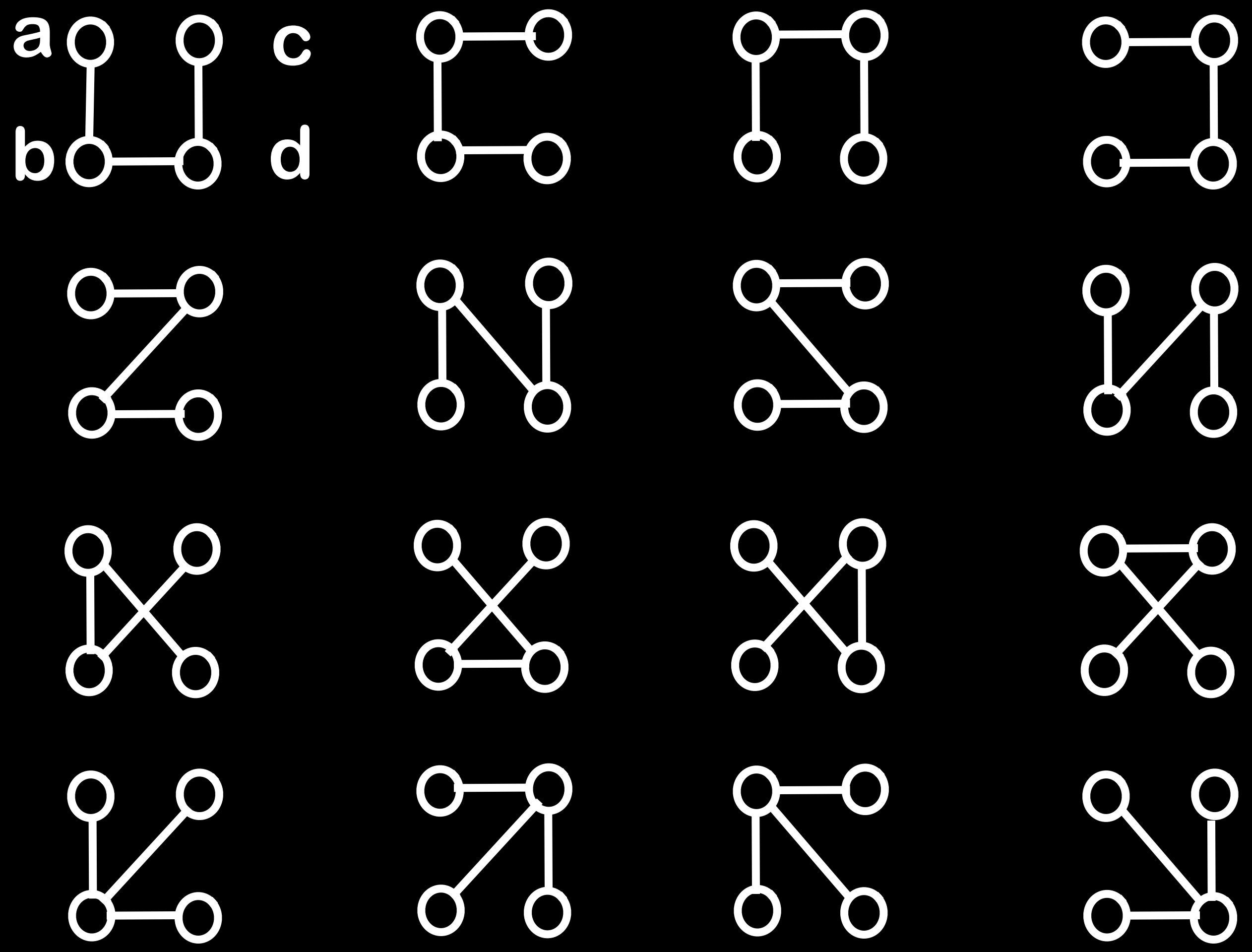


How many **labeled** trees are there with four nodes?



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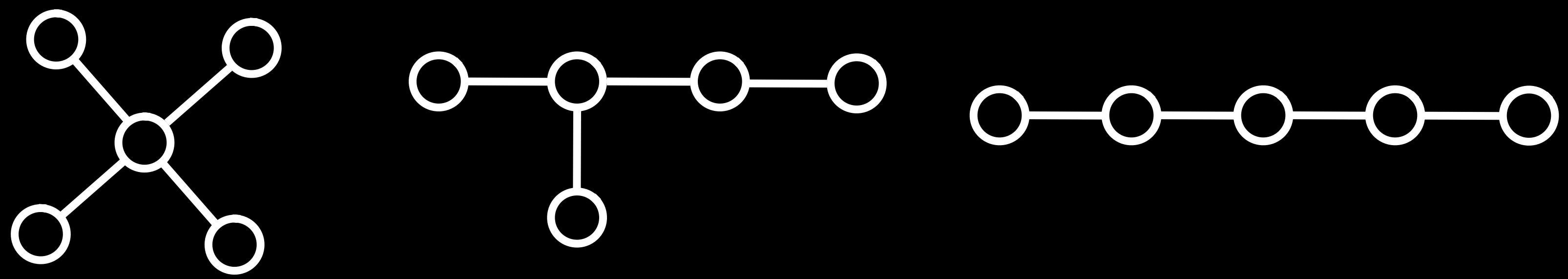

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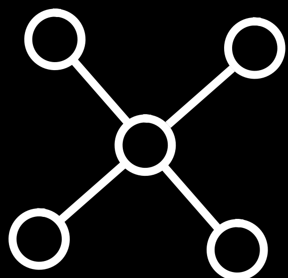
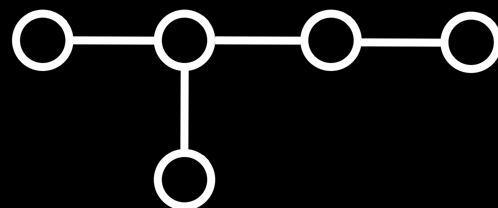


5
labelings



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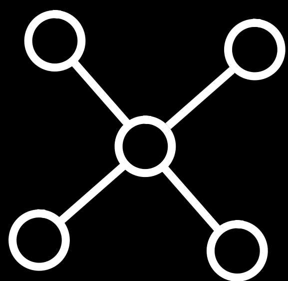

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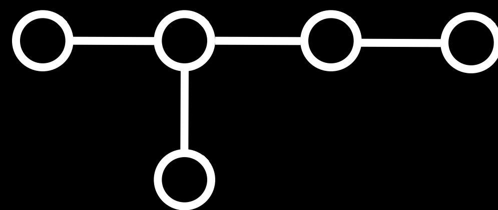

5 x 4 x 3

labelings

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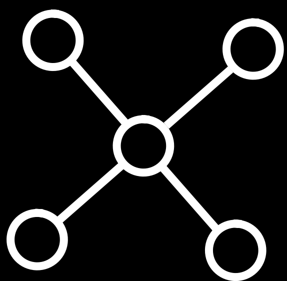

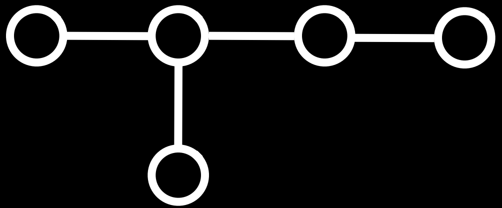

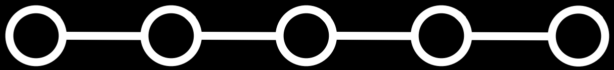

5
labelings

5 x 4 x 3

labelings

5!/ 2

labelings

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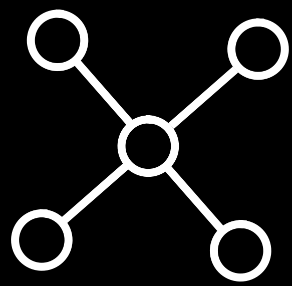

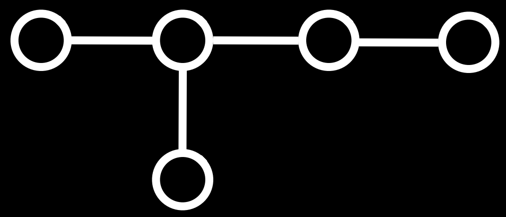

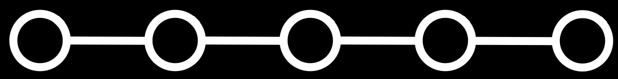

5
labelings

5 x 4 x 3

labelings

5!/ 2

labelings

125 labeled trees

How many labeled trees are there with n nodes?

