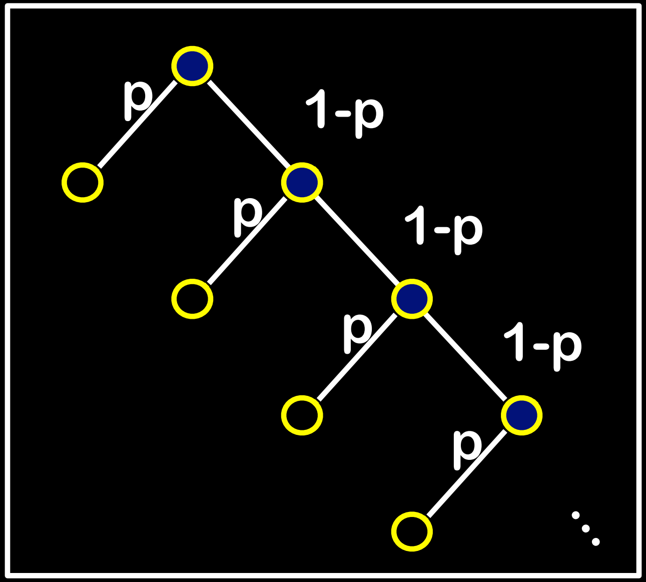
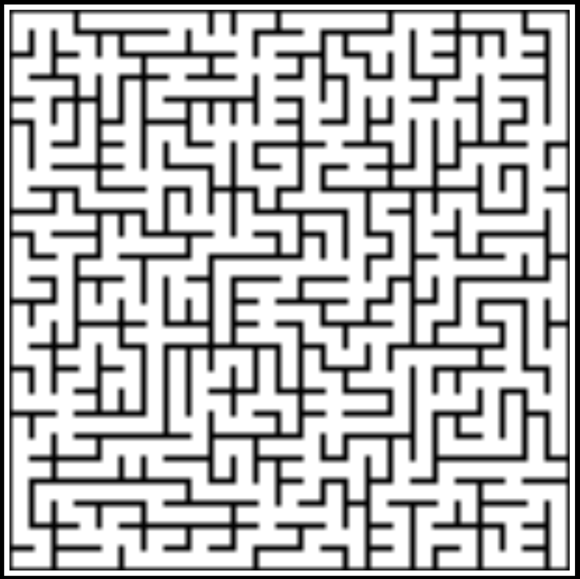
15-251

Great Theoretical Ideas in Computer Science

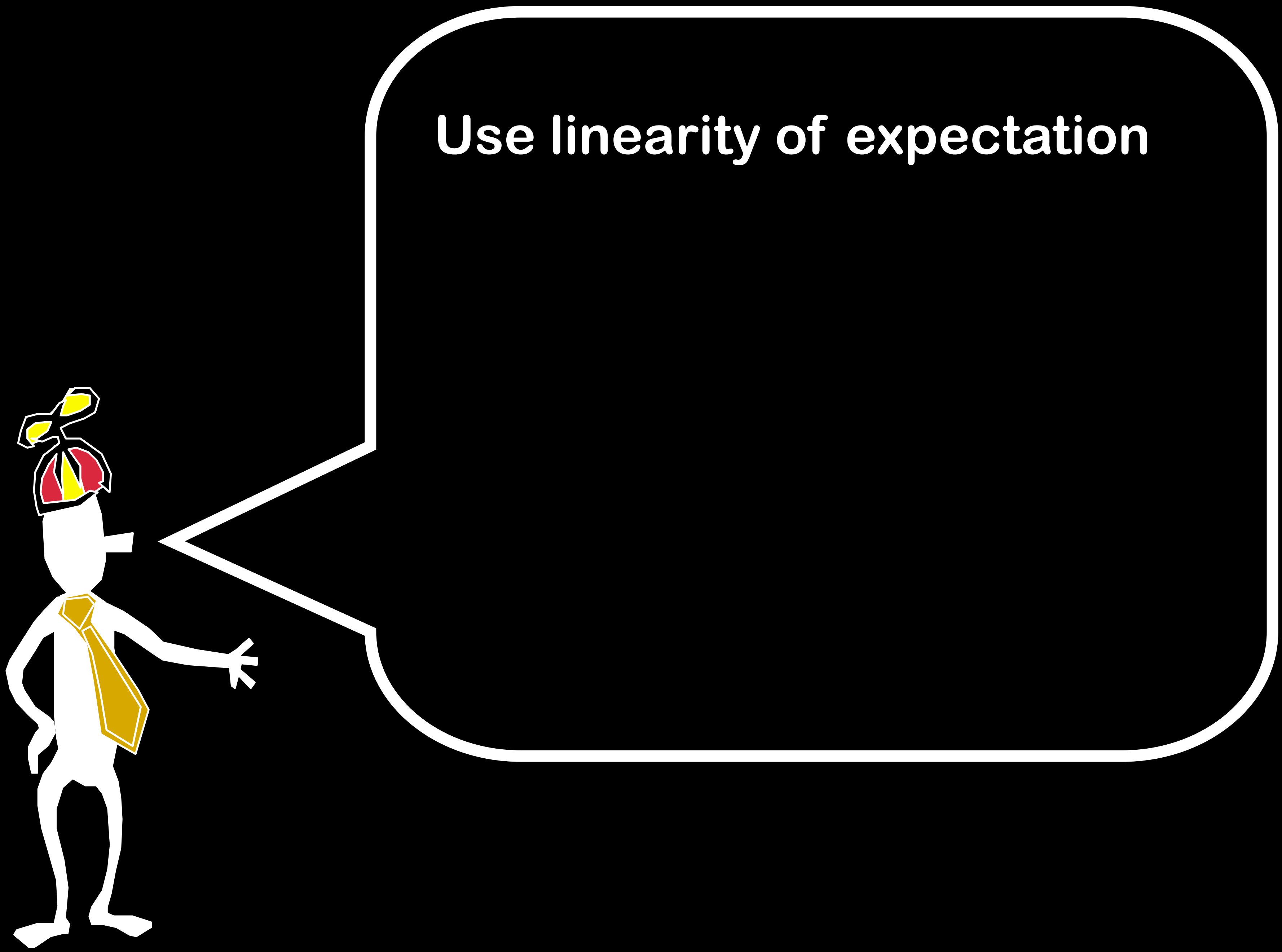
Infinite Sample Spaces and Random Walks

Lecture 11, September 30, 2008





See handout for probability review and some new stuff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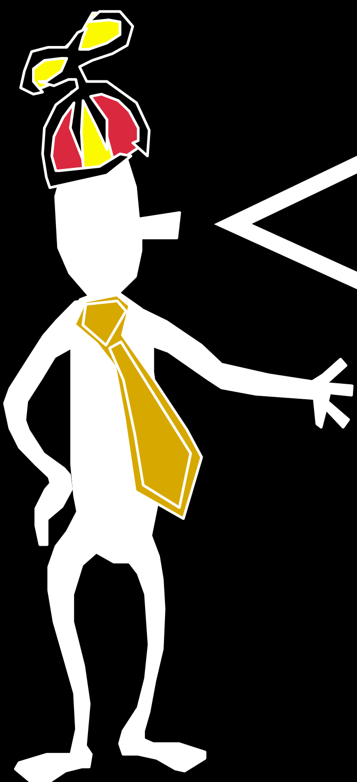

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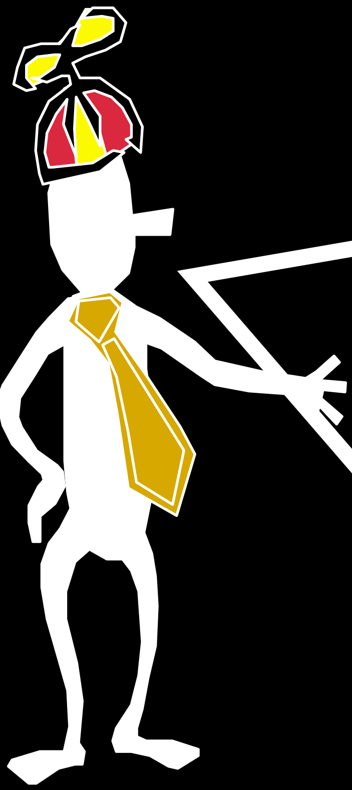
Use m(m-1)/2 indicator variables, one for each pair of people



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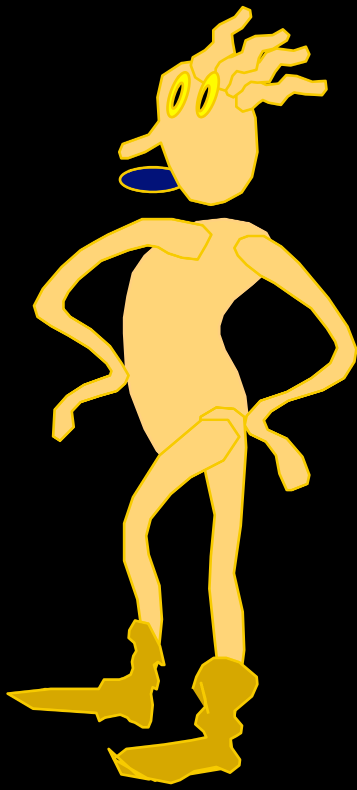
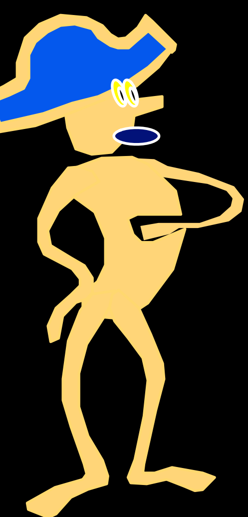
$$E[X] = E[\textstyle\sum_{j \leq k \leq m} X_{jk}] = \textstyle\sum_{j \leq k \leq m} E[X_{jk}] = m(m-1)/2 \times 1/366$$

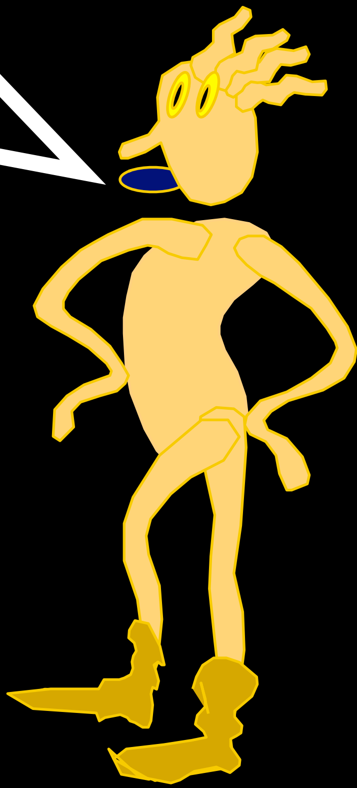
Step Right Up…



Step Right Up…

You pick a number **n** $\in$ [1..6]. You roll 3 dice. If any match **n**, you win \$1. Else you pay me \$1. Want to play?





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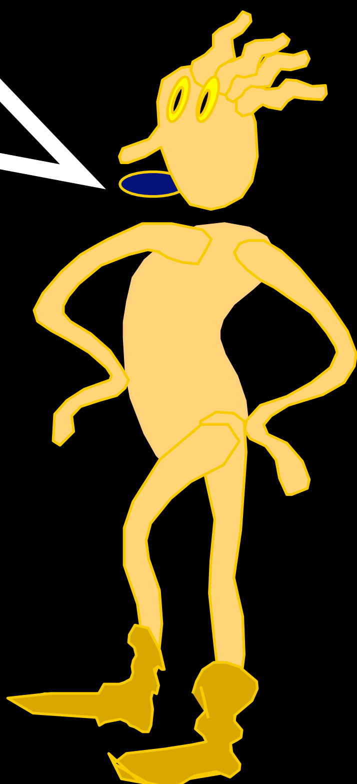
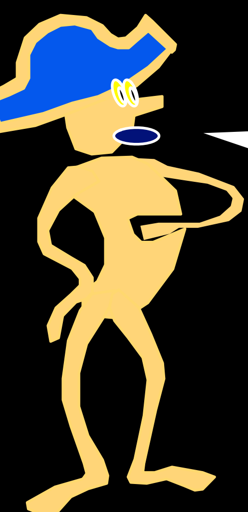


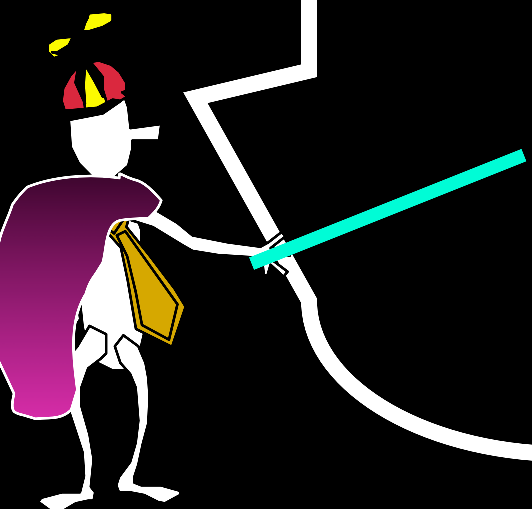


Hmm… let's see

Analysis

Analysis

A_i = event that i-th die matches



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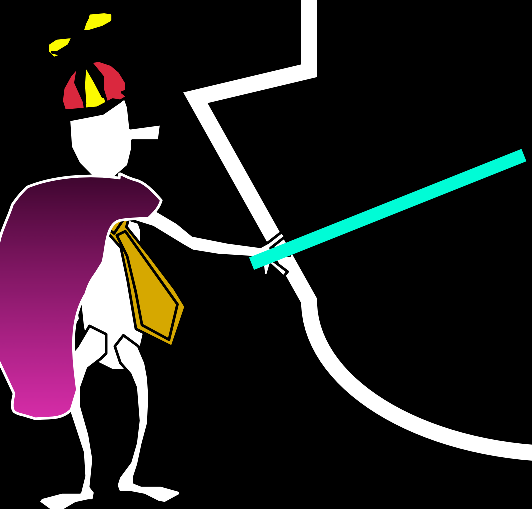

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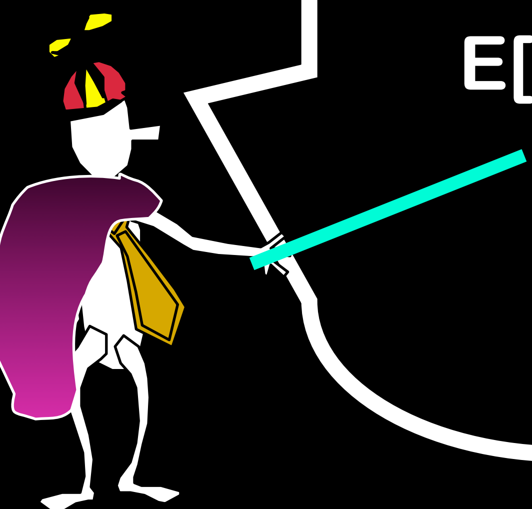
Expected number of dice that match:

$E[X_1+X_2+X_3] = 1/6+1/6+1/6 = ½$



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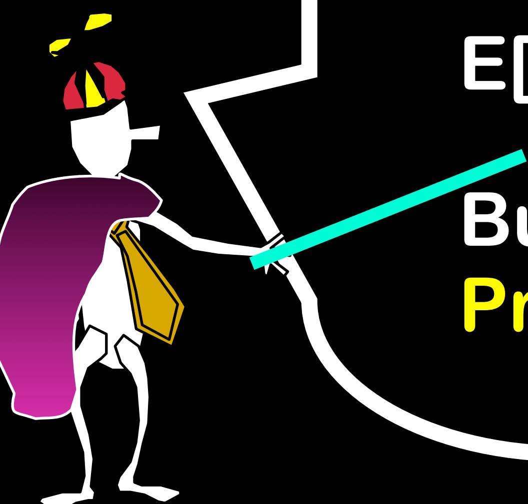

