What’s a tree?

A tree is a connected graph with no cycles

Upcoming Events
Review Session on Saturday
(5 pm, Wean 5409)
Test on Monday
Election Day
How Many n-Node Trees?

1: •
2: •—•
3: •••
4: •••—•
5: •••••—•

We’ll pass around a piece of paper. Draw a new 8-node tree, and put your name next to it. (There are 23 of them…)

The Shy People Party

At the shy people party, people enter one-by-one, and as a person comes in, (s)he shakes hand with only one person already at the party.

Prove that at a shy party with n people (n >= 2), at least two people have shaken hands with only one other person.
Notation
In this lecture:
n will denote the number of nodes in a graph
e will denote the number of edges in a graph

Theorem: Let G be a graph with n nodes and e edges
The following are equivalent:
1. G is a tree (connected, acyclic)
2. Every two nodes of G are joined by a unique path
3. G is connected and n = e + 1
4. G is acyclic and n = e + 1
5. G is acyclic and if any two non-adjacent points are joined by a line, the resulting graph has exactly one cycle

To prove this, it suffices to show
1 => 2 => 3 => 4 => 5 => 1

Proof: (by contradiction)
Assume G is a tree that has two nodes connected by two different paths:

Then there exists a cycle!
2. Every two nodes of G are joined by a unique path

3. G is connected and \( n = e + 1 \)

Proof: (by induction)

Assume true for every graph with \( < n \) nodes

Let \( G \) have \( n \) nodes and let \( x \) and \( y \) be adjacent

Let \( n_1, e_1 \) be number of nodes and edges in \( G_1 \)
Then \( n = n_1 + n_2 = e_1 + e_2 + 2 = e + 1 \)

3. G is connected and \( n = e + 1 \)

4. G is acyclic and \( n = e + 1 \)

Proof: (by contradiction)

Assume \( G \) is connected with \( n = e + 1 \), and \( G \) has a cycle containing \( k \) nodes

In any graph, sum of the degrees = \( 2e \)

Then the total number of edges in the tree is at least \( (2n-1)/2 = n - 1/2 > n - 1 \)

How many labeled trees are there with three nodes?

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 3 & 2 \\
2 & 1 & 3 \\
\end{array}
\]

Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

Proof:

Assume all but one of the points in the tree have degree at least 2

Number of edges in the graph will be at least \( n \)
How many labeled trees are there with four nodes?

How many labeled trees are there with five nodes?

125 labeled trees

How many labeled trees are there with n nodes?

3 labeled trees with 3 nodes
16 labeled trees with 4 nodes
125 labeled trees with 5 nodes

$n^{n-2}$ labeled trees with n nodes

Cayley’s Formula

The number of labeled trees on n nodes is $n^{n-2}$
The proof will use the correspondence principle.

Each labeled tree on n nodes corresponds to
A sequence in \{1,2,...,n\}^{n-2} (that is, n-2 numbers, each in the range [1..n])

Example:

How to make a sequence from a tree?
Loop through i from 1 to n-2
Let L be the degree-1 node with the lowest label
Define the i^th element of the sequence as the label of the node adjacent to L
Delete the node L from the tree

Example:

How to reconstruct the unique tree from a sequence S:
Let I = \{1, 2, 3, ..., n\}
Loop until S is empty
Let i = smallest # in I but not in S
Let s = first label in sequence S
Add edge \{i, s\} to the tree
Delete i from I
Delete s from S
Add edge \{a,b\}, where I = \{a,b\}

Spanning Trees

A spanning tree of a graph G is a tree that touches every node of G and uses only edges from G

Every connected graph has a spanning tree
A graph is planar if it can be drawn in the plane without crossing edges.

Examples of Planar Graphs

Faces
A planar graph splits the plane into disjoint faces.

http://www.planarity.net
Euler’s Formula
If \( G \) is a connected planar graph with \( n \) vertices, \( e \) edges and \( f \) faces, then \( n - e + f = 2 \).

Rather than using induction, we’ll use the important notion of the dual graph.

Dual = put a node in every face, and an edge between every adjacent face.

Let \( G^* \) be the dual graph of \( G \).
Let \( T \) be a spanning tree of \( G \).
Let \( T^* \) be the graph where there is an edge in dual graph for each edge in \( G - T \).

Then \( T^* \) is a spanning tree for \( G^* \).

\[
\begin{align*}
n &= e_T + 1 \\
n + f &= e_T + e_{T^*} + 2 \\
f &= e_{T^*} + 1 \\
&= e + 2
\end{align*}
\]

Corollary: Let \( G \) be a simple planar graph with \( n > 2 \) vertices. Then:
1. \( G \) has a vertex of degree at most 5
2. \( G \) has at most \( 3n - 6 \) edges

Proof of 1:
In any graph, (sum of degrees) = 2\( e \)
Assume all vertices have degree \( \geq 6 \)
Then \( e \geq 3n \)
Furthermore, since \( G \) is simple, \( 3f \leq 2e \)
So \( 3n + 3f \leq 3e \Rightarrow 3(n-e+f) \leq 0 \), contradiction.
A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color.

Graph Coloring

Arises surprisingly often in CS

Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register.

Theorem: Every planar graph can be 6-colored

Proof Sketch (by induction):
Assume every planar graph with less than n vertices can be 6-colored
Assume G has n vertices
Since G is planar, it has some node v with degree at most 5
Remove v and color by Induction Hypothesis
Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree $\leq 5$.

4-color theorem remains challenging!

Here’s What You Need to Know…

Trees
- Counting Trees
- Different Characterizations
- Cayley’s formula

Planar Graphs
- Definition
- Euler’s Theorem
- Coloring Planar Graphs