How many labeled trees are there with n nodes?

3 labeled trees with 3 nodes

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3 labeled trees with 3 nodes

16 labeled trees with 4 nodes

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16 labeled trees with 4 nodes

125 labeled trees with 5 nodes

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3 labeled trees with 3 nodes

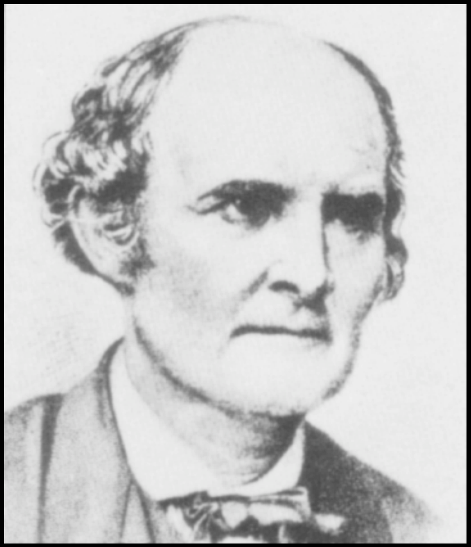
16 labeled trees with 4 nodes

125 labeled trees with 5 nodes

n^{n-2} labeled trees with n nodes

Cayley's Formula

The number of labeled trees on n nodes is n^{n-2}



The proof will use the correspondence principle

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Each labeled tree on n nodes

corresponds to

A sequence in $\{1,2,\ldots,n\}^{n-2}$ (that is, n-2 numbers, each in the range [1..n])

How to make a sequence from a tree?

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Loop through i from 1 to $n-2$

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Let L be the degree-1 node with the lowest label

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Define the i^{th} element of the sequence as the label of the node adjacent to L

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Loop through i from 1 to $n-2$

Let L be the degree-1 node with the lowest label

Define the i^{th} element of the sequence as the label of the node adjacent to L

Delete the node L from the tree

How to make a sequence from a tree?

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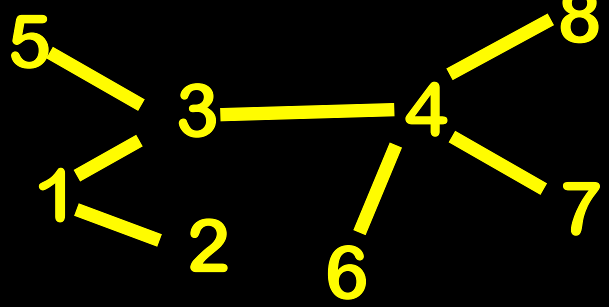

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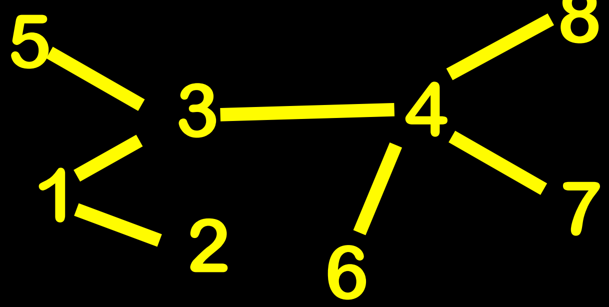

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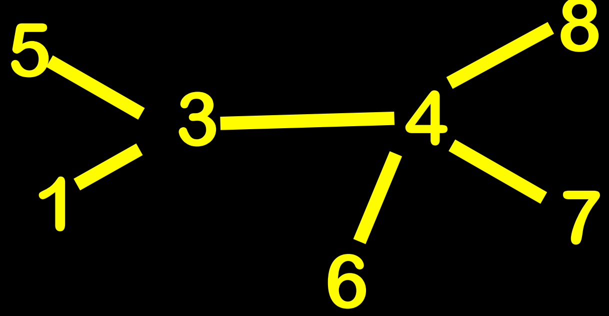

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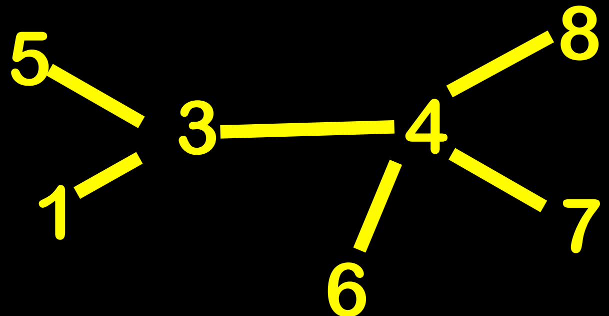

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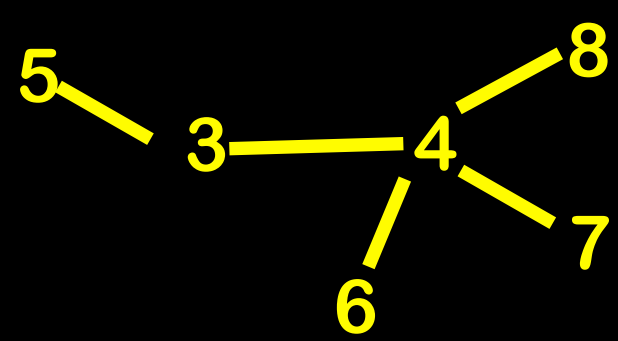

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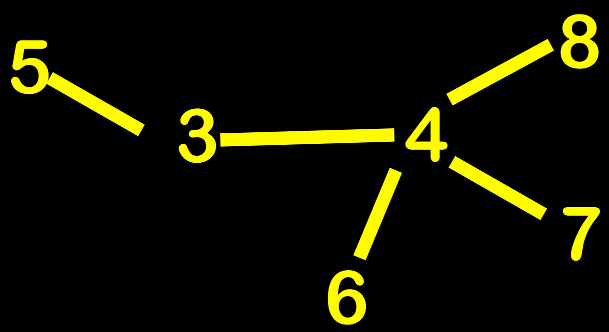

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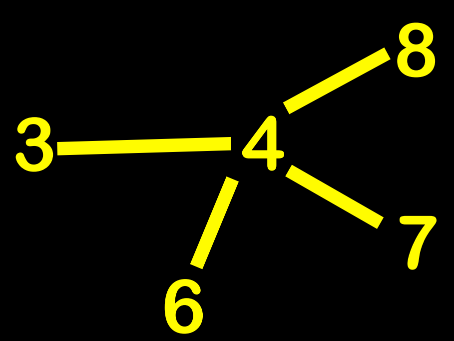

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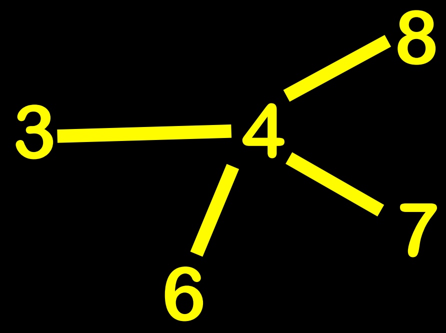

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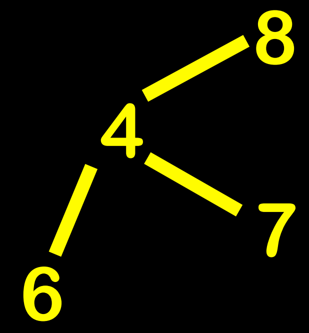