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But this is not the same as

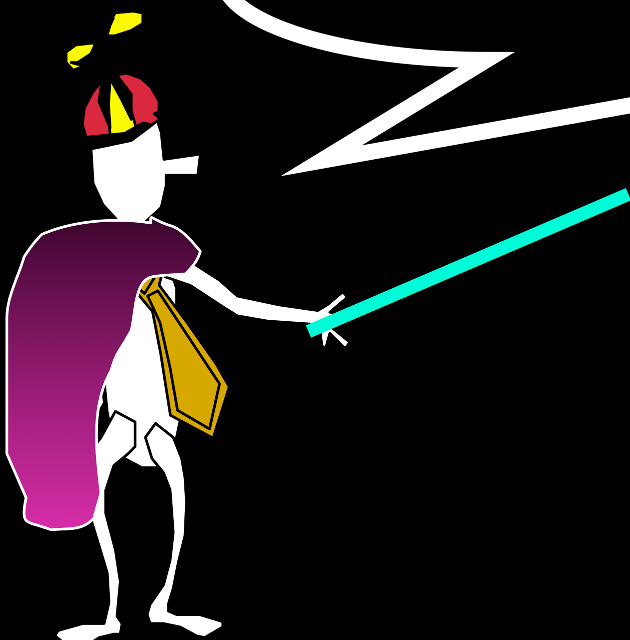
Pr(at least one die matches)

Analysis

Pr(at least one die matches)

$= 1 - \Pr(\text{none match})$

$= 1 - (5/6)^3 = 0.416$



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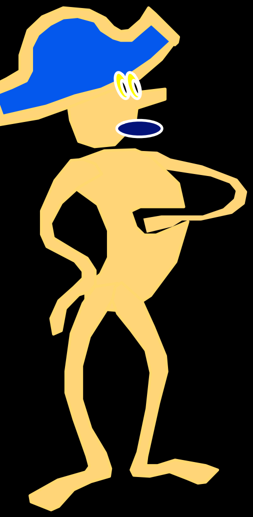

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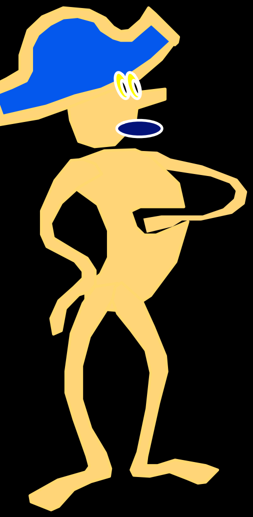
Answer: 2



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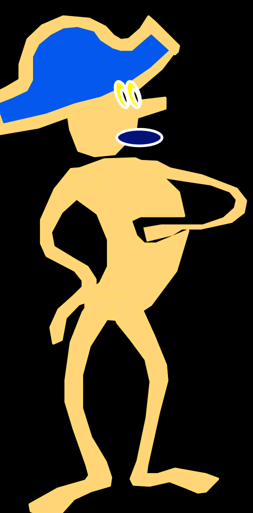

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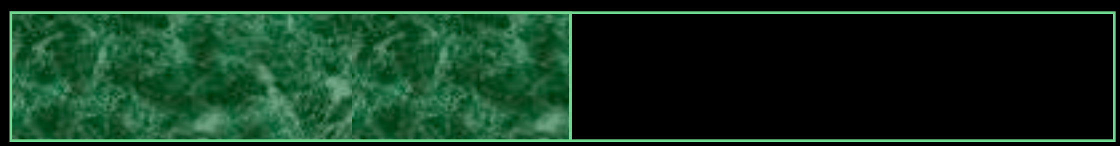

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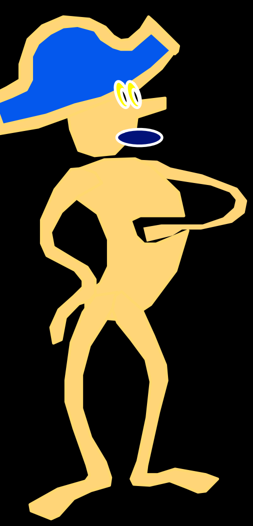

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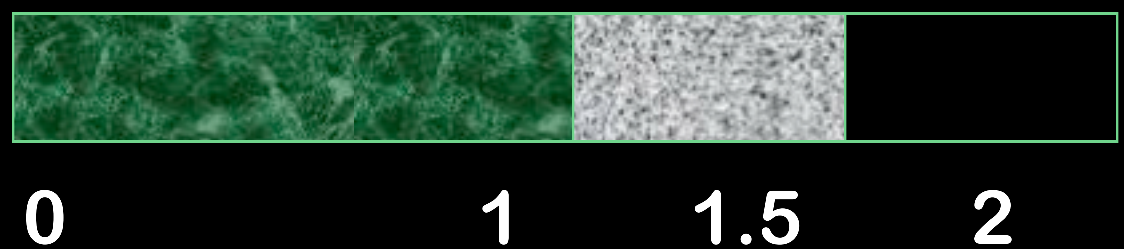

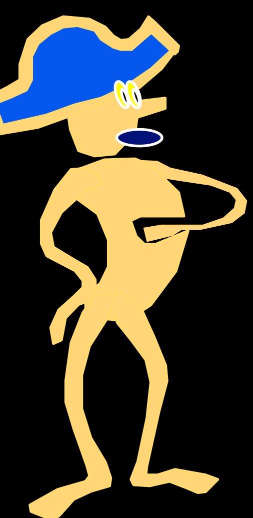

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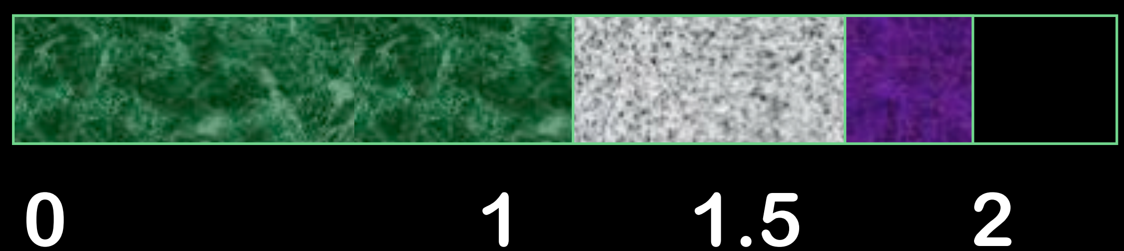

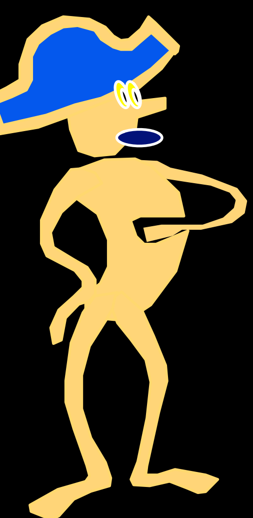

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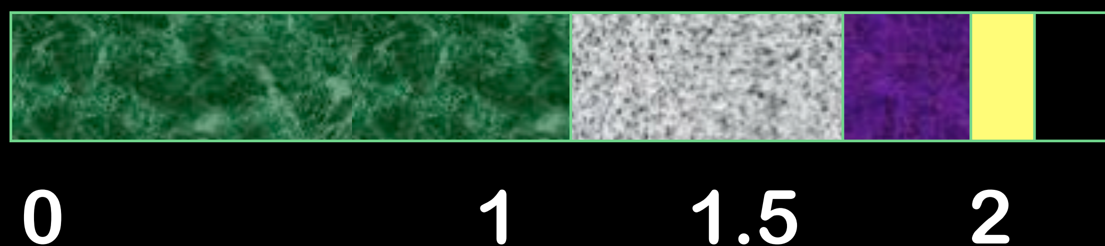
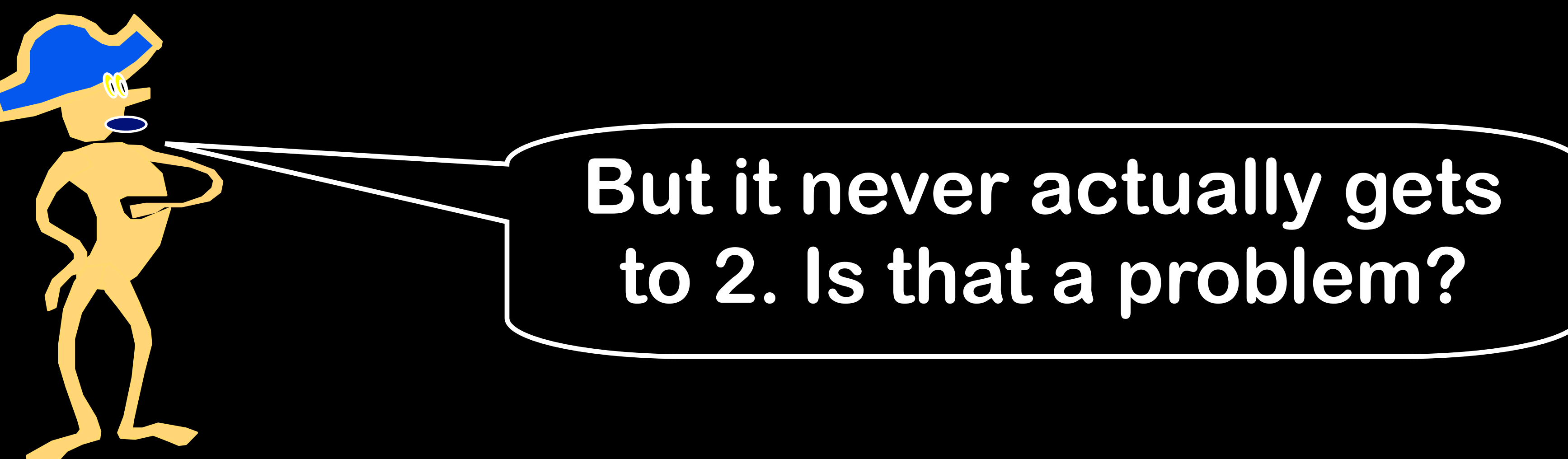
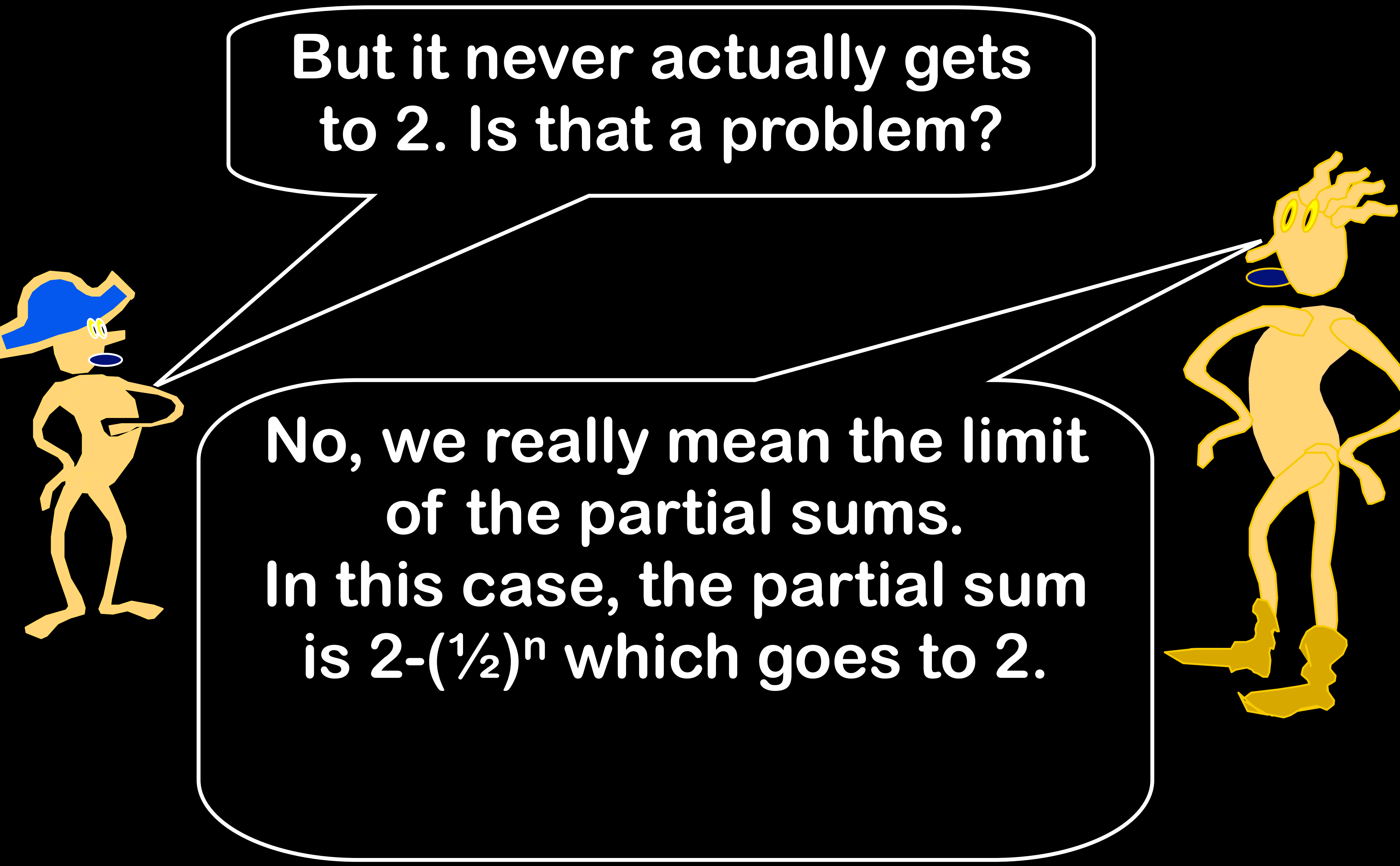
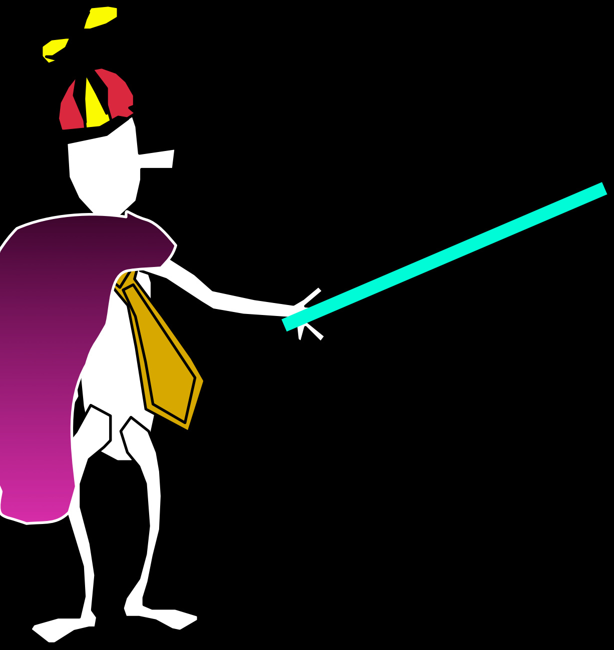





But it never actually gets to 2. Is that a problem?
No, we really mean the limit of the partial sums. In this case, the partial sum is $2-(\frac{1}{2})^n$ which goes to 2.

