15-251
Great Theoretical Ideas in Computer Science
Upcoming Events

Review Session on Saturday
(5 pm, Wean 5409)

Test on Monday

Election Day
Graphs

Lecture 18, October 23, 2008
What’s a tree?
What’s a tree?

A **tree** is a connected graph with no cycles.
Tree
Not a Tree
Not a Tree
Tree
How Many n-Node Trees?
How Many n-Node Trees?
How Many n-Node Trees?

1: 0
How Many n-Node Trees?

1: 0

2:
How Many n-Node Trees?

1:  

2:  

<p>| | |</p>
<table>
<thead>
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How Many n-Node Trees?

1: O

2: O—O

3:
How Many n-Node Trees?

1:  

2:  

3:  
How Many n-Node Trees?

1: 〇

2: 〇〇

3: 〇〇〇

4:
How Many n-Node Trees?

1: 〇

2: 〇〇

3: 〇〇〇

4: 〇〇〇〇
How Many n-Node Trees?

1:  

2:  

3:  

4:  

4:  

How Many n-Node Trees?

1: 〇

2: 〇〇

3: 〇〇〇

4: 〇〇〇〇 〇〇〇〇

5:
How Many n-Node Trees?

1: 〇
2: 〇〇
3: 〇〇〇
4: 〇〇〇〇 〇〇〇
5: 〇〇〇〇〇
How Many n-Node Trees?

1: 〇

2: 〇〇

3: 〇〇〇

4: 〇〇〇〇 〇〇〇

5: 〇〇〇〇〇 〇〇〇〇〇
How Many n-Node Trees?

1:  

2:  

3:  

4:  

5:  

---
We’ll pass around a piece of paper. Draw a new 8-node tree, and put your name next to it. (There are 23 of them...)
At the shy people party, people enter one-by-one, and as a person comes in, (s)he shakes hand with only one person already at the party.
At the shy people party, people enter one-by-one, and as a person comes in, (s)he shakes hands with only one person already at the party.

Prove that at a shy party with \( n \) people \( (n \geq 2) \), at least two people have shaken hands with only one other person.
The Shy People Party
Notation
Notation

In this lecture:
Notation

In this lecture:

$n$ will denote the number of nodes in a graph
Notation

In this lecture:

n will denote the number of nodes in a graph

e will denote the number of edges in a graph
Theorem: Let G be a graph with n nodes and e edges
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The following are equivalent:
Theorem: Let $G$ be a graph with $n$ nodes and $e$ edges. The following are equivalent:

1. $G$ is a tree (connected, acyclic)
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The following are equivalent:

1. $G$ is a tree (connected, acyclic)
2. Every two nodes of $G$ are joined by a unique path
Theorem: Let $G$ be a graph with $n$ nodes and $e$ edges

The following are equivalent:

1. $G$ is a tree (connected, acyclic)
2. Every two nodes of $G$ are joined by a unique path
3. $G$ is connected and $n = e + 1$
Theorem: Let $G$ be a graph with $n$ nodes and $e$ edges

The following are equivalent:

1. $G$ is a tree (connected, acyclic)
2. Every two nodes of $G$ are joined by a unique path
3. $G$ is connected and $n = e + 1$
4. $G$ is acyclic and $n = e + 1$
Theorem: Let $G$ be a graph with $n$ nodes and $e$ edges

The following are equivalent:

1. $G$ is a tree (connected, acyclic)

2. Every two nodes of $G$ are joined by a unique path

3. $G$ is connected and $n = e + 1$

4. $G$ is acyclic and $n = e + 1$

5. $G$ is acyclic and if any two non-adjacent points are joined by a line, the resulting graph has exactly one cycle
To prove this, it suffices to show

\[ 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1 \]
1 => 2

1. G is a tree (connected, acyclic)
2. Every two nodes of G are joined by a unique path
1 => 2

1. G is a tree (connected, acyclic)

2. Every two nodes of G are joined by a unique path

Proof: (by contradiction)
1 $\Rightarrow$ 2

1. G is a tree (connected, acyclic)

2. Every two nodes of G are joined by a unique path

Proof: (by contradiction)

Assume G is a tree that has two nodes connected by two different paths:
1 => 2

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Proof: (by contradiction)

Assume G is a tree that has two nodes connected by two different paths:
1 $\Rightarrow$ 2

1. G is a tree (connected, acyclic)

2. Every two nodes of G are joined by a unique path

Proof: (by contradiction)