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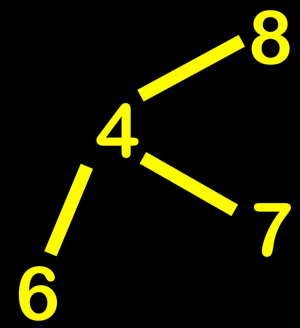

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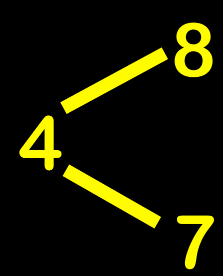

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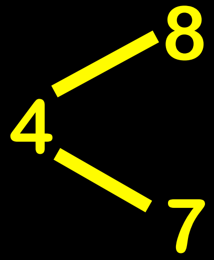

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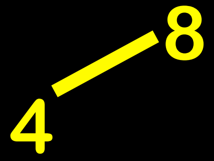

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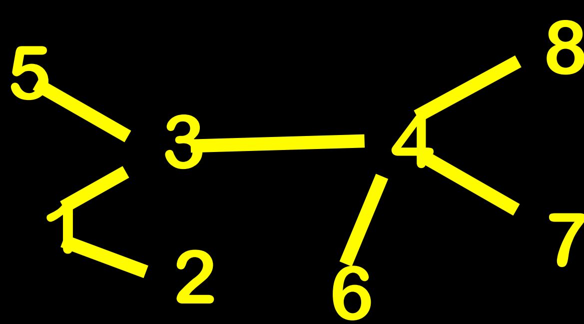

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Spanning Trees

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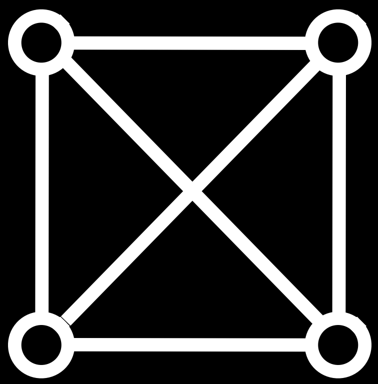

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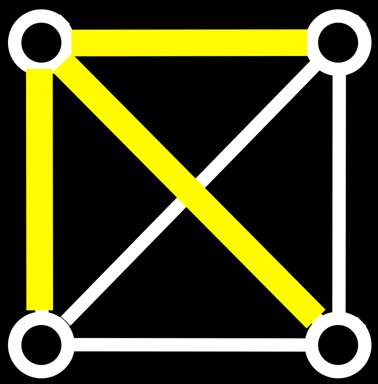

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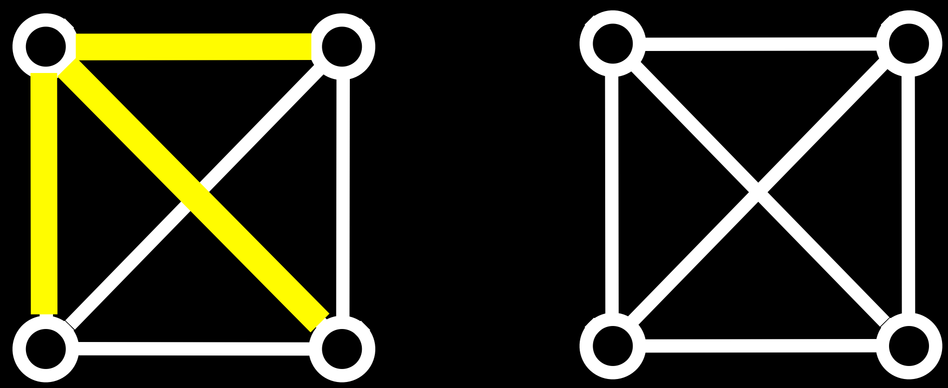

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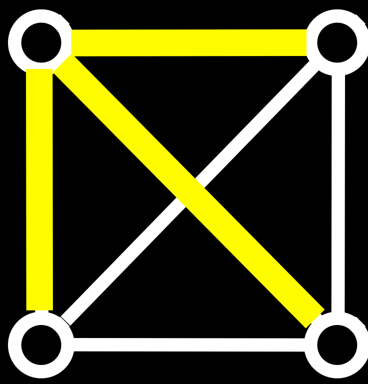

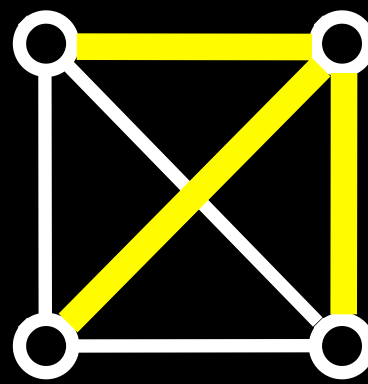

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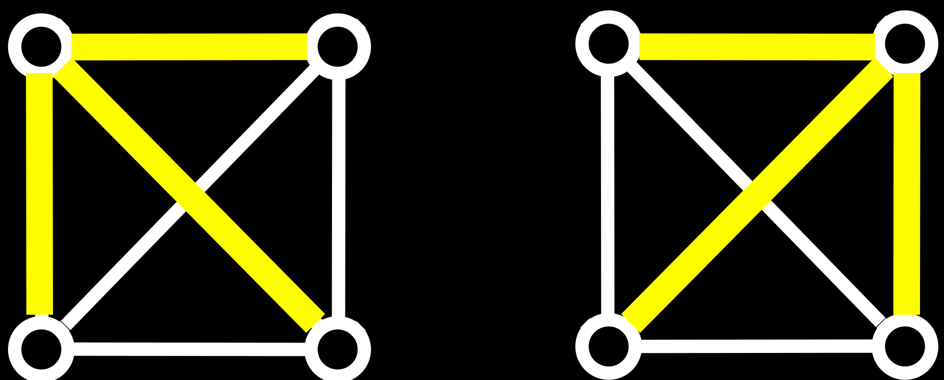
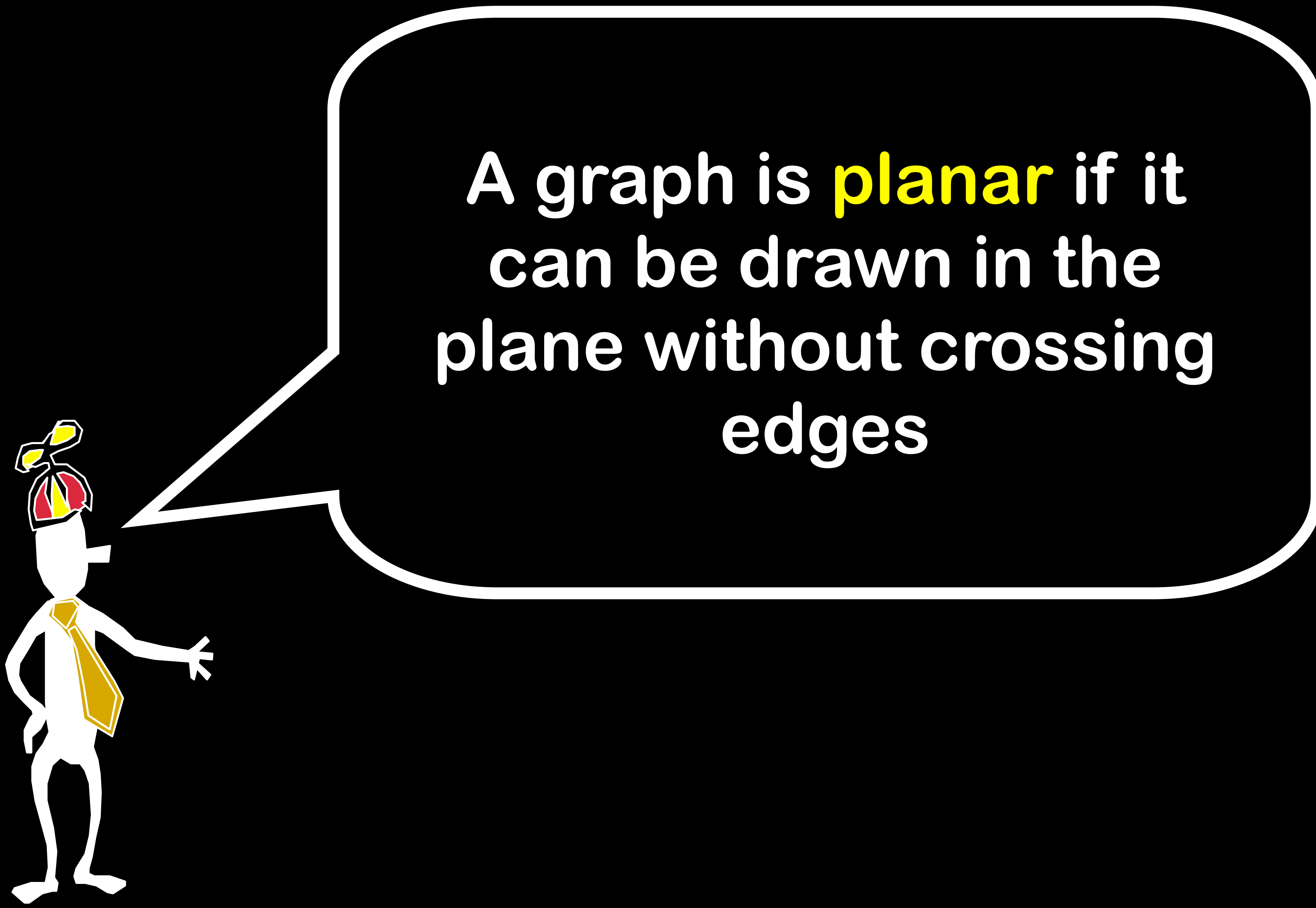