A related question



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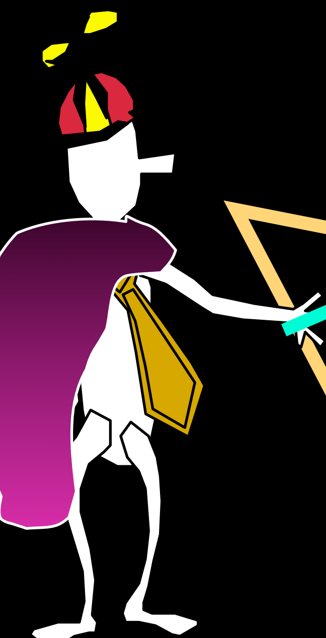
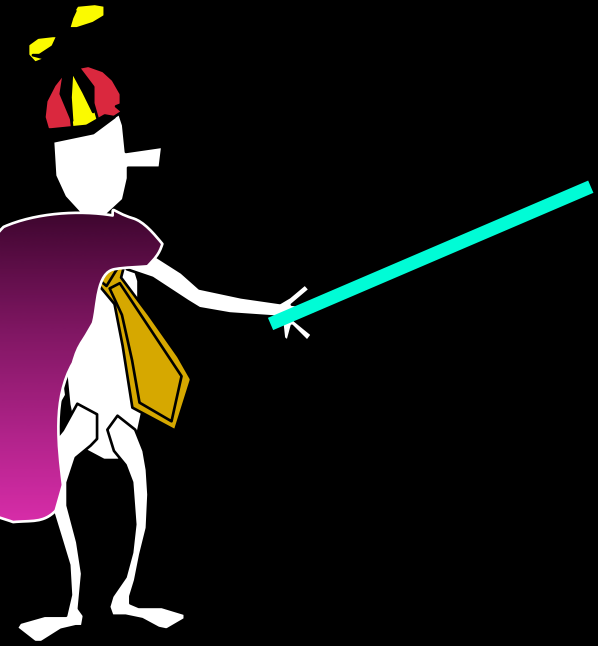


Suppose I flip a coin of bias p, stopping when I first get heads.

What's the chance that I:

- Flip exactly once?
  - Ans: p
- Flip exactly two times?
  - Ans: $(1-p)p$
- Flip exactly k times?
  - Ans: $(1-p)^{k-1}p$
- Eventually stop?
  - Ans: 1. (assuming $p>0$)

A related question



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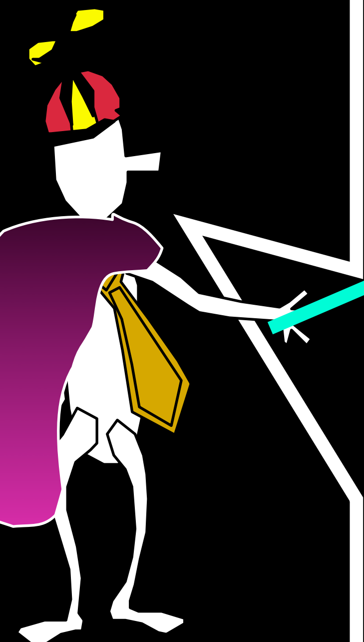
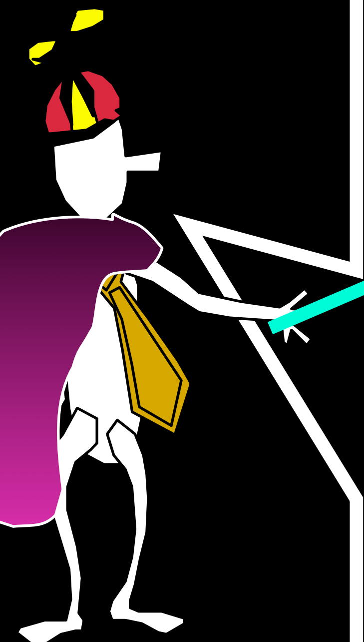


Pr(flip once) + Pr(flip 2 times) +
Pr(flip 3 times) + ... = 1:

$$p + (1-p)p + (1-p)^2p + (1-p)^3p + \ldots = 1.$$

Or, using $q = 1-p$,

A related question



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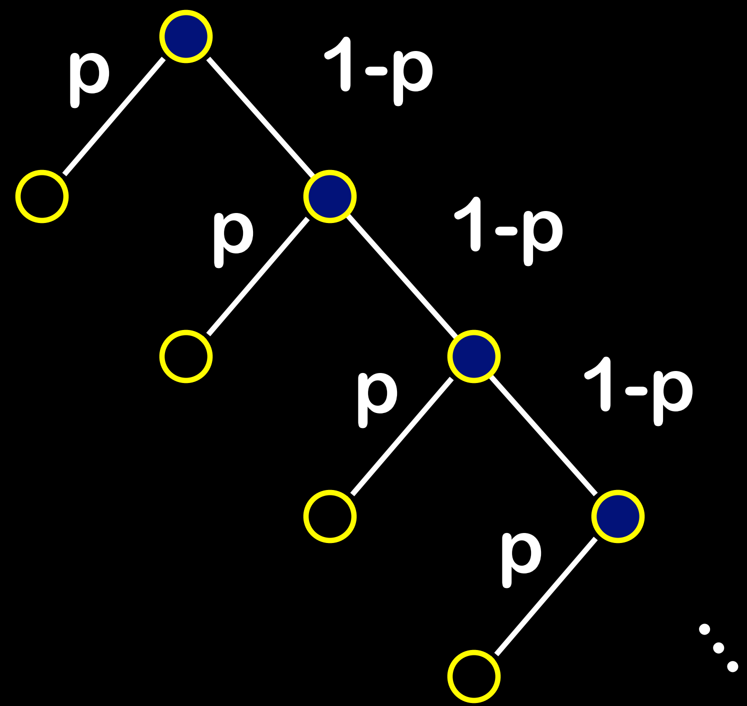
Or, using q = 1-p,

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q} = \frac{1}{p}$$

Pictorial view



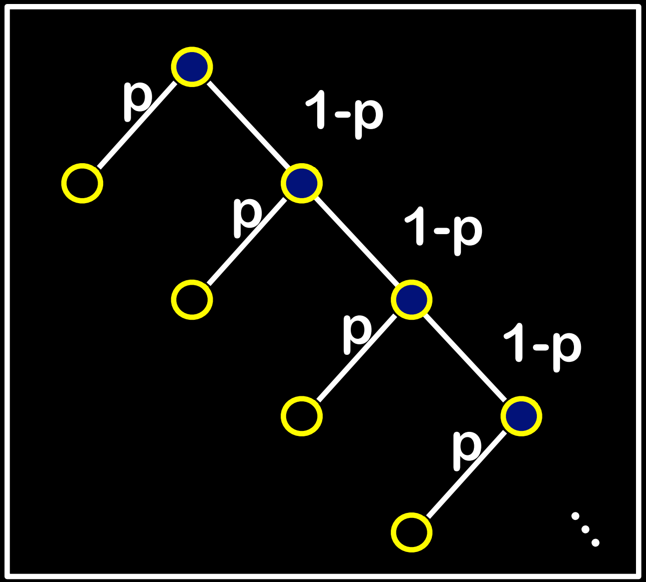
Sample space **S** = leaves in this tree.

Pr(x) = product of edges on path to x.

If **p>0**, Pr(not halting by time n) goes to zero.

Reason about expectations too!

Suppose A is a node in this tree



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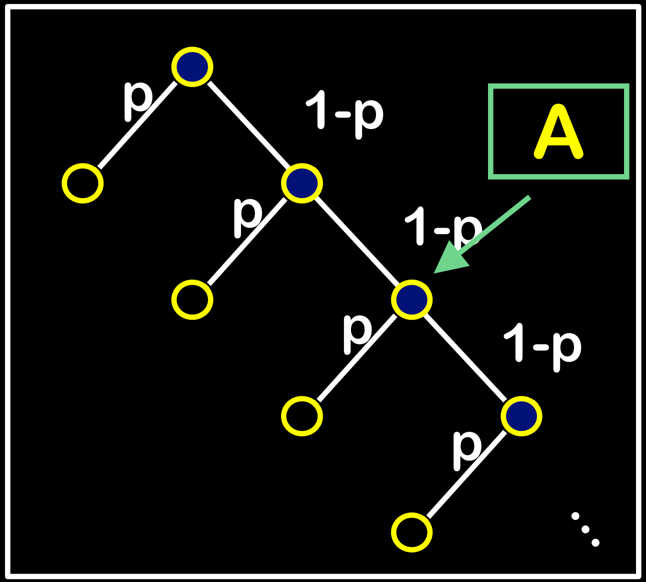

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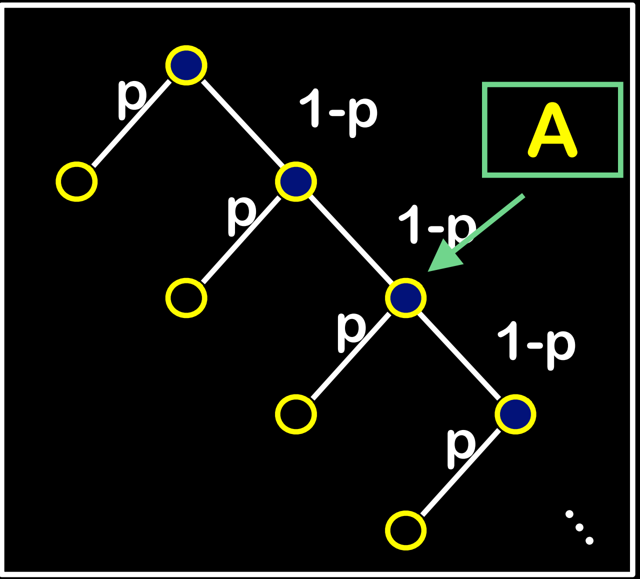

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Pr(x|A)=product of edges on path from A to x.

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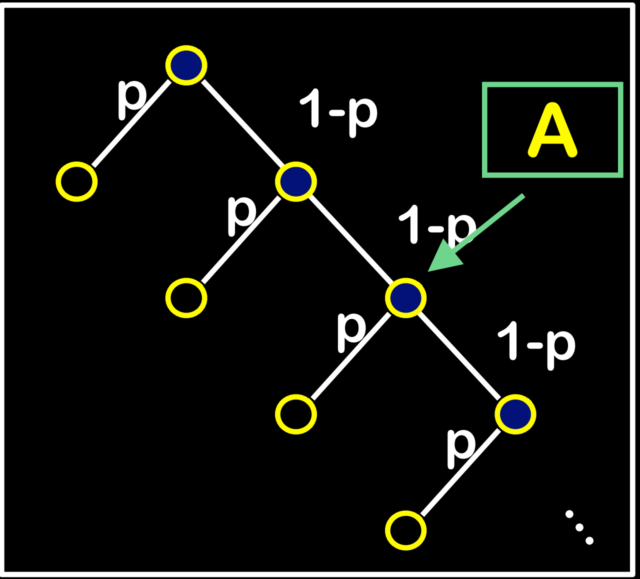


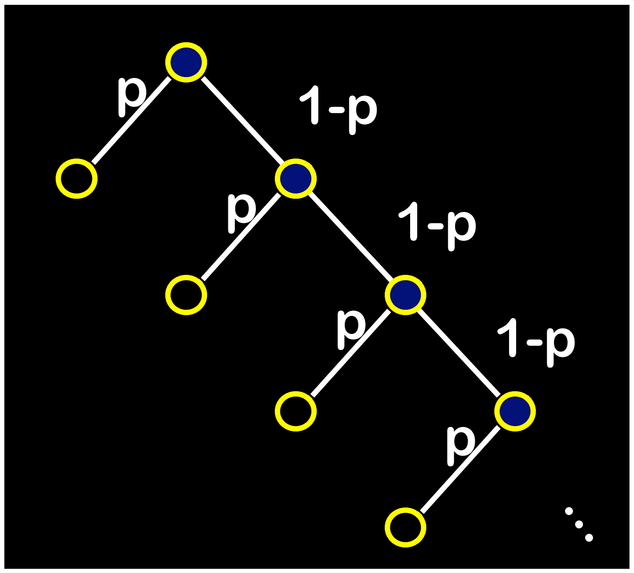
Suppose A is a node in this tree

Pr(x|A)=product of edges on path from A to x.

$E[X] = \sum_x \Pr(x)X(x)$.

$E[X|A] = \sum_{x \text{ in } A} \Pr(x|A)X(x)$.
I.e., it is as if we started the game at A.

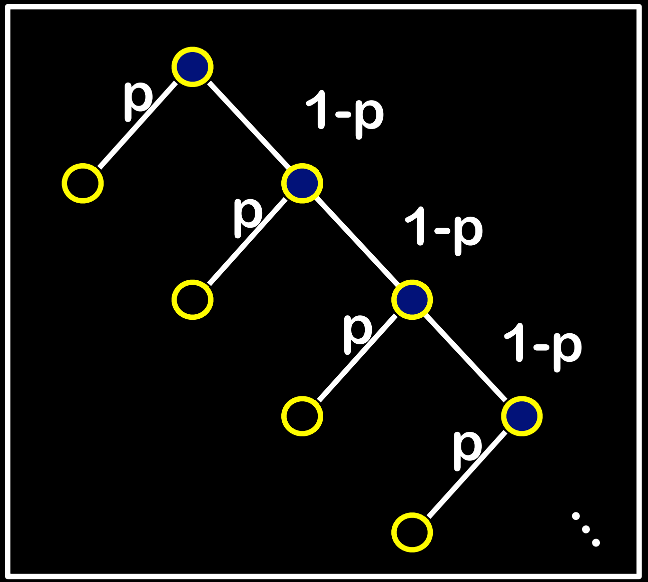
Expected number of heads



Flip bias-p coin until heads.

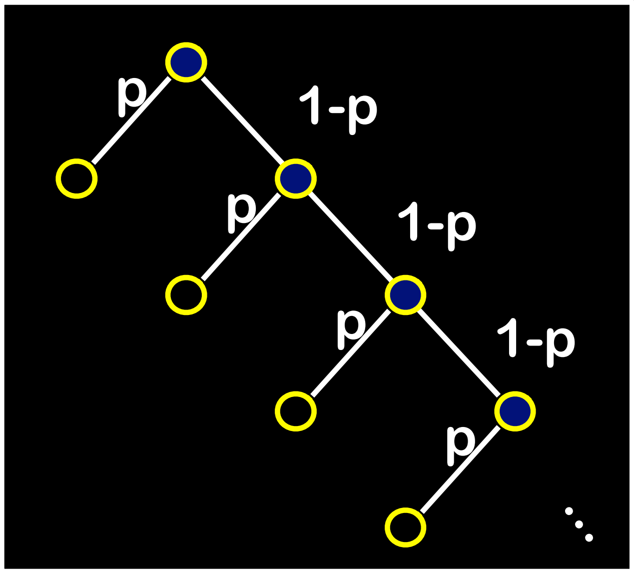
What is expected number of flips?

Expected number of heads



Let X = # flips.