Assume G is a tree that has two nodes connected by two different paths:

Then there exists a cycle!
2 \Rightarrow 3 \quad 2. \text{ Every two nodes of } G \text{ are joined by a unique path}

3. \ G \text{ is connected and } n = e + 1
2 => 3  2. Every two nodes of G are joined by a unique path

3. G is connected and $n = e + 1$

Proof: (by induction)
2 => 3  2. Every two nodes of G are joined by a unique path

3. G is connected and \( n = e + 1 \)

Proof: (by induction)

Assume true for every graph with < \( n \) nodes
2 => 3 2. Every two nodes of G are joined by a unique path

3. G is connected and \( n = e + 1 \)

**Proof: (by induction)**

Assume true for every graph with \( < n \) nodes

Let G have \( n \) nodes and let \( x \) and \( y \) be adjacent
2 => 3 2. Every two nodes of $G$ are joined by a unique path

3. $G$ is connected and $n = e + 1$

Proof: (by induction)

Assume true for every graph with $< n$ nodes
Let $G$ have $n$ nodes and let $x$ and $y$ be adjacent
2 \implies 3 \quad 2. \text{Every two nodes of } G \text{ are joined by a unique path}

3. \text{G is connected and } n = e + 1

\textbf{Proof: (by induction)}

Assume true for every graph with < n nodes.
Let G have n nodes and let x and y be adjacent.

Let \( n_1, e_1 \) be number of nodes and edges in \( G_1 \).
2 => 3 2. Every two nodes of G are joined by a unique path

3. G is connected and \( n = e + 1 \)

Proof: (by induction)

Assume true for every graph with \(< n\) nodes

Let G have \( n \) nodes and let x and y be adjacent

Let \( n_1, e_1 \) be number of nodes and edges in \( G_1 \)

Then \( n = n_1 + n_2 = e_1 + e_2 + 2 = e + 1 \)
3 => 4
3. G is connected and \( n = e + 1 \)
4. G is acyclic and \( n = e + 1 \)
3 => 4 3. G is connected and n = e + 1
4. G is acyclic and n = e + 1

Proof: (by contradiction)
3 => 4  3. G is connected and n = e + 1
4. G is acyclic and n = e + 1

Proof: (by contradiction)

Assume G is connected with n = e + 1, and G has a cycle containing k nodes
3 => 4  
3. G is connected and \( n = e + 1 \)
4. G is acyclic and \( n = e + 1 \)

**Proof: (by contradiction)**

Assume G is connected with \( n = e + 1 \), and G has a cycle containing \( k \) nodes.
3 => 4  3. G is connected and \( n = e + 1 \)
4. G is acyclic and \( n = e + 1 \)

**Proof: (by contradiction)**

Assume G is connected with \( n = e + 1 \), and G has a cycle containing \( k \) nodes

Note that the cycle has \( k \) nodes and \( k \) edges
3 => 4

3. G is connected and \( n = e + 1 \)

4. G is acyclic and \( n = e + 1 \)

**Proof: (by contradiction)**

Assume G is connected with \( n = e + 1 \), and G has a cycle containing \( k \) nodes

Start adding nodes and edges until you cover the whole graph

Note that the cycle has \( k \) nodes and \( k \) edges

Start adding nodes and edges until you cover the whole graph
3 --> 4
3. G is connected and $n = e + 1$
4. G is acyclic and $n = e + 1$

Proof: (by contradiction)

Assume G is connected with $n = e + 1$, and G has a cycle containing k nodes

Note that the cycle has k nodes and k edges

Start adding nodes and edges until you cover the whole graph

Number of edges in the graph will be at least n
Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)
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Proof:
Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

Proof:

Assume all but one of the points in the tree have degree at least 2
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Proof:

Assume all but one of the points in the tree have degree at least 2

In any graph, sum of the degrees =
Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

Proof:
Assume all but one of the points in the tree have degree at least 2

In any graph, sum of the degrees = 2e
Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

Proof:

Assume all but one of the points in the tree have degree at least 2

In any graph, sum of the degrees = 2e

Then the total number of edges in the tree is at least \((2n-1)/2 = n - 1/2 > n - 1\)
How many labeled trees are there with three nodes?
How many labeled trees are there with three nodes?
How many labeled trees are there with three nodes?

1. 1 2 3
2. 1 3 2
How many labeled trees are there with three nodes?