Every connected graph has a spanning tree


A graph is planar if it can be drawn in the plane without crossing edges

Examples of Planar Graphs

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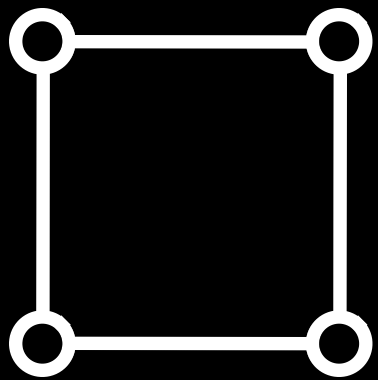

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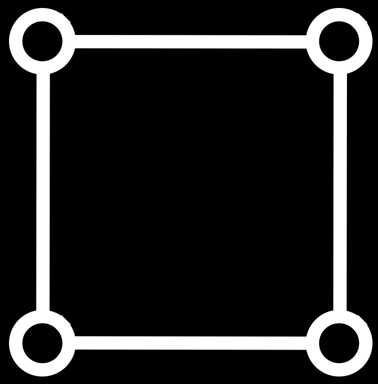

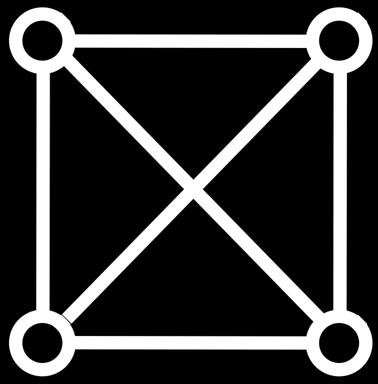

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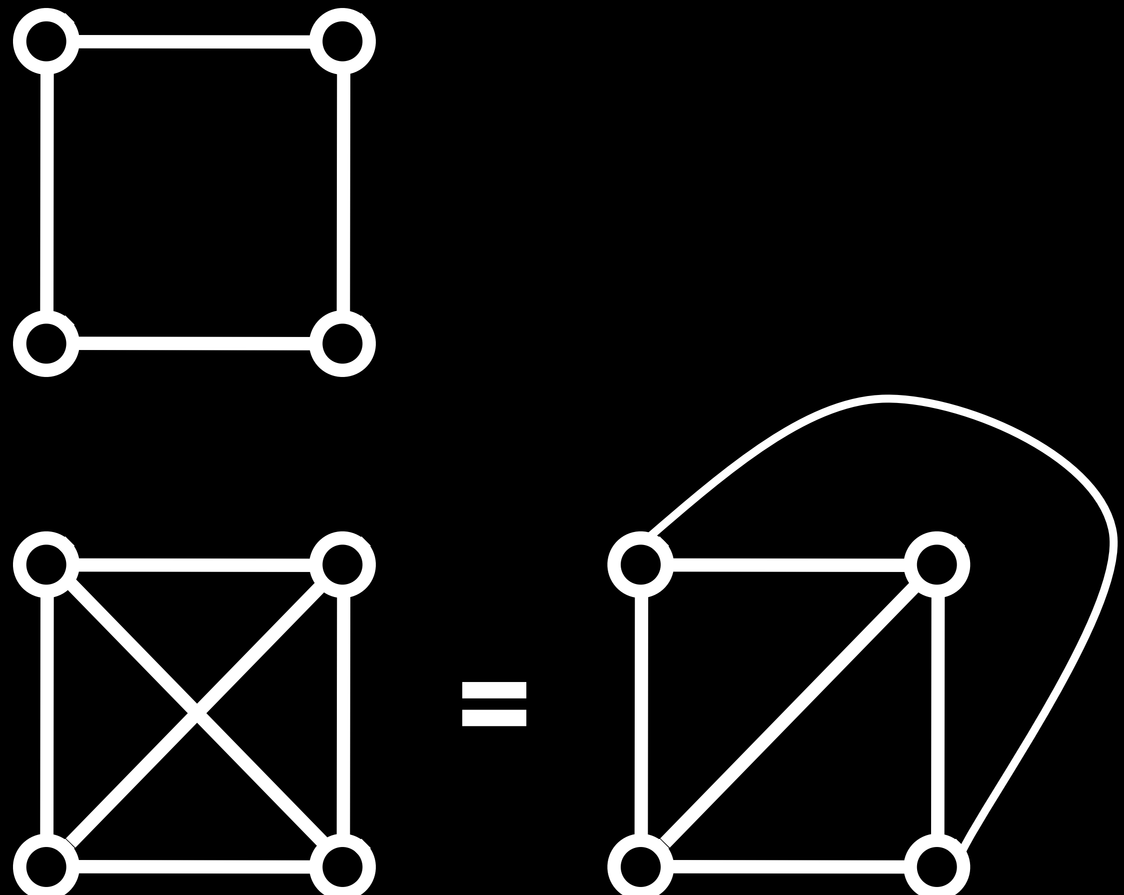
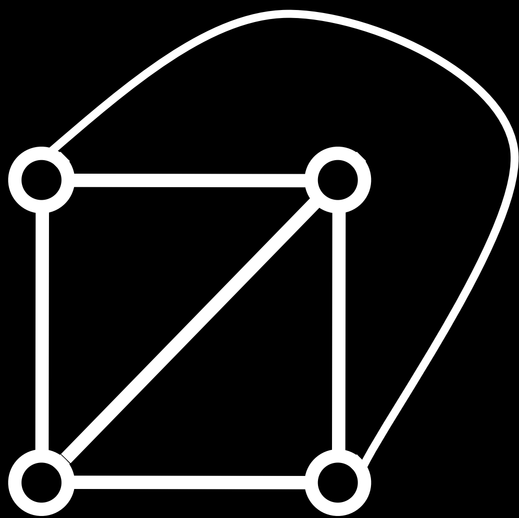