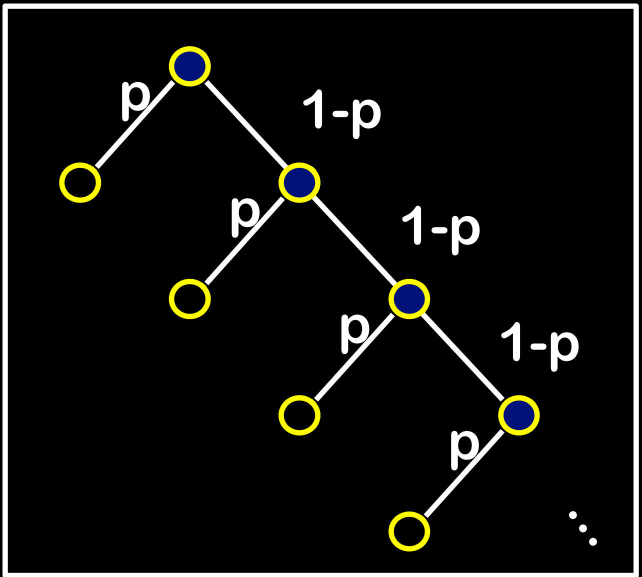
Expected number of heads



Let X = # flips.

B = event "1st flip is heads"

Expected number of heads

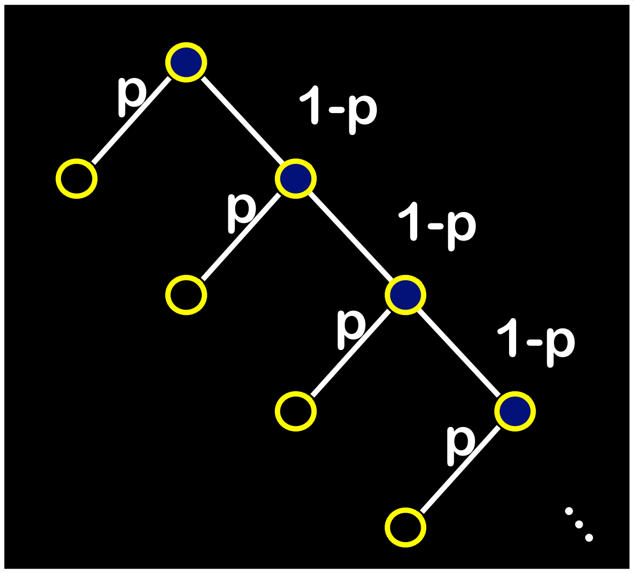


Let X = # flips.

B = event "1st flip is heads"

$$E[X] = E[X|B] \times \Pr(B) + E[X|\text{ not } B] \times \Pr(\text{not } B)$$

Expected number of heads



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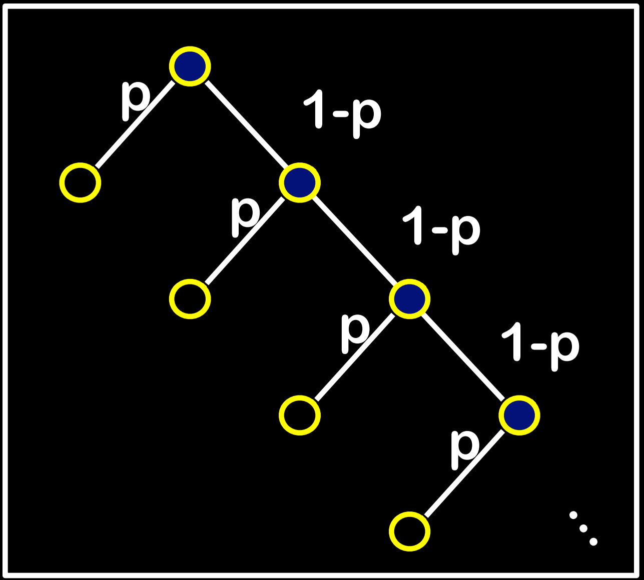
$$E[X] = E[X|B] \times \Pr(B) + E[X|\text{ not } B] \times \Pr(\text{not } B)$$

$$= 1 \times p + (1 + E[X]) \times (1-p).$$

Expected number of heads



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$$= 1 \times p + (1 + E[X]) \times (1-p).$$

Solving: $p \times E[X] = p + (1-p)$

$\Rightarrow E[X] = 1/p.$

Infinite Probability spaces

Notice we are using infinite probability spaces here, but we really only defined things for <u>finite</u> spaces so far.

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Notice we are using infinite probability spaces here, but we really only defined things for <u>finite</u> spaces so far.

Infinite probability spaces can sometimes be weird.

Luckily, in CS we will almost always be looking at spaces that can be viewed as choice trees where

Pr(haven't halted by time t) goes to 0 as t gets large

General picture

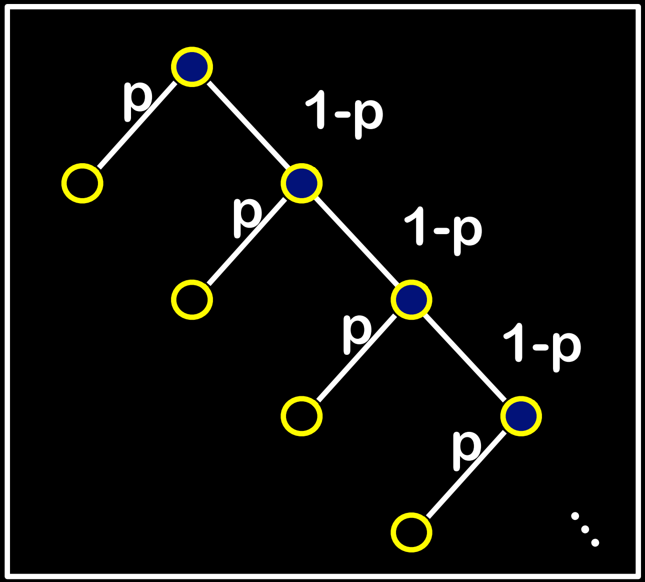
Let sample space S be leaves of a choice tree.

Let S_n = {leaves at depth $\leq$ n}.

For event A, let $A_n = A \cap S_n$.

If $\lim_n \Pr(S_n)=1$, can define:

$$\Pr(A)=\lim_n \Pr(A_n).$$



Setting that doesn't fit our model

Event: "Flip coin until #heads > 2*#tails."

There's a reasonable chance
this will <u>never</u> stop...

How to walk home drunk

Drunk man will find way home, but drunk bird may get lost forever

- Shizuo Kakutani

Abstraction of Student Life

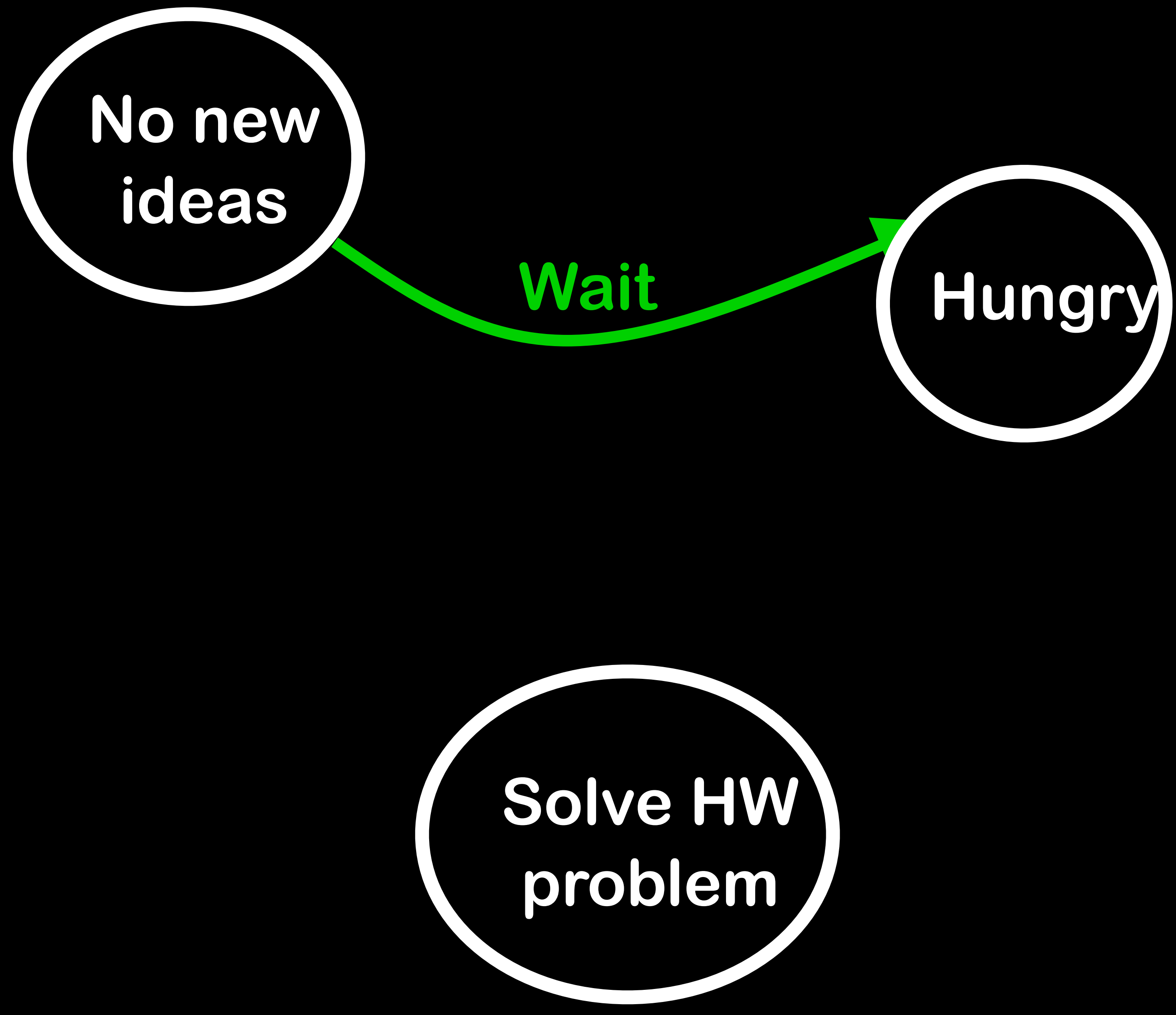
Abstraction of Student Life

No new ideas

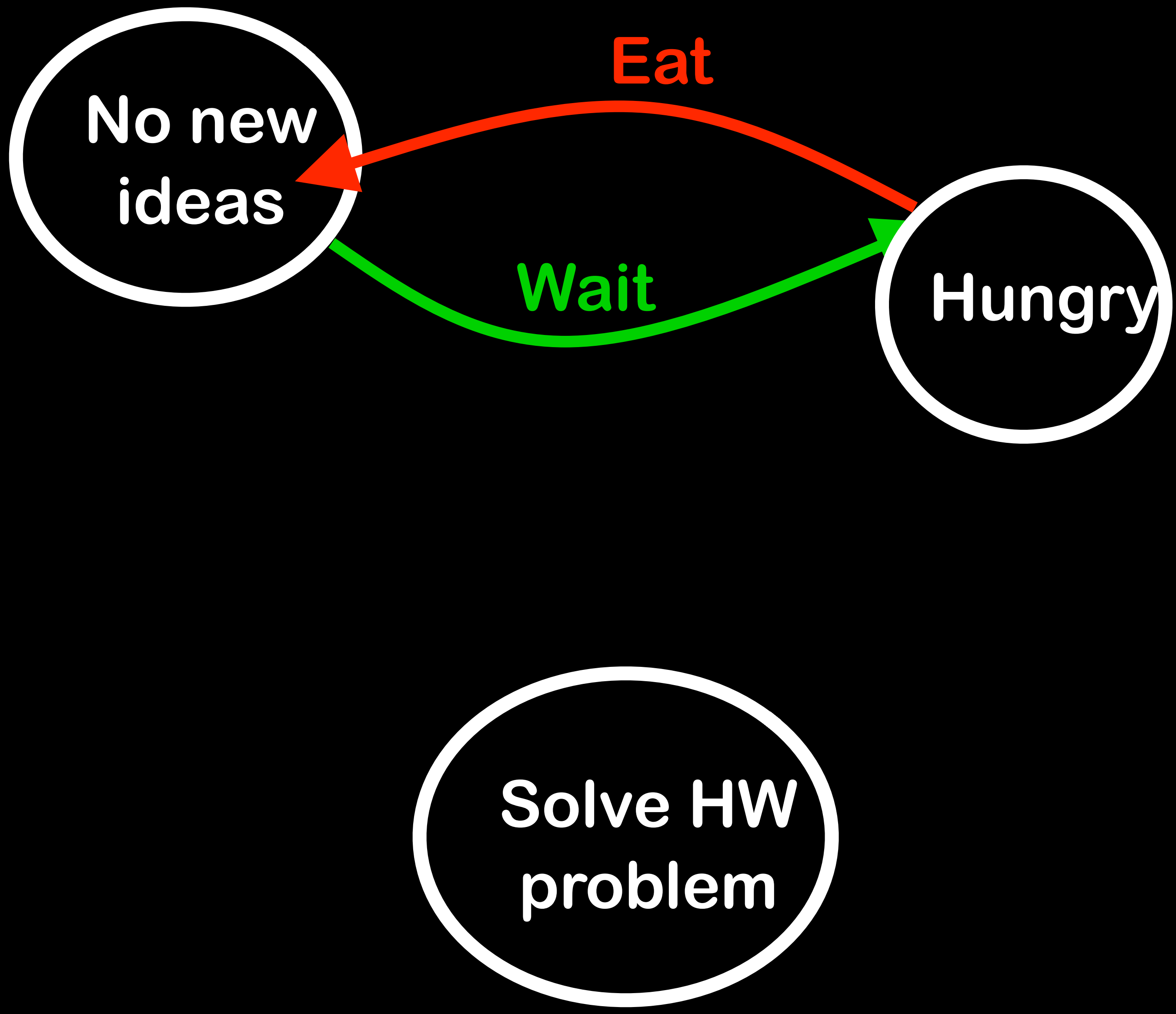
Hungry

Solve HW problem

Abstraction of Student Life



Abstraction of Student Life



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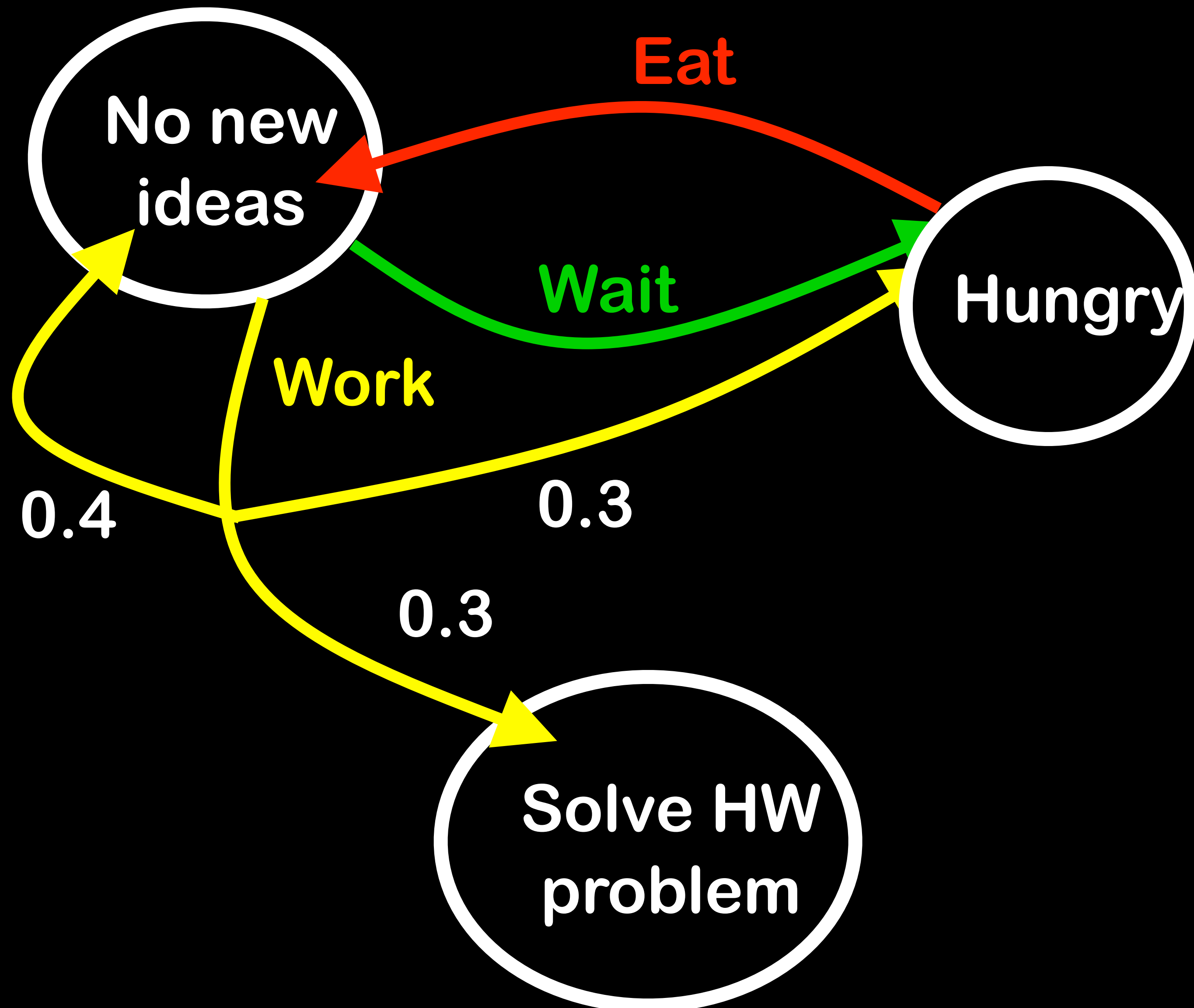