1. O---O---O
   1  2  3

2. O---O---O
   1  3  2

3. O---O---O
   2  1  3
How many **labeled** trees are there with four nodes?

a
b
c
d
How many labeled trees are there with five nodes?
How many **labeled** trees are there with five nodes?
How many **labeled** trees are there with five nodes?

5 labelings
How many **labeled** trees are there with five nodes?

5 labelings

5 x 4 x 3 labelings
How many **labeled** trees are there with five nodes?

- 5 labelings
- \(5 \times 4 \times 3\) labelings
- \(\frac{5!}{2}\) labelings
How many **labeled** trees are there with five nodes?

5 labeled trees

5 labelings

5 x 4 x 3 labelings

5!/2 labelings

125 labeled trees
How many **labeled** trees are there with $n$ nodes?
How many labeled trees are there with $n$ nodes?

3 labeled trees with 3 nodes
How many labeled trees are there with \( n \) nodes?

- 3 labeled trees with 3 nodes
- 16 labeled trees with 4 nodes
How many labeled trees are there with n nodes?

- 3 labeled trees with 3 nodes
- 16 labeled trees with 4 nodes
- 125 labeled trees with 5 nodes
How many labeled trees are there with \( n \) nodes?

- 3 labeled trees with 3 nodes
- 16 labeled trees with 4 nodes
- 125 labeled trees with 5 nodes

\( n^{n-2} \) labeled trees with \( n \) nodes
Cayley’s Formula

The number of labeled trees on $n$ nodes is $n^{n-2}$
The proof will use the correspondence principle
The proof will use the correspondence principle

Each labeled tree on $n$ nodes corresponds to

A sequence in $\{1, 2, \ldots, n\}^{n-2}$ (that is, $n-2$ numbers, each in the range $[1..n]$)
How to make a sequence from a tree?
How to make a sequence from a tree?
Loop through $i$ from 1 to $n-2$
How to make a sequence from a tree?
Loop through $i$ from 1 to $n-2$

Let L be the degree-1 node with the lowest label
How to make a sequence from a tree?
Loop through $i$ from 1 to $n-2$

Let $L$ be the degree-1 node with the lowest label

Define the $i^{th}$ element of the sequence as the label of the node adjacent to $L$
How to make a sequence from a tree?
Loop through $i$ from 1 to $n-2$

Let $L$ be the degree-1 node with the lowest label

Define the $i^{th}$ element of the sequence as the label of the node adjacent to $L$

Delete the node $L$ from the tree
How to make a sequence from a tree?

Loop through $i$ from 1 to $n-2$

Let $L$ be the degree-1 node with the lowest label

Define the $i^{th}$ element of the sequence as the label of the node adjacent to $L$

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Example:
How to make a sequence from a tree?

Loop through $i$ from 1 to $n-2$

Let $L$ be the degree-1 node with the lowest label

Define the $i^{th}$ element of the sequence as the label of the node adjacent to $L$

Delete the node $L$ from the tree

Example:
How to make a sequence from a tree?

Loop through i from 1 to n-2

Let L be the degree-1 node with the lowest label

Define the i\textsuperscript{th} element of the sequence as the label of the node adjacent to L

Delete the node L from the tree

Example:
How to make a sequence from a tree?

Loop through $i$ from 1 to $n-2$

Let $L$ be the degree-1 node with the lowest label

Define the $i^{th}$ element of the sequence as the label of the node adjacent to $L$

Delete the node $L$ from the tree

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Loop through $i$ from 1 to $n-2$

Let $L$ be the degree-1 node with the lowest label

Define the $i^{th}$ element of the sequence as the label of the node adjacent to $L$

Delete the node $L$ from the tree

Example:
How to make a sequence from a tree?

Loop through \( i \) from 1 to \( n-2 \)

Let L be the degree-1 node with the lowest label

Define the \( i^{\text{th}} \) element of the sequence as the label of the node adjacent to L

Delete the node L from the tree

Example:
How to make a sequence from a tree?

Loop through \( i \) from 1 to \( n-2 \):

1. Let \( L \) be the degree-1 node with the lowest label.
2. Define the \( i^{th} \) element of the sequence as the label of the node adjacent to \( L \).
3. Delete the node \( L \) from the tree.

Example:
How to make a sequence from a tree?

Loop through $i$ from $1$ to $n-2$

Let $L$ be the degree-1 node with the lowest label

Define the $i^{th}$ element of the sequence as the label of the node adjacent to $L$

Delete the node $L$ from the tree

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How to make a sequence from a tree?

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Example:
How to make a sequence from a tree?