http://www.planarity.net

Faces



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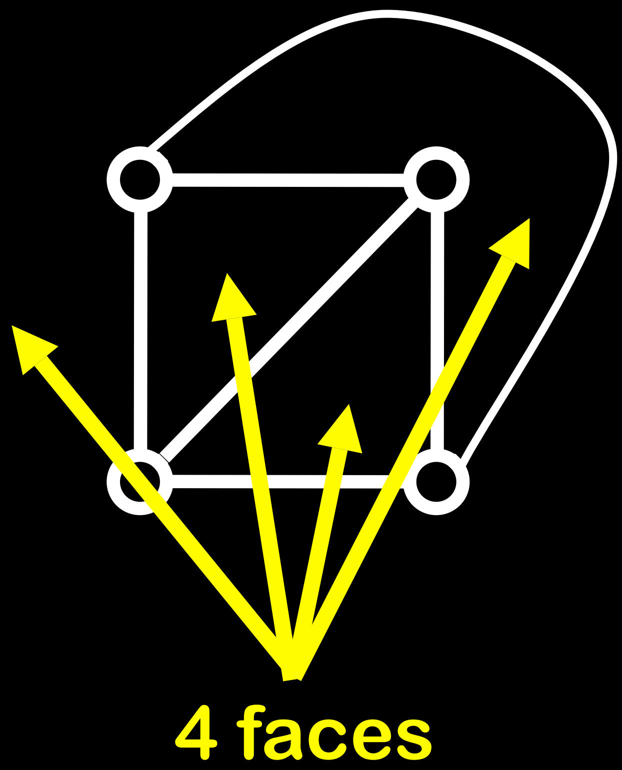
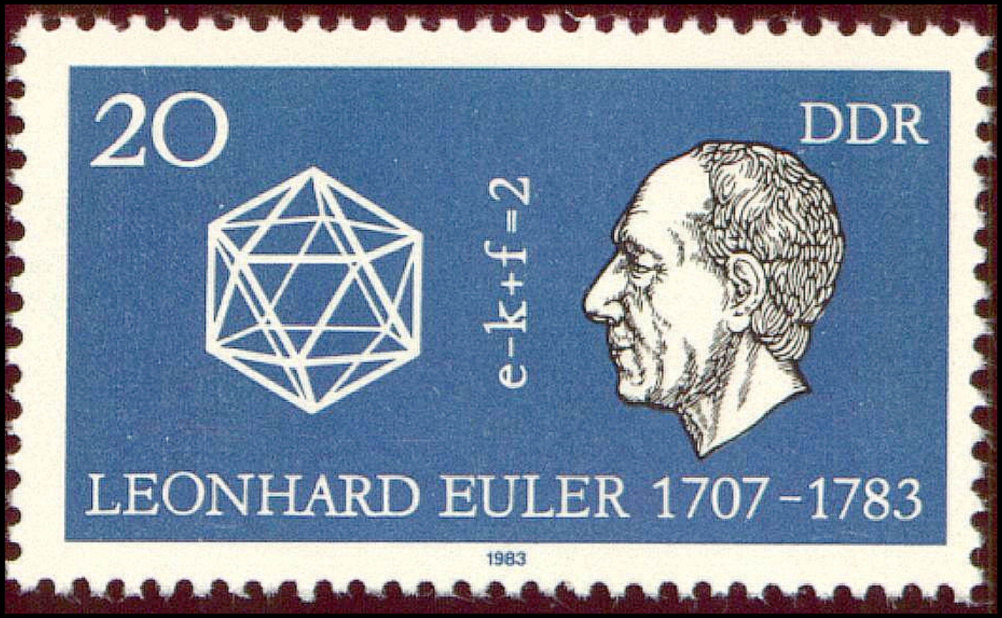


Euler’s Formula

If G is a connected planar graph with n vertices, e edges and f faces, then $n - e + f = 2$



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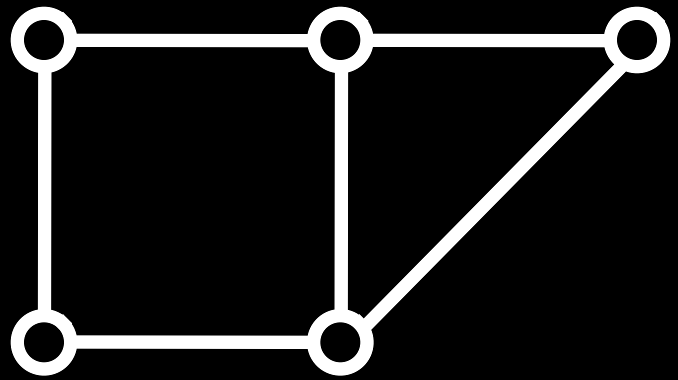

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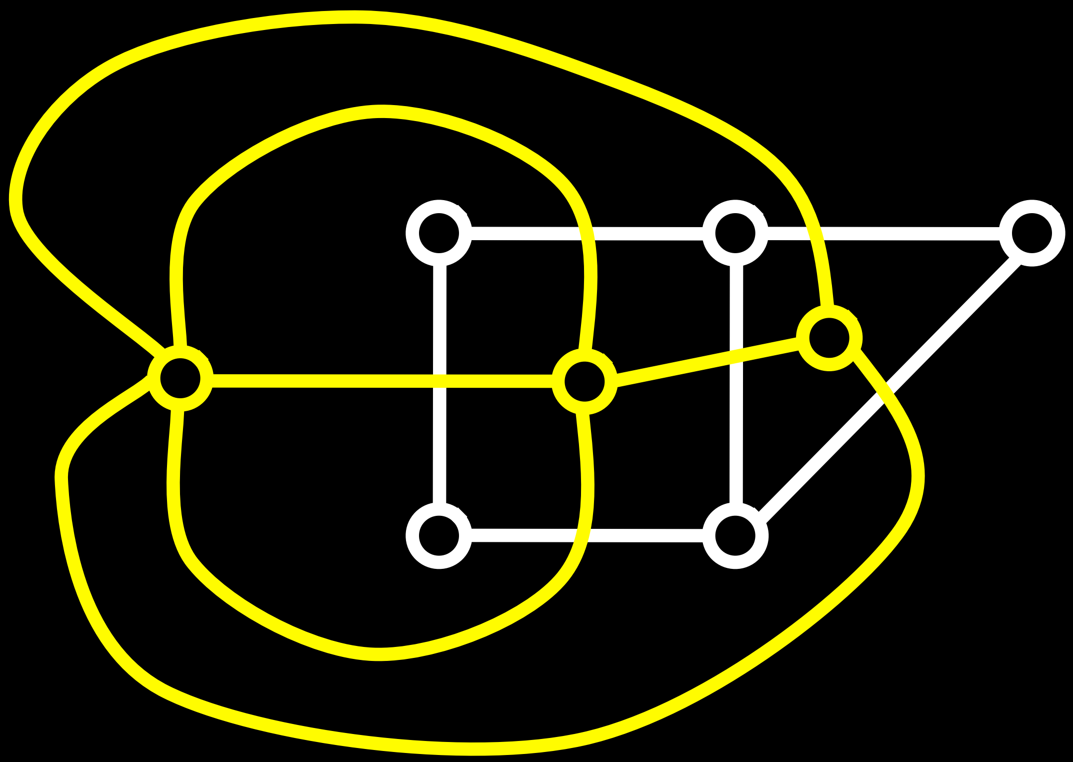