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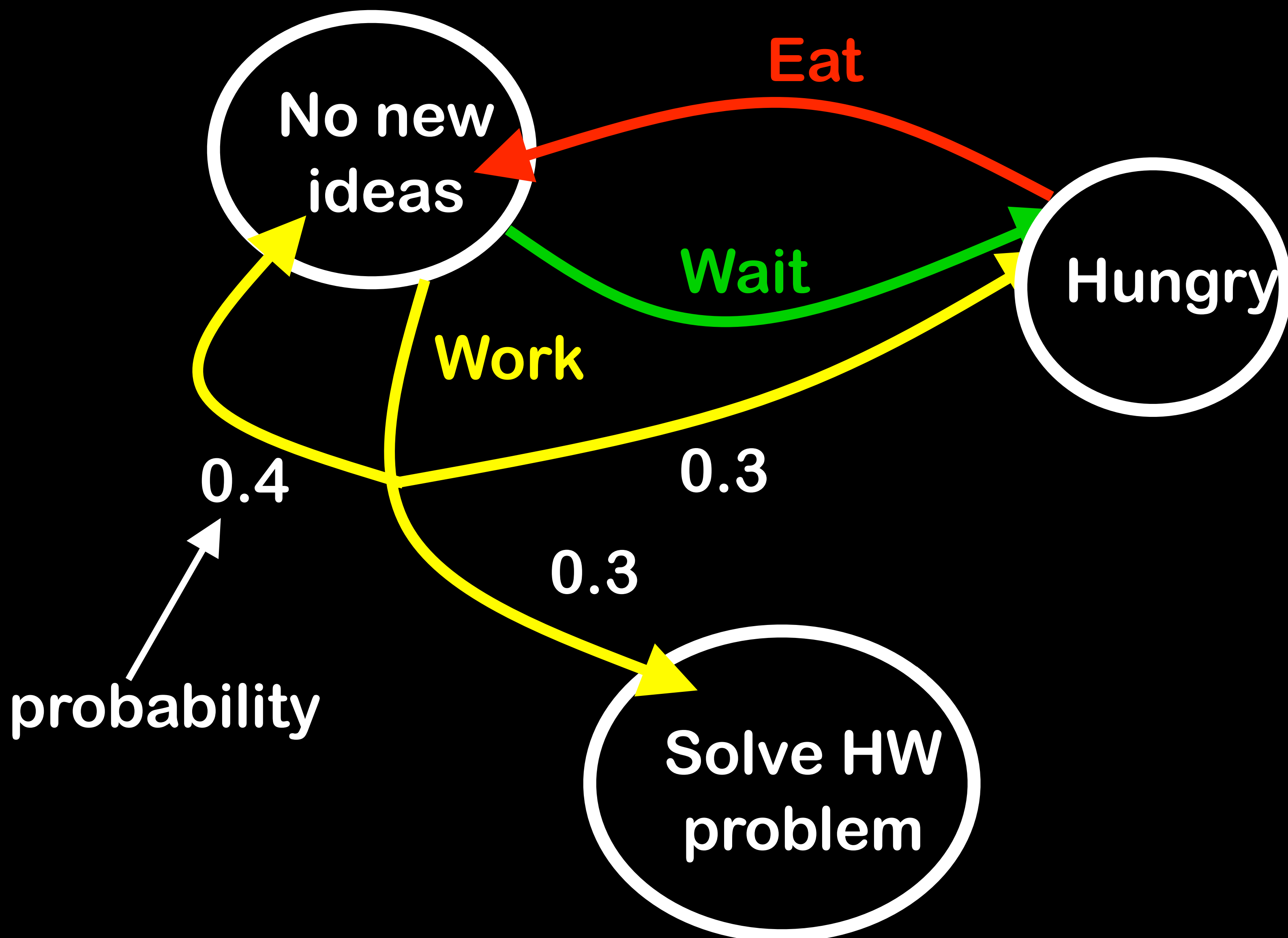

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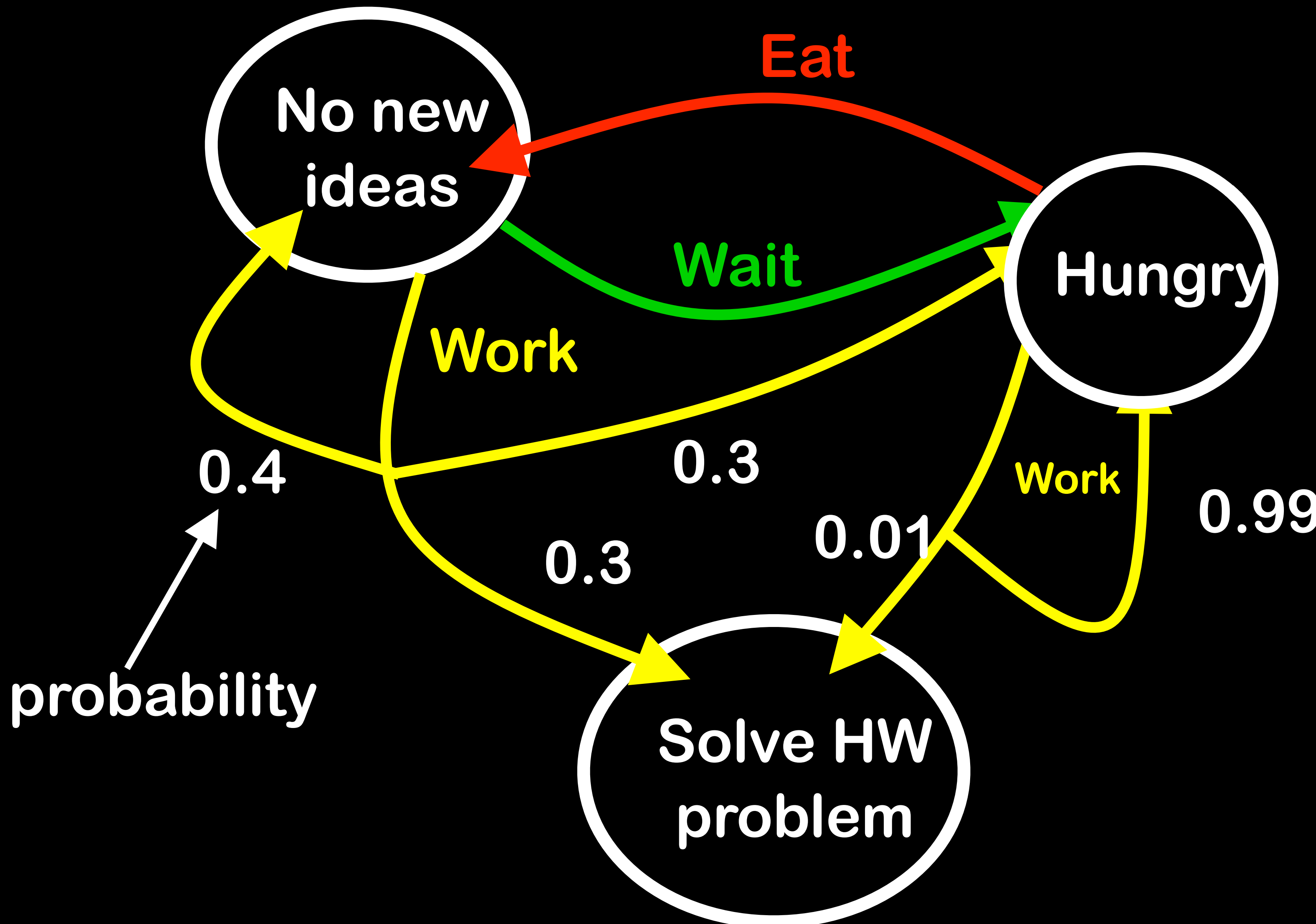

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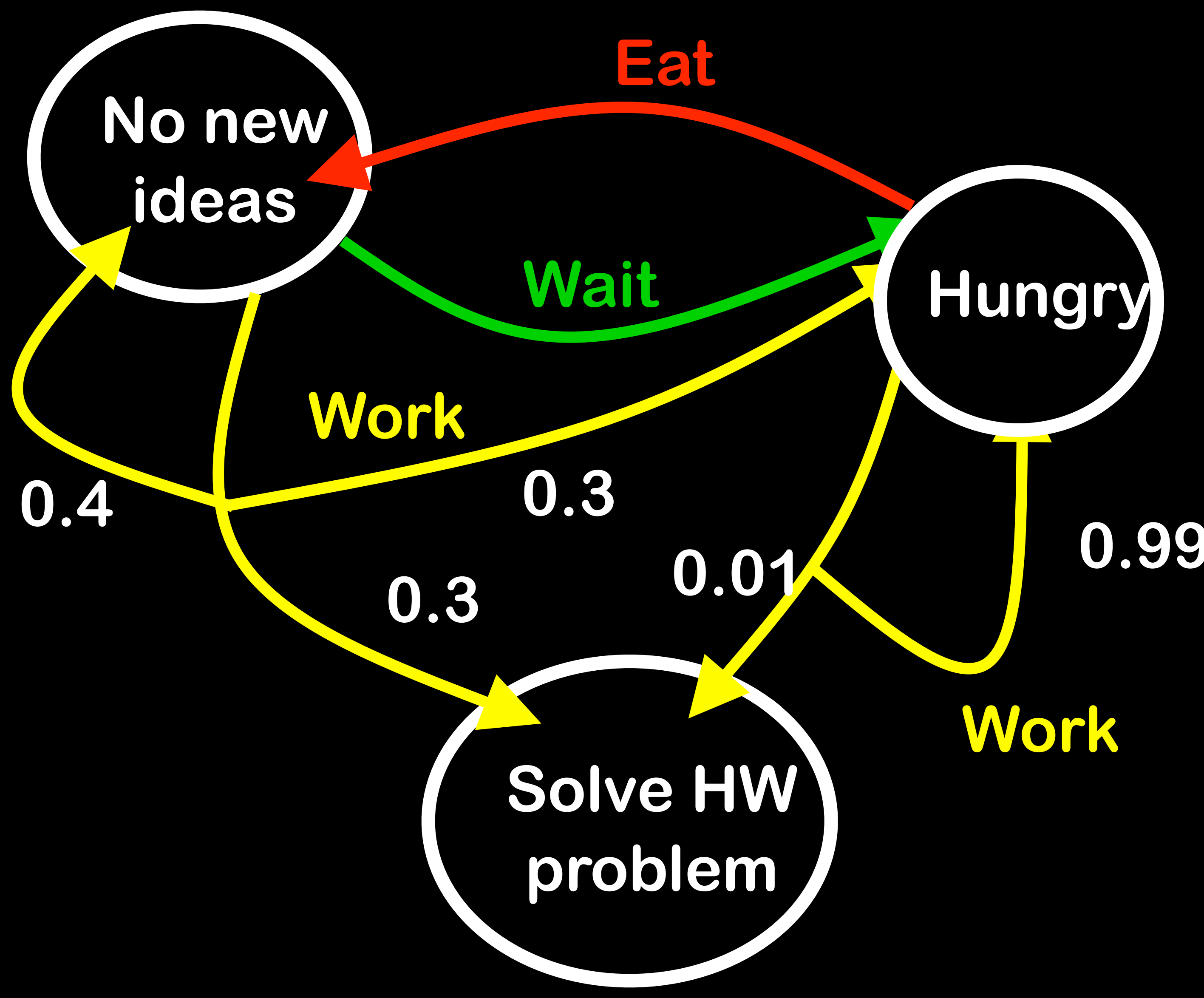

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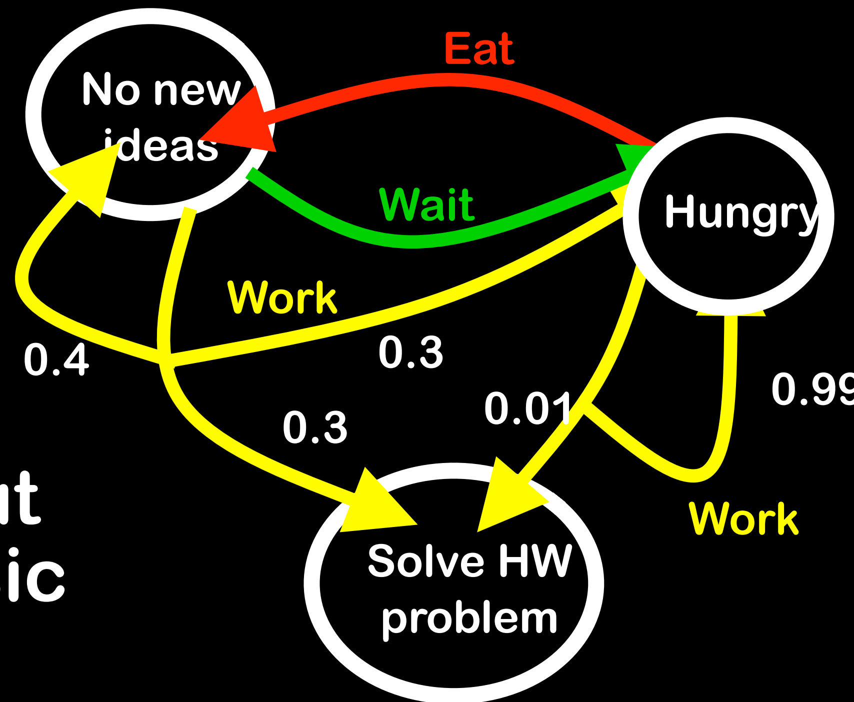
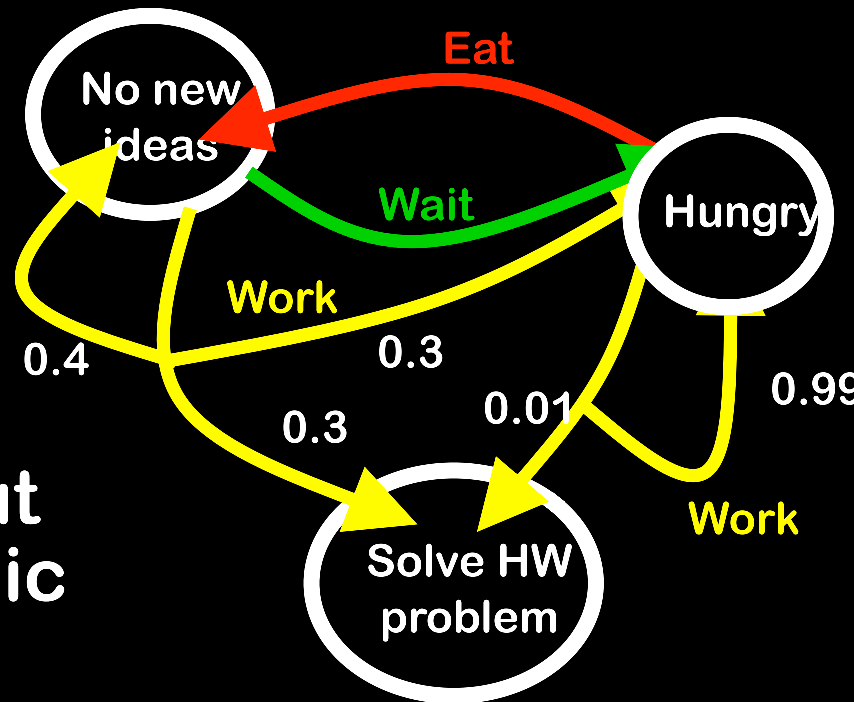


Like finite automata, but instead of a determinisic or non-deterministic action, we have a probabilistic action

Abstraction of Student Life



Like finite automata, but instead of a determinisic or non-deterministic action, we have a probabilistic action

Example questions: “What is the probability of reaching goal on string Work,Eat,Work?”