Loop through $i$ from 1 to $n-2$

Let L be the degree-1 node with the lowest label

Define the $i^{th}$ element of the sequence as the label of the node adjacent to L

Delete the node L from the tree

Example:

1 3 3 4 4 4
How to reconstruct the unique tree from a sequence S:
How to reconstruct the unique tree from a sequence $S$:
Let $I = \{1, 2, 3, \ldots, n\}$
How to reconstruct the unique tree from a sequence S:
Let I = \{1, 2, 3, \ldots, n\}
Loop until S is empty
How to reconstruct the unique tree from a sequence S:

Let $I = \{1, 2, 3, \ldots, n\}$

Loop until S is empty

Let $i =$ smallest # in $I$ but not in $S$
Let $s =$ first label in sequence $S$
Add edge $\{i, s\}$ to the tree
Delete $i$ from $I$
Delete $s$ from $S$
How to reconstruct the unique tree from a sequence S:

Let $I = \{1, 2, 3, \ldots, n\}$

Loop until S is empty

1. Let $i =$ smallest # in $I$ but not in S
2. Let $s =$ first label in sequence S
3. Add edge $\{i, s\}$ to the tree
4. Delete $i$ from $I$
5. Delete $s$ from S

Add edge $\{a, b\}$, where $I = \{a, b\}$
How to reconstruct the unique tree from a sequence $S$:

Let $I = \{1, 2, 3, \ldots, n\}$

Loop until $S$ is empty

Let $i =$ smallest # in $I$ but not in $S$

Let $s =$ first label in sequence $S$

Add edge $\{i, s\}$ to the tree

Delete $i$ from $I$

Delete $s$ from $S$

Add edge $\{a, b\}$, where $I = \{a, b\}$
Spanning Trees
Spanning Trees

A spanning tree of a graph $G$ is a tree that touches every node of $G$ and uses only edges from $G$.
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A spanning tree of a graph $G$ is a tree that touches every node of $G$ and uses only edges from $G$

Every connected graph has a spanning tree
A graph is **planar** if it can be drawn in the plane without crossing edges.
Examples of Planar Graphs
Examples of Planar Graphs

[Diagram of a planar graph with four vertices and four edges forming a square]

1. Two vertices are connected by a single edge.
2. Several vertices are connected in a way that no edges cross each other within the plane.
Examples of Planar Graphs
Examples of Planar Graphs

- Two simple planar graphs
- A more complex planar graph
http://www.planarity.net
Faces
A planar graph splits the plane into disjoint faces
Faces

A planar graph splits the plane into disjoint faces.

4 faces
Euler’s Formula

If G is a connected planar graph with n vertices, e edges and f faces, then \( n - e + f = 2 \)
Rather than using induction, we’ll use the important notion of the dual graph.
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Rather than using induction, we’ll use the important notion of the **dual graph**.

**Dual** = put a node in every face, and an edge between every adjacent face.
Let $G^*$ be the dual graph of $G$. 
Let $G^*$ be the dual graph of $G$

Let $T$ be a spanning tree of $G$
Let $G^*$ be the dual graph of $G$.

Let $T$ be a spanning tree of $G$.

Let $T^*$ be the graph where there is an edge in the dual graph for each edge in $G - T$. 
Let $G^*$ be the dual graph of $G$

Let $T$ be a spanning tree of $G$

Let $T^*$ be the graph where there is an edge in dual graph for each edge in $G - T$
Let $G^*$ be the dual graph of $G$

Let $T$ be a spanning tree of $G$

Let $T^*$ be the graph where there is an edge in dual graph for each edge in $G - T$

Then $T^*$ is a spanning tree for $G^*$
Let $G^*$ be the dual graph of $G$.

Let $T$ be a spanning tree of $G$.

Let $T^*$ be the graph where there is an edge in the dual graph for each edge in $G - T$.