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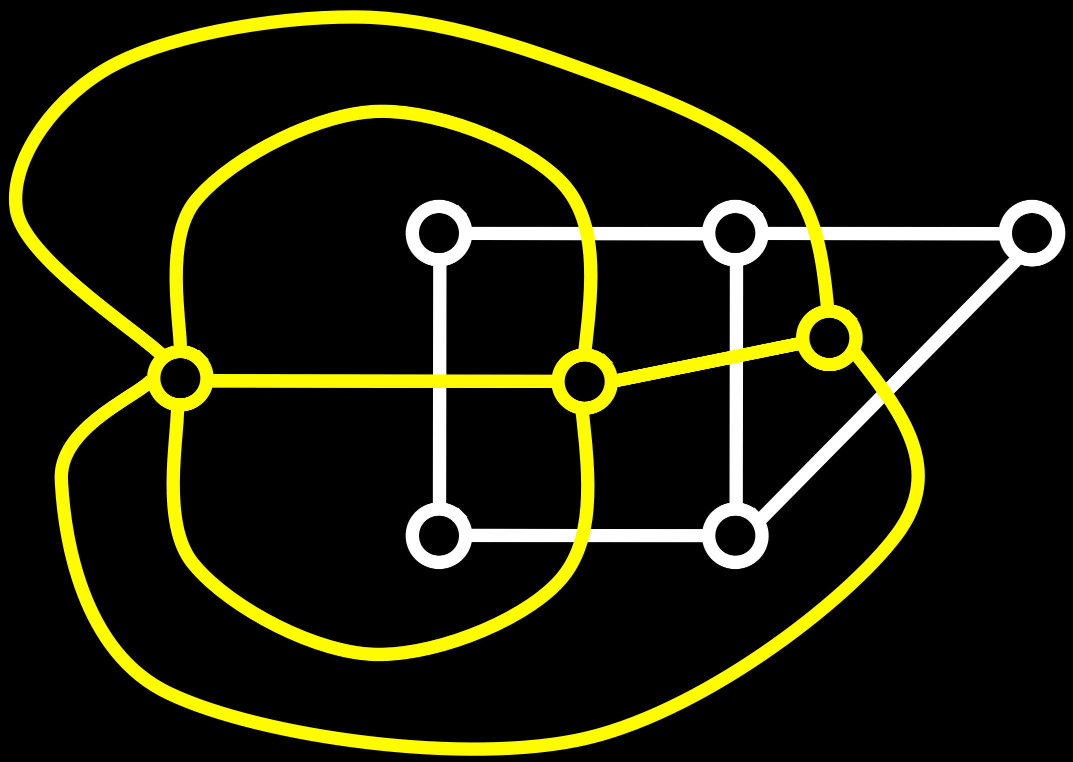
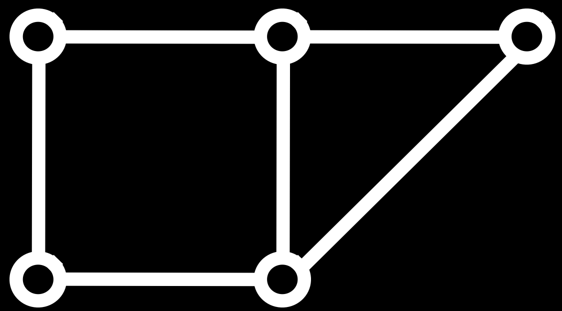
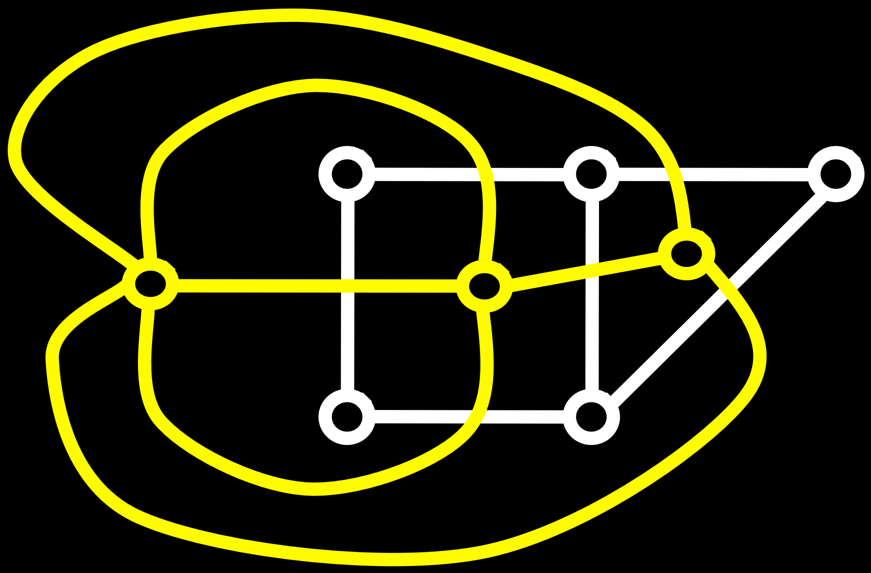
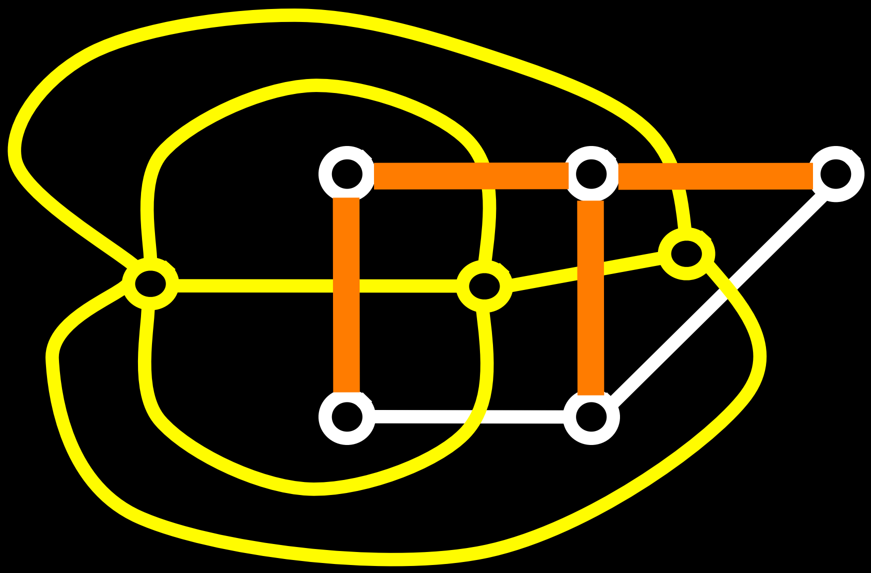


Dual = put a node in every face, and an edge between every adjacent face





Let G* be the dual graph of G



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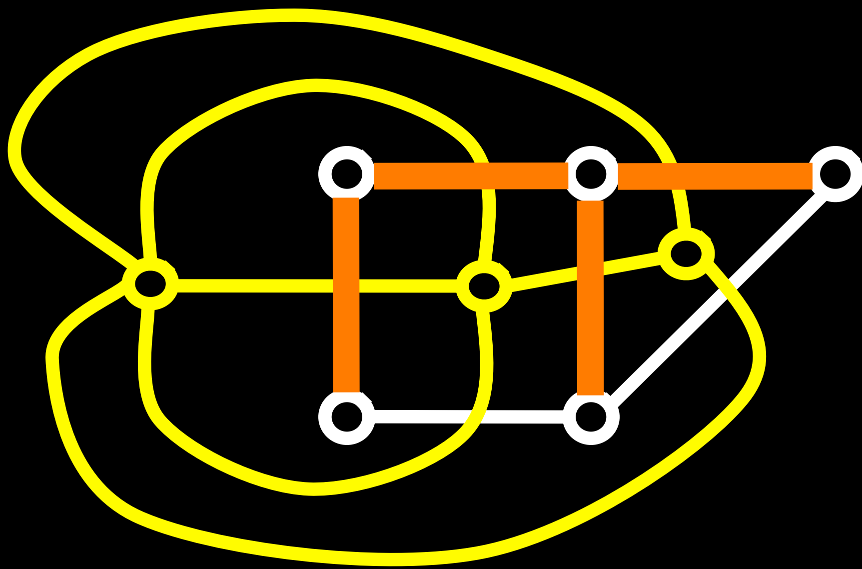