Simpler:
Random Walks on Graphs

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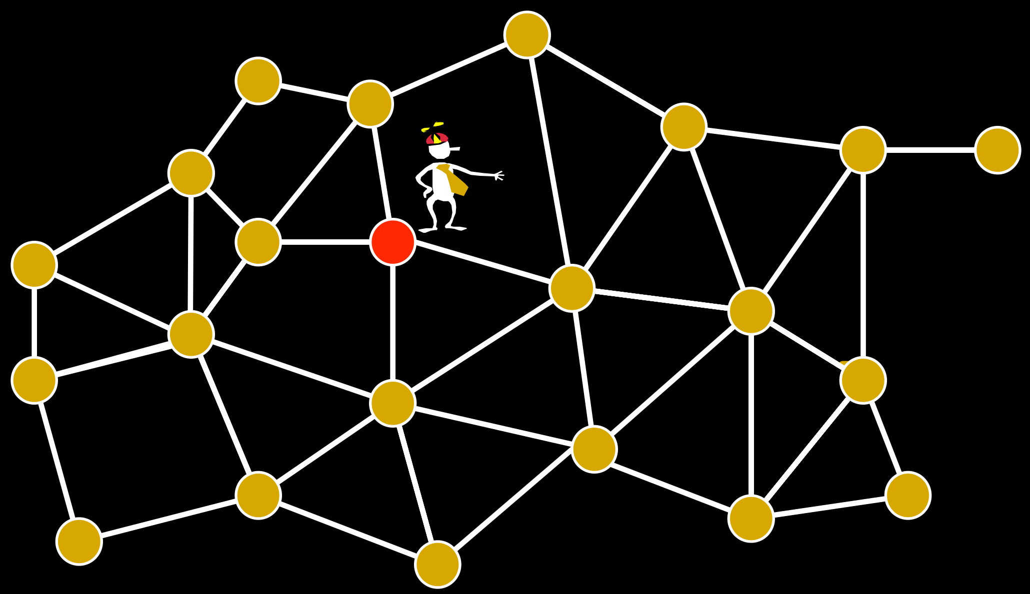

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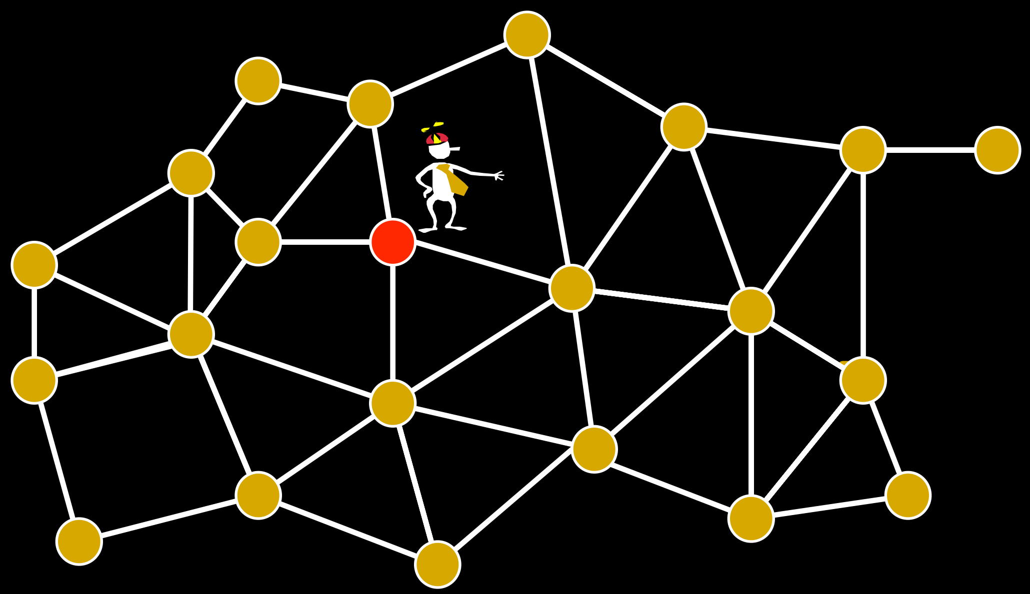
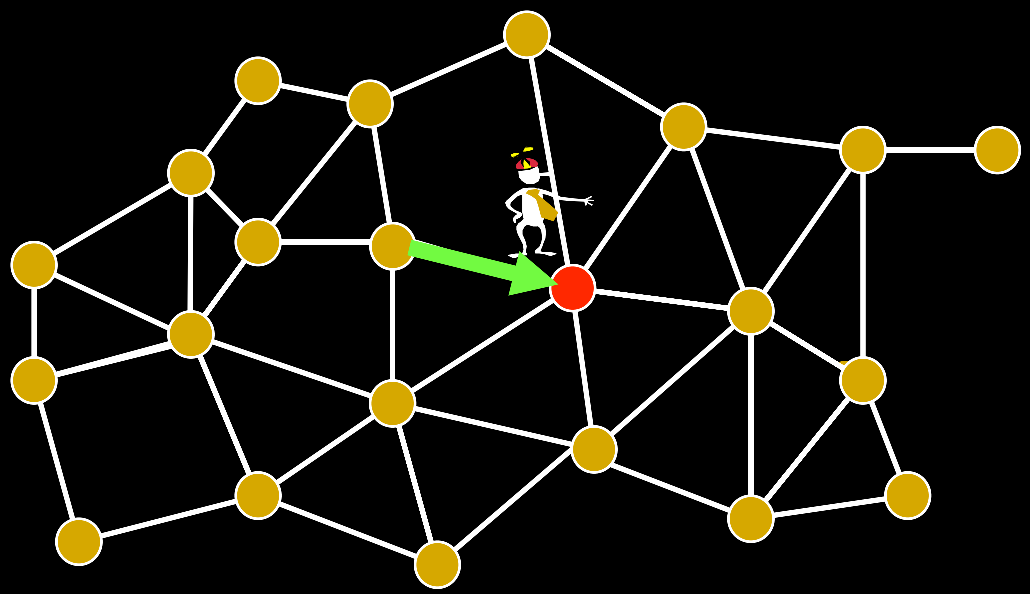


At any node, go to one of the neighbors of the node with equal probability

Simpler: Random Walks on Graphs



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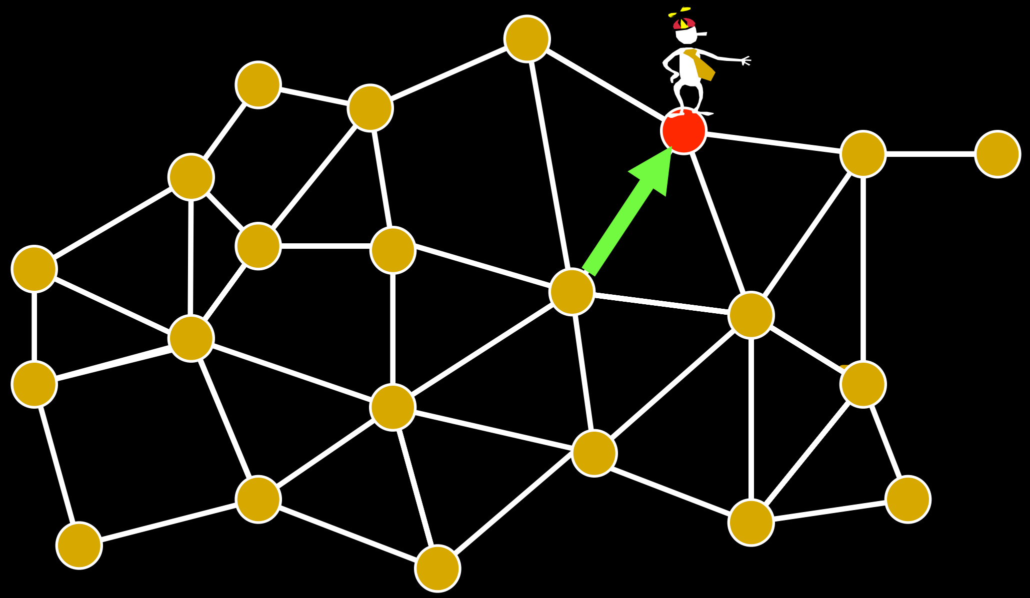

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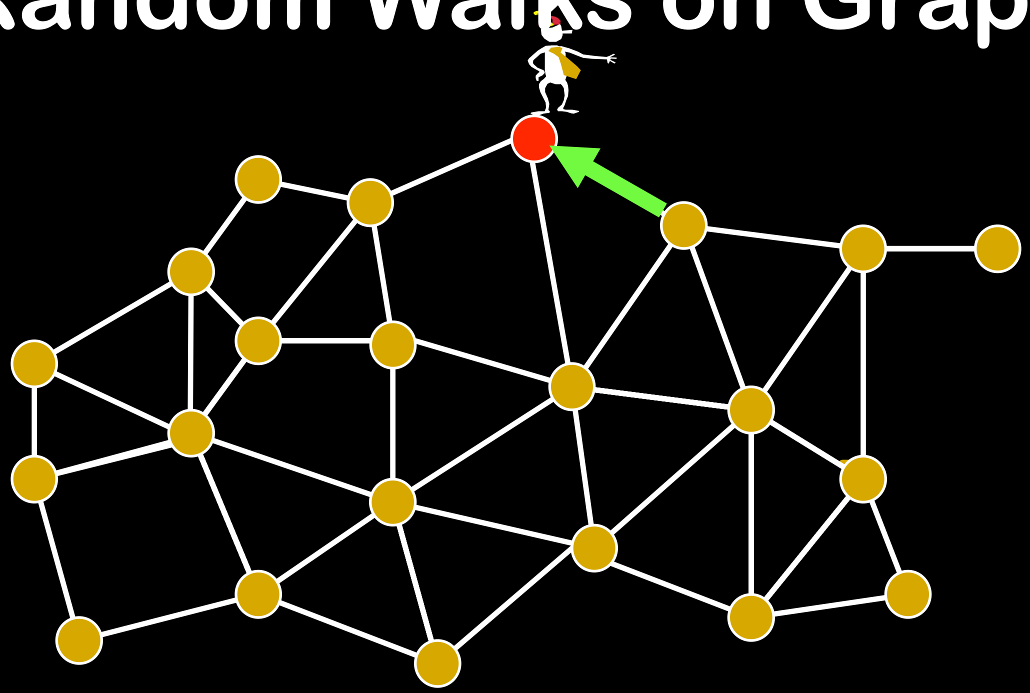

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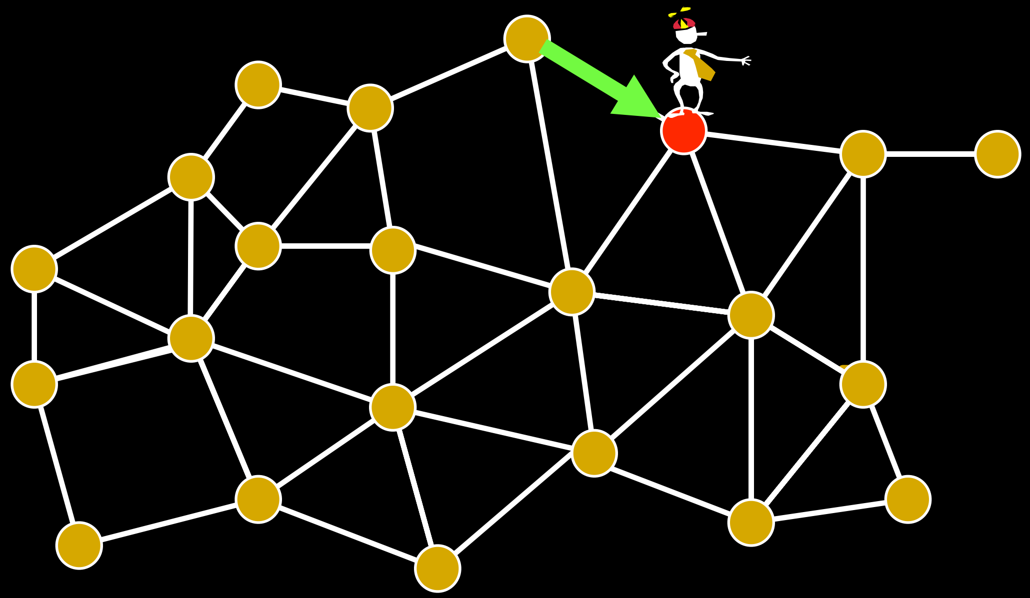

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Random Walk on a Line

Random Walk on a Line

You go into a casino with $k, and at each time step, you bet $1 on a fair game

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You leave when you are broke or have $n

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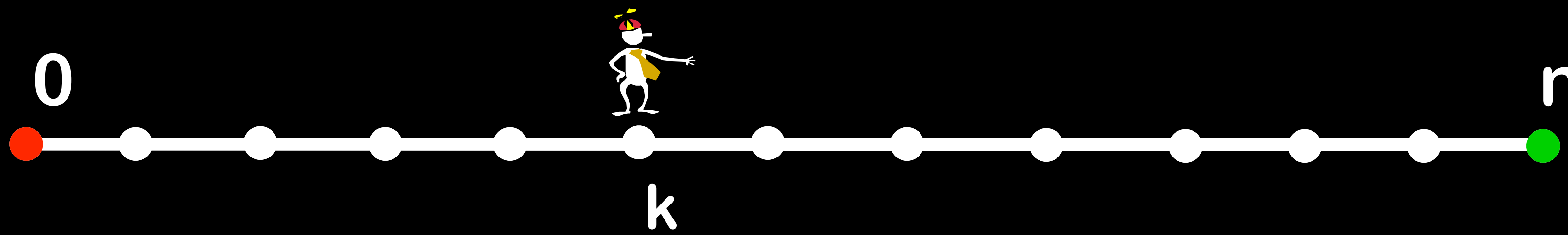

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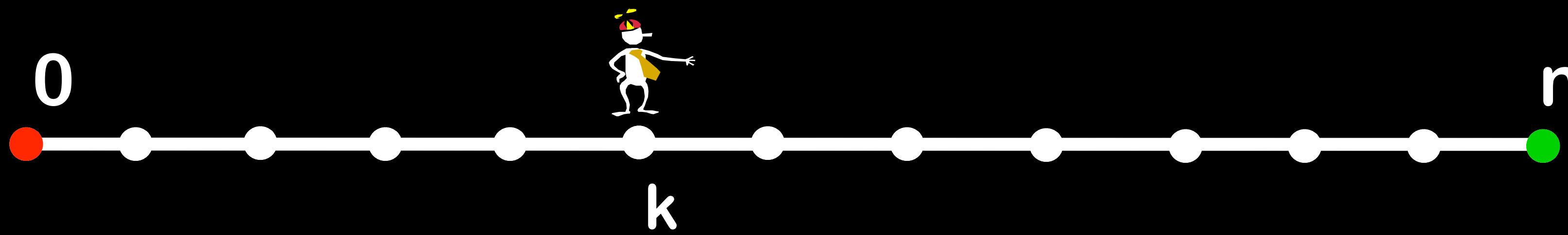

Question 1: what is your expected amount of money at time t?