Then $T^*$ is a spanning tree for $G^*$.

$$n = e_T + 1$$
Let $G^*$ be the dual graph of $G$

Let $T$ be a spanning tree of $G$

Let $T^*$ be the graph where there is an edge in dual graph for each edge in $G-T$

Then $T^*$ is a spanning tree for $G^*$

$n = e_T + 1$

$f = e_{T^*} + 1$
Let \( G^* \) be the dual graph of \( G \)

Let \( T \) be a spanning tree of \( G \)

Let \( T^* \) be the graph where there is an edge in dual graph for each edge in \( G - T \)

Then \( T^* \) is a spanning tree for \( G^* \)

\[
\begin{align*}
n &= e_T + 1 \\
n + f &= e_T + e_{T^*} + 2 \\
f &= e_{T^*} + 1
\end{align*}
\]
Let $G^*$ be the dual graph of $G$

Let $T$ be a spanning tree of $G$

Let $T^*$ be the graph where there is an edge in dual graph for each edge in $G - T$

Then $T^*$ is a spanning tree for $G^*$

$$n = e_T + 1$$

$$f = e_{T^*} + 1$$

$$n + f = e_T + e_{T^*} + 2 = e + 2$$
Corollary: Let $G$ be a simple planar graph with $n > 2$ vertices. Then:
Corollary: Let G be a simple planar graph with \( n > 2 \) vertices. Then:

1. G has a vertex of degree at most 5
Corollary: Let $G$ be a simple planar graph with $n > 2$ vertices. Then:

1. $G$ has a vertex of degree at most 5
2. $G$ has at most $3n - 6$ edges
Corollary: Let G be a simple planar graph with \( n > 2 \) vertices. Then:

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Proof of 1:
Corollary: Let $G$ be a simple planar graph with $n > 2$ vertices. Then:

1. $G$ has a vertex of degree at most 5
2. $G$ has at most $3n - 6$ edges

**Proof of 1:**

In any graph, \((\text{sum of degrees}) = 2e\)
Corollary: Let G be a simple planar graph with \( n > 2 \) vertices. Then:

1. G has a vertex of degree at most 5
2. G has at most \( 3n - 6 \) edges

**Proof of 1:**

In any graph, \( \text{(sum of degrees)} = 2e \)

Assume all vertices have degree \( \geq 6 \)
Corollary: Let G be a simple planar graph with $n > 2$ vertices. Then:

1. G has a vertex of degree at most 5
2. G has at most $3n - 6$ edges

Proof of 1:

In any graph, (sum of degrees) = $2e$
Assume all vertices have degree $\geq 6$
Then $e \geq 3n$
Corollary: Let G be a simple planar graph with \( n > 2 \) vertices. Then:

1. G has a vertex of degree at most 5
2. G has at most \( 3n - 6 \) edges

Proof of 1:

In any graph, \((\text{sum of degrees}) = 2e\)

Assume all vertices have degree \( \geq 6 \)

Then \( e \geq 3n \)

Furthermore, since G is simple, \( 3f \leq 2e \)
Corollary: Let G be a simple planar graph with \( n > 2 \) vertices. Then:

1. G has a vertex of degree at most 5
2. G has at most \( 3n - 6 \) edges

Proof of 1:

In any graph, \((\text{sum of degrees}) = 2e\)
Assume all vertices have degree \( \geq 6 \)
Then \( e \geq 3n \)
Furthermore, since G is simple, \( 3f \leq 2e \)
So \( 3n + 3f \leq 3e \Rightarrow 3(n-e+f) \leq 0\), contradiction.
Graph Coloring
Graph Coloring

A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color.
A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color.
Graph Coloring

Arises surprisingly often in CS
Graph Coloring

Arises surprisingly often in CS

Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register
Instructions

b = a+2

c = b*b

b = c+1

return a*b

Live variables

a

a,b

a,c

a,b

a,b

a,b

a,b
Theorem: Every planar graph can be 6-colored
Theorem: Every planar graph can be 6-colored

Proof Sketch (by induction):
Theorem: Every planar graph can be 6-colored

Proof Sketch (by induction):

Assume every planar graph with less than n vertices can be 6-colored.
Theorem: Every planar graph can be 6-colored

Proof Sketch (by induction):

Assume every planar graph with less than n vertices can be 6-colored

Assume G has n vertices
Theorem: Every planar graph can be 6-colored

Proof Sketch (by induction):

Assume every planar graph with less than $n$ vertices can be 6-colored

Assume $G$ has $n$ vertices

Since $G$ is planar, it has some node $v$ with degree at most 5
Theorem: Every planar graph can be 6-colored

Proof Sketch (by induction):

Assume every planar graph with less than n vertices can be 6-colored.

Assume G has n vertices.

Since G is planar, it has some node v with degree at most 5.

Remove v and color by Induction Hypothesis.
Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree $\leq 5$
Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree $\leq 5$

4-color theorem remains challenging!
Here’s What You Need to Know…

Trees
- Counting Trees
- Different Characterizations
- Cayley’s formula

Planar Graphs
- Definition
- Euler’s Theorem
- Coloring Planar Graphs