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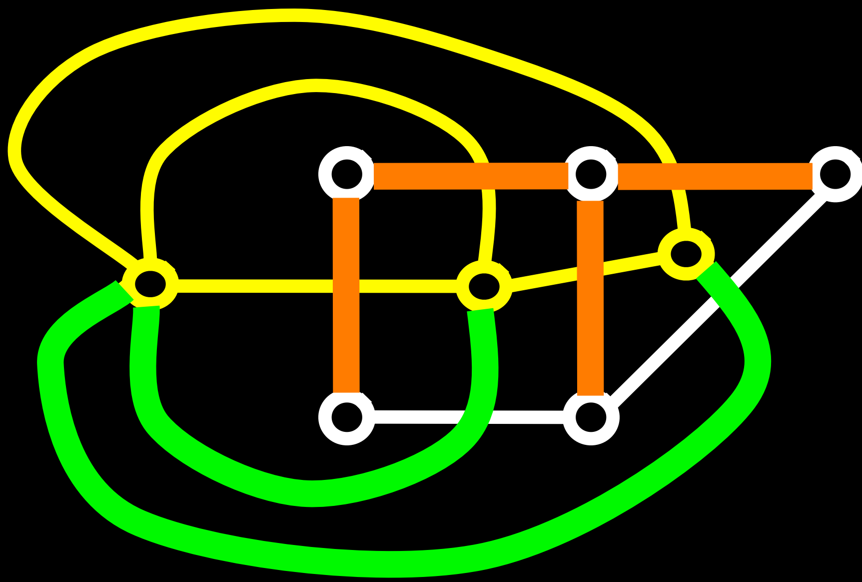

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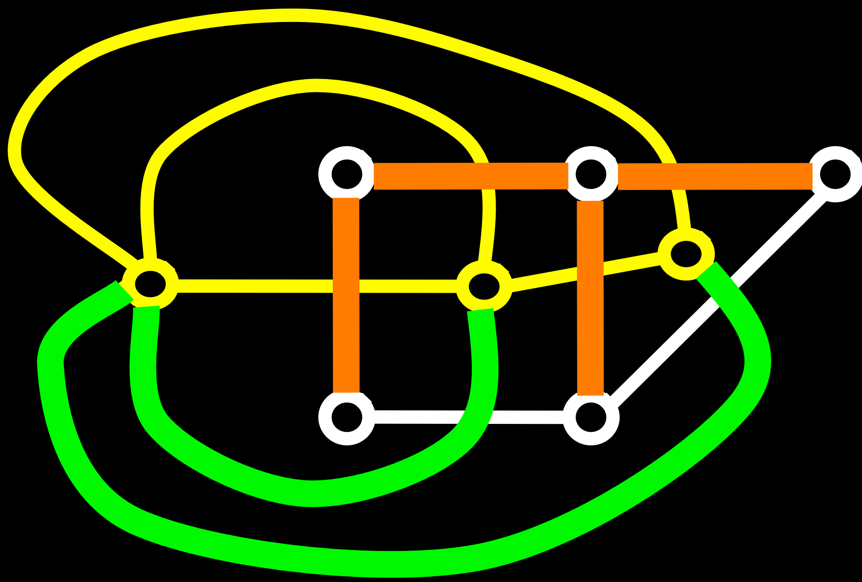

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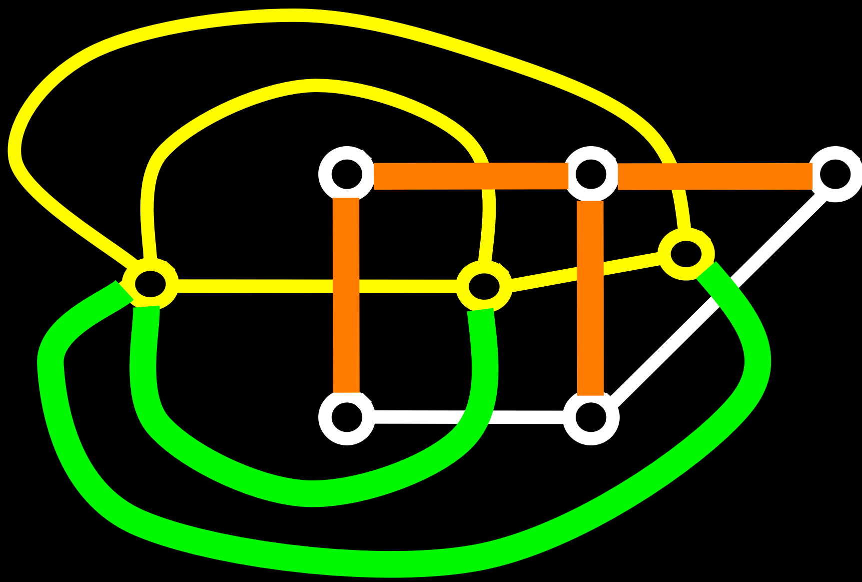

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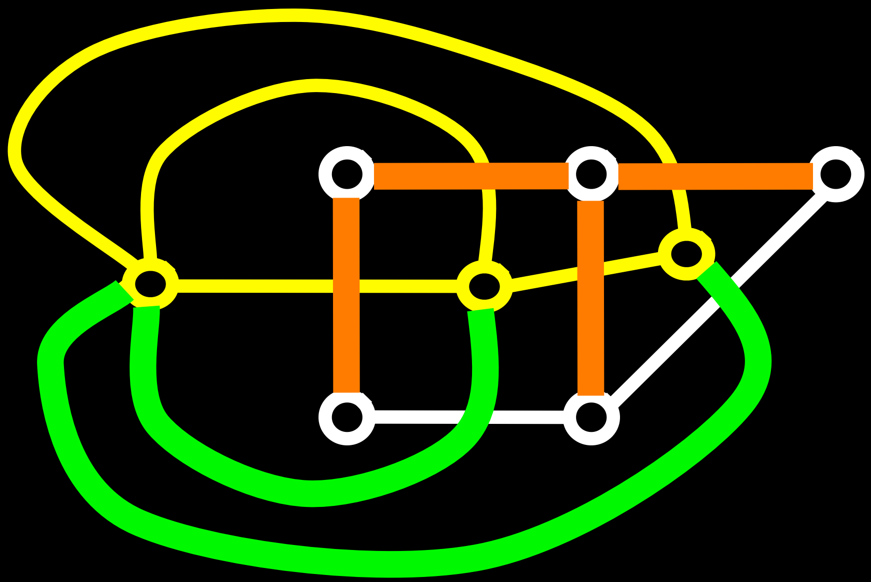

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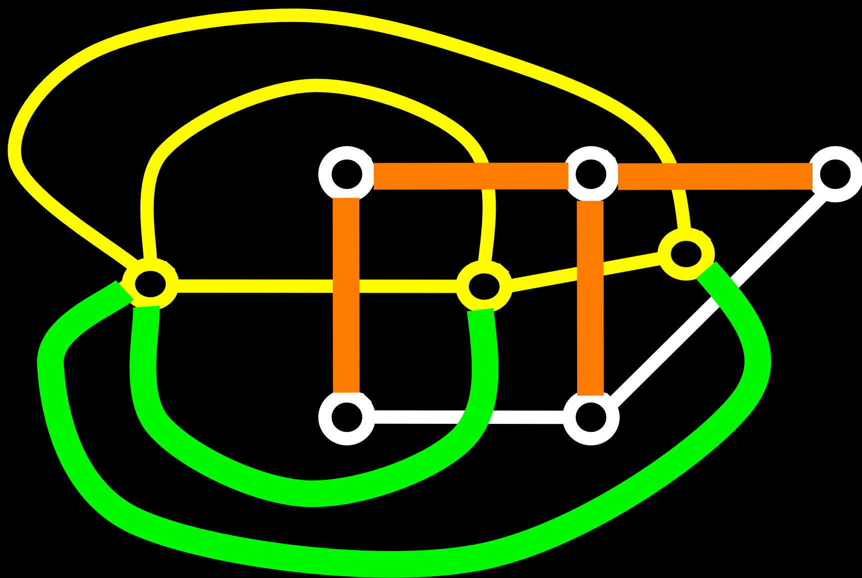