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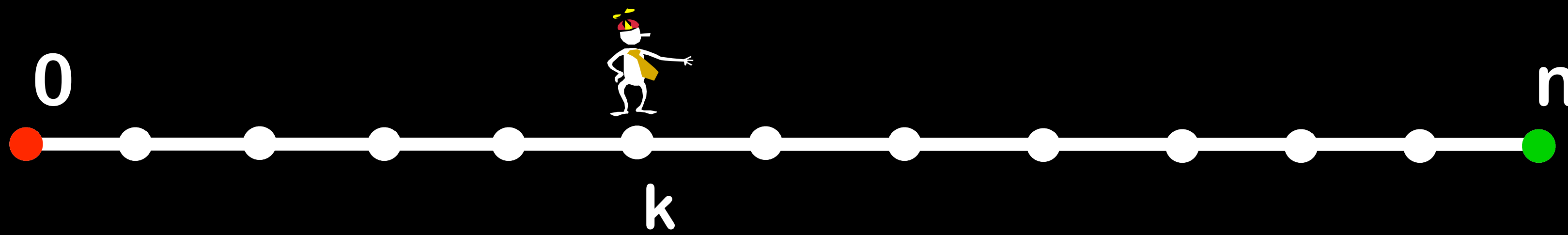

Question 1: what is your expected amount of money at time t?

Let X_t be a R.V. for the amount of \$\$\$ at time t

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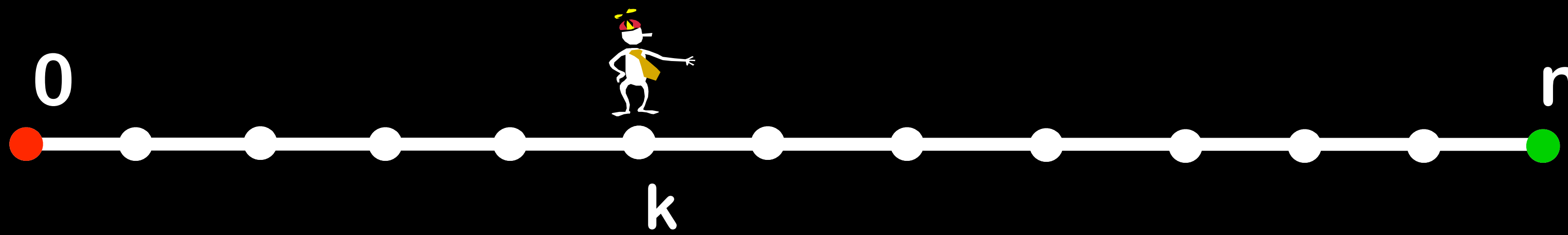

$X_t = k + \delta_1 + \delta_2 + \ldots + \delta_t$,

(δ_i is RV for change in your money at time i)

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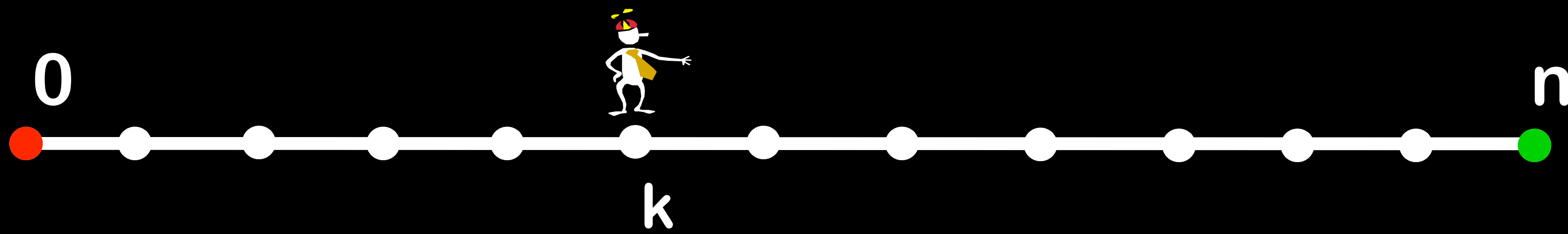

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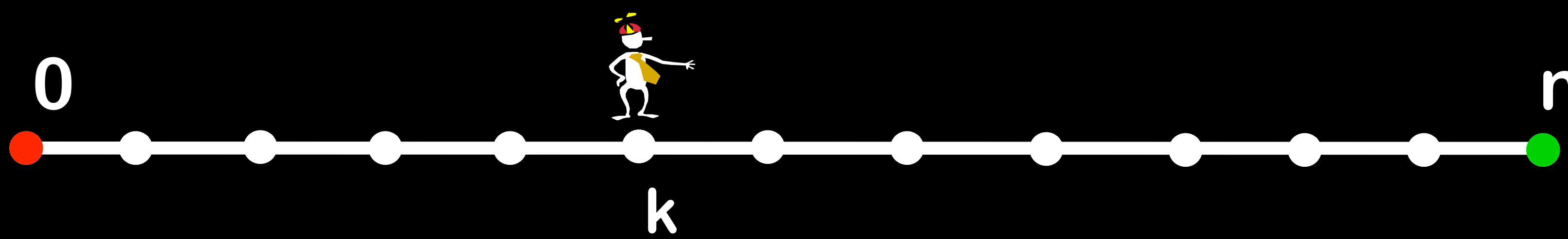

$X_t = k + \delta_1 + \delta_2 + \ldots + \delta_t,$

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$E[\delta_i] = 0$

So, $E[X_t] = k$

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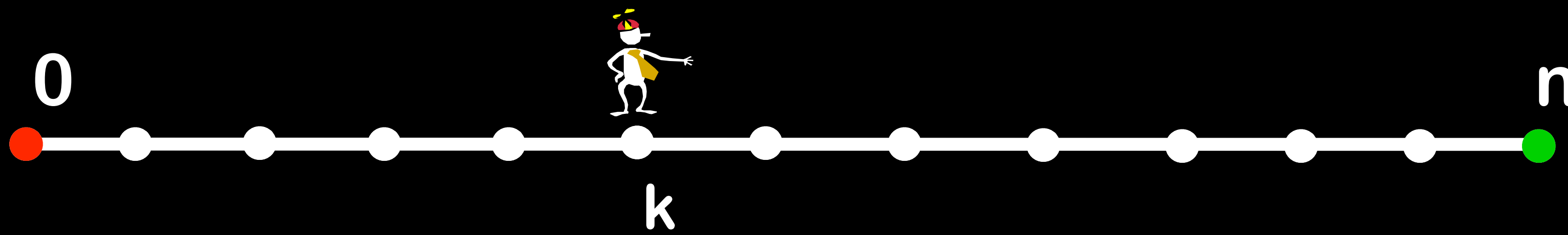

Question 2: what is the probability that you leave with \$n?

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Random Walk on a Line

Question 2: what is the probability that you leave with $n?

$E[X_t] = k$

$$E[X_t] = E[X_t \mid X_t = 0] \times \Pr(X_t = 0)$$
$$+ E[X_t \mid X_t = n] \times \Pr(X_t = n)$$
$$+ E[X_t \mid \text{neither}] \times \Pr(\text{neither})$$

Random Walk on a Line

Question 2: what is the probability that you leave with $n?

$$E[X_t] = k$$

$$\begin{aligned} E[X_t] = {} & E[X_t \mid X_t = 0] \times \Pr(X_t = 0) \\ & + E[X_t \mid X_t = n] \times \Pr(X_t = n) \\ & + E[X_t \mid \text{neither}] \times \Pr(\text{neither}) \end{aligned}$$

$$\begin{aligned} k = {} & n \times \Pr(X_t = n) \\ & + (\text{something}_t) \times \Pr(\text{neither}) \end{aligned}$$

Random Walk on a Line

Question 2: what is the probability that you leave with \$n?

$E[X_t] = k$

$$E[X_t] = E[X_t \mid X_t = 0] \times \Pr(X_t = 0) + E[X_t \mid X_t = n] \times \Pr(X_t = n) + E[X_t \mid \text{neither}] \times \Pr(\text{neither})$$

$$k = n \times \Pr(X_t = n) + (\text{something}_t) \times \Pr(\text{neither})$$

As $t \to \infty$, $\Pr(\text{neither}) \to 0$, also $\text{something}_t < n$

Hence $\Pr(X_t = n) \to k/n$

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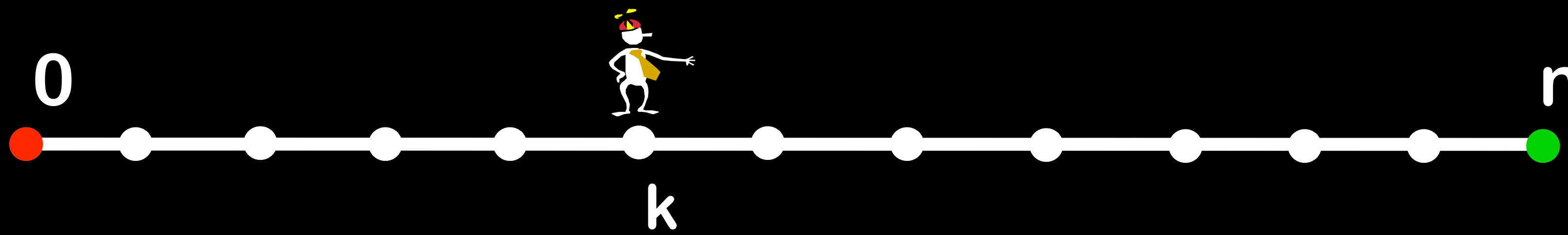

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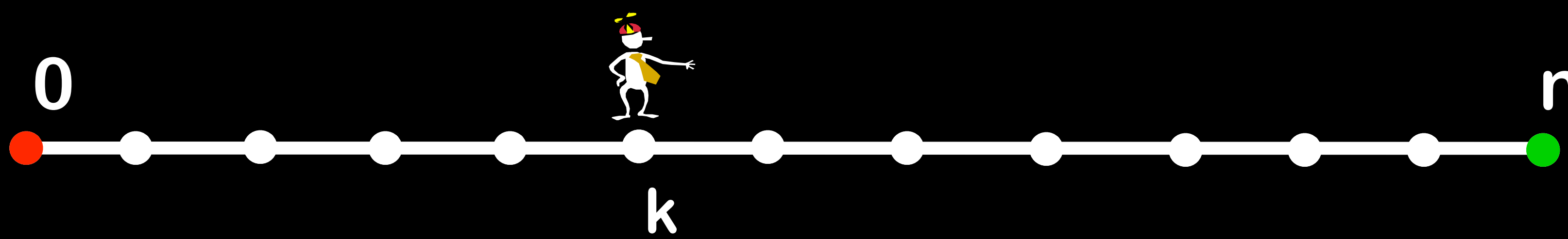
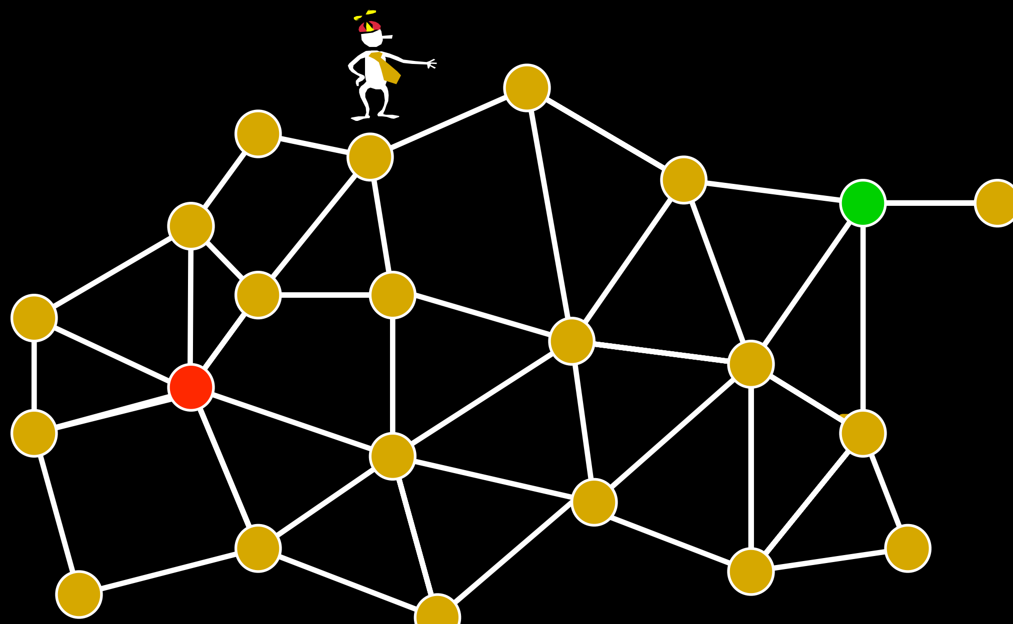


Question 2: what is the probability that you leave with \$n?

= probability that I hit green before I hit red

Random Walks and Electrical Networks

What is chance I reach green before red?