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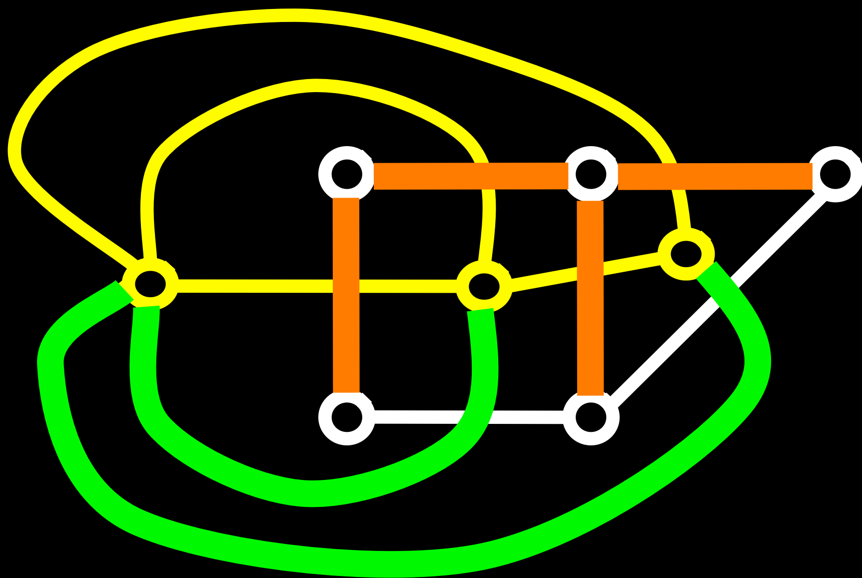

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So $3n + 3f \leq 3e \Rightarrow 3(n-e+f) \leq 0$, contradiction.

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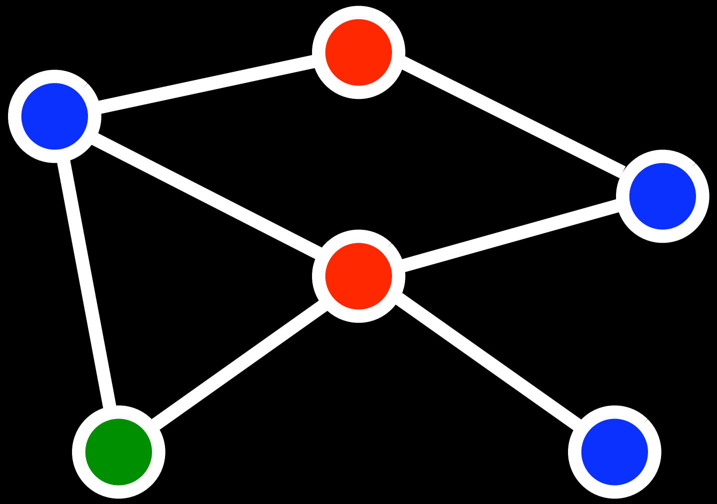

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Arises surprisingly often in CS

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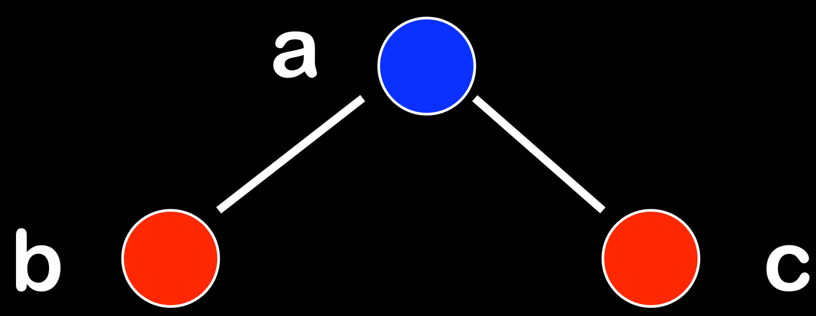
Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register

Instructions	Live variables
	a
b = a+2	
	a,b
c = b*b	
	a,c
b = c+1	
	a,b
return a*b	



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Remove v and color by Induction Hypothesis

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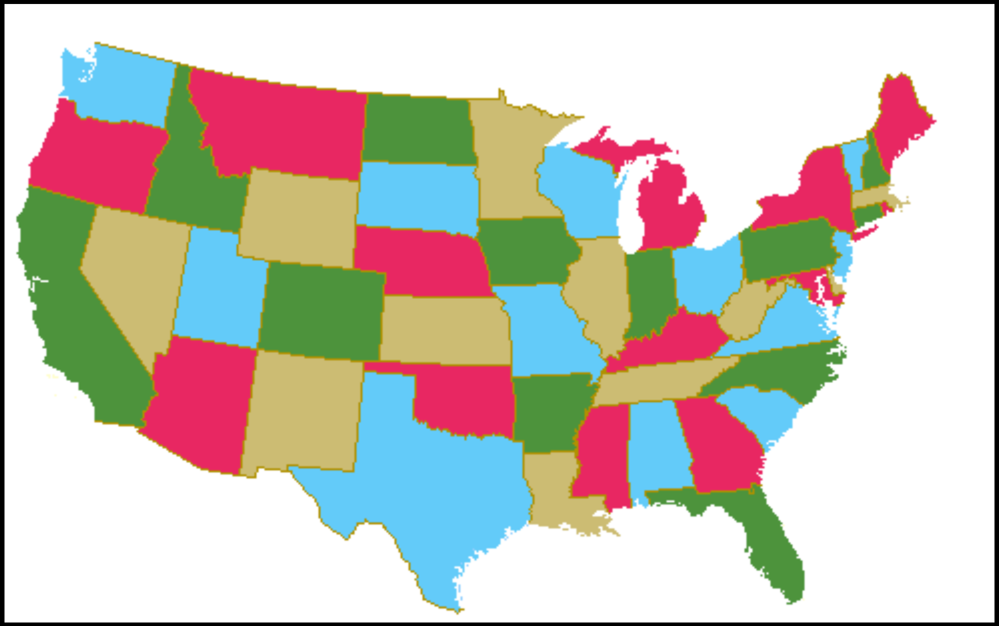
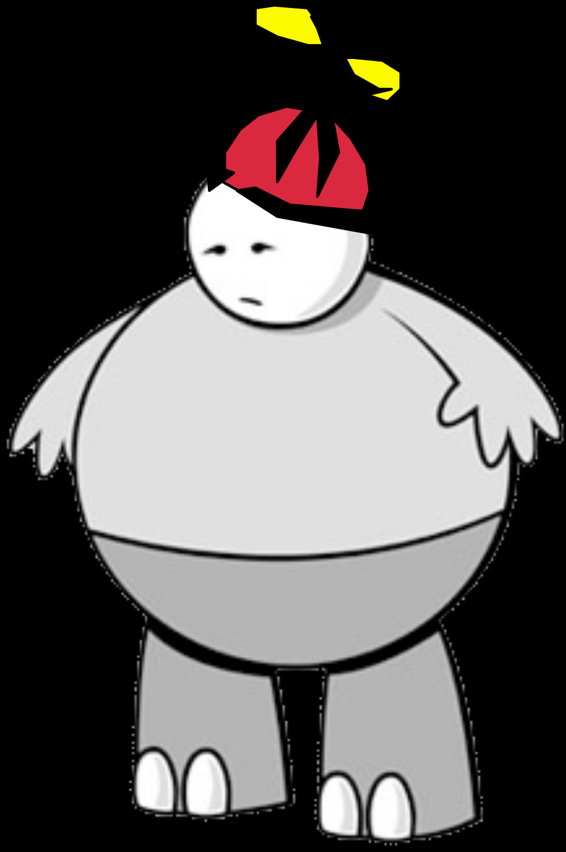
4-color theorem remains challenging!





Here's What You Need to Know…

Trees

- Counting Trees
- Different Characterizations
- Cayley's formula

Planar Graphs

- Definition
- Euler's Theorem
- Coloring Planar Graphs