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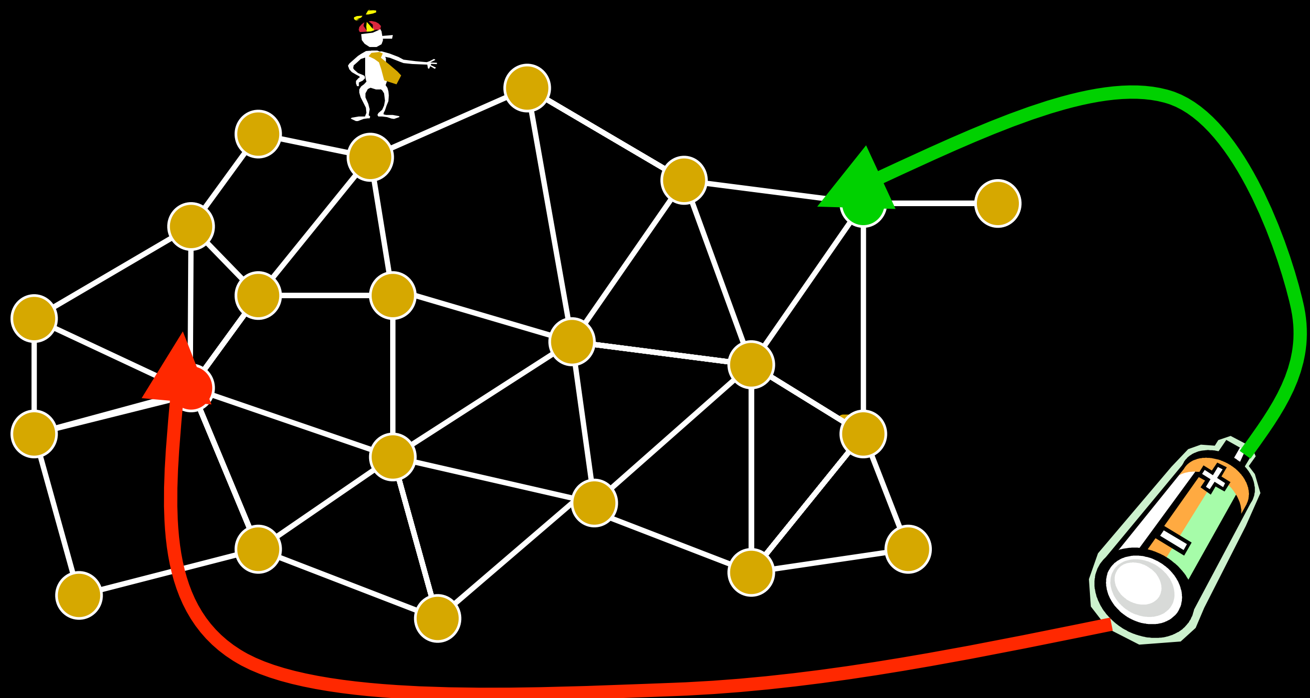
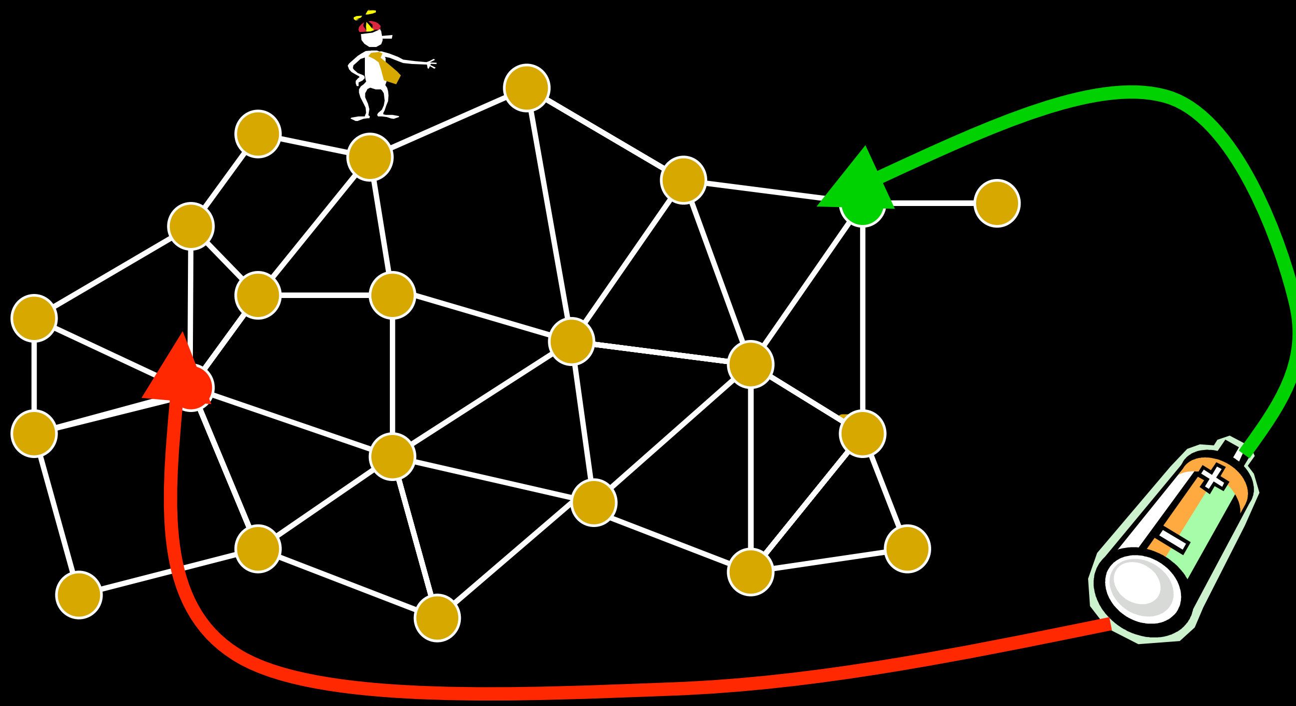


Same as voltage if edges are resistors and we put 1-volt battery between green and red

Random Walks and Electrical Networks



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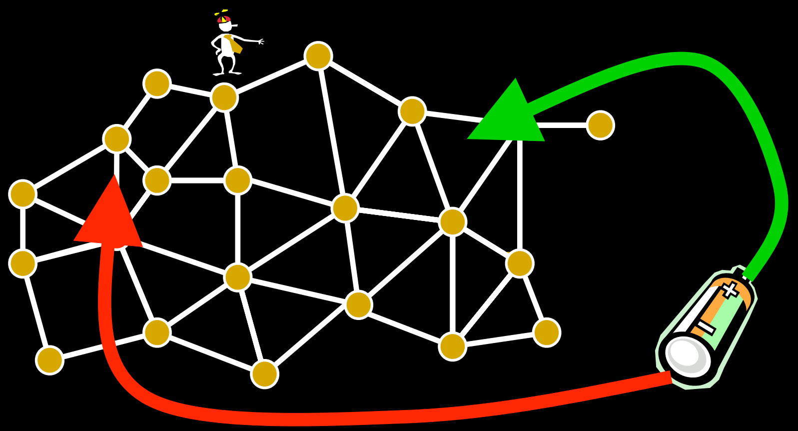

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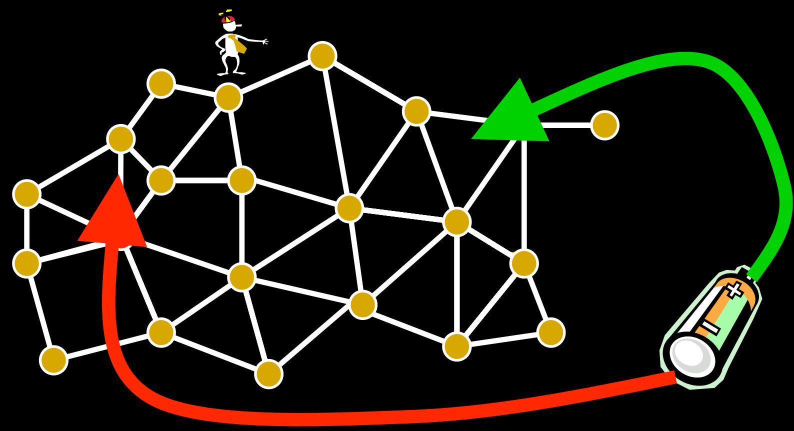
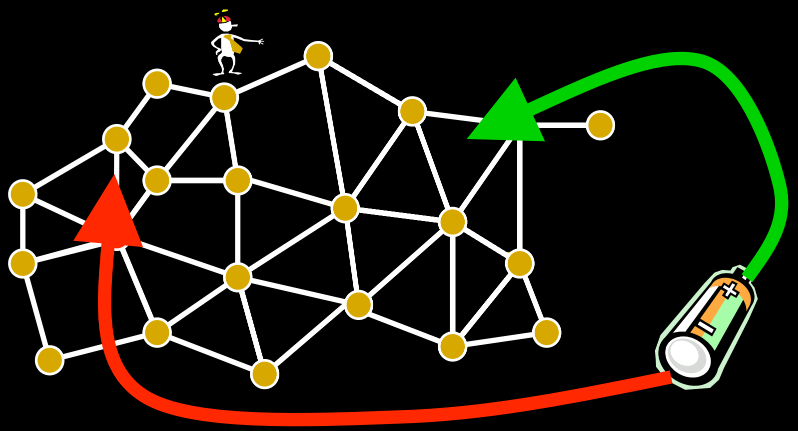


p_x = Pr(reach green first starting from x)

Random Walks and Electrical Networks



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$p_{green} = 1$, $p_{red} = 0$

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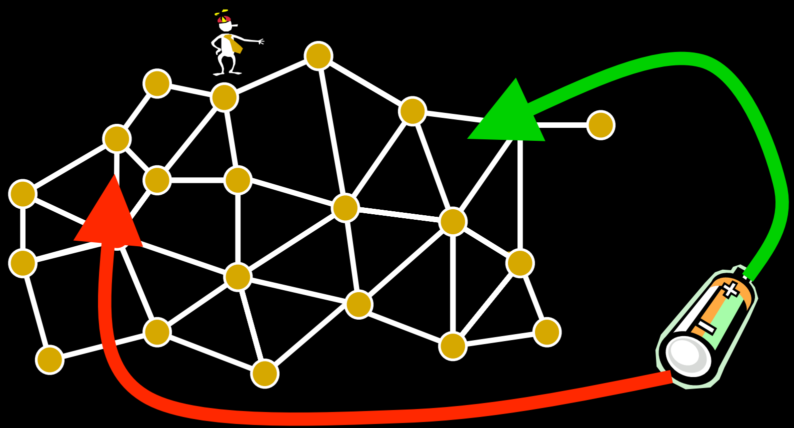

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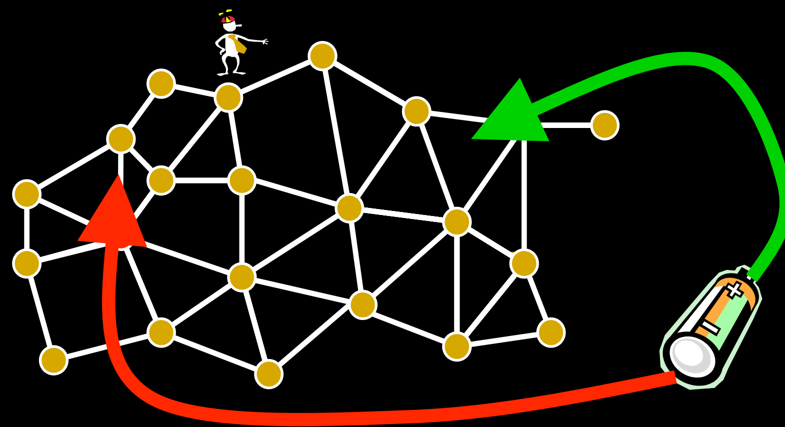
And for the rest $p_x = \text{Average}_{y \in Nbr(x)}(p_y)$

Random Walks and Electrical Networks



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Same as equations for voltage if edges all have same resistance!

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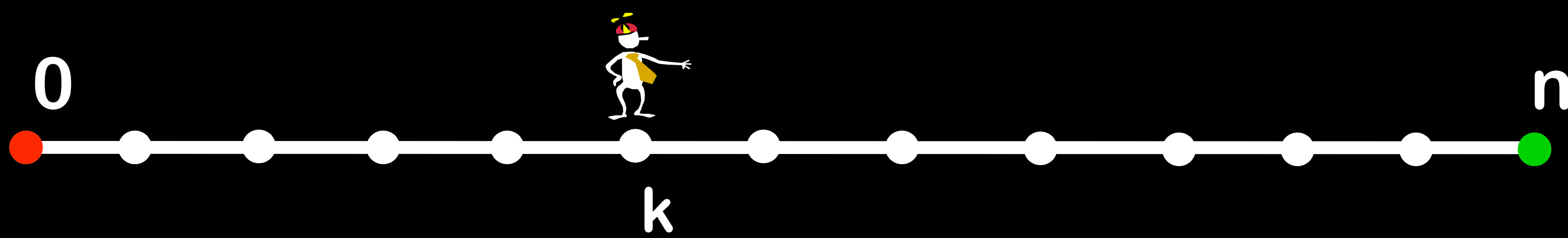

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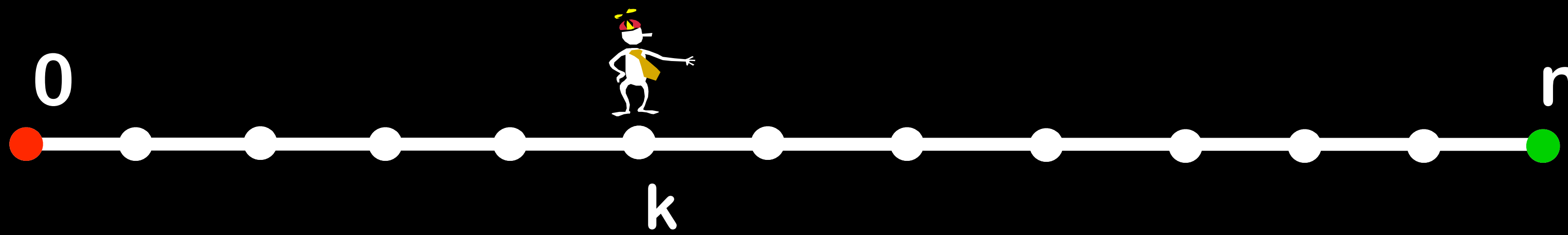
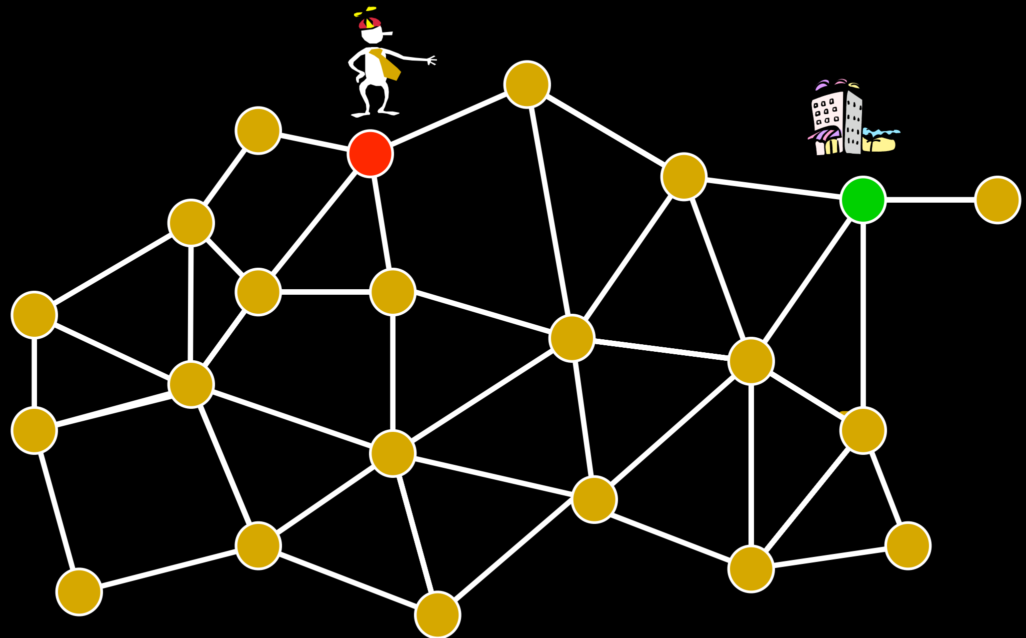


Question 2: what is the probability that you leave with $n?

voltage(k) = k/n
= Pr[hitting n before 0 starting at k] !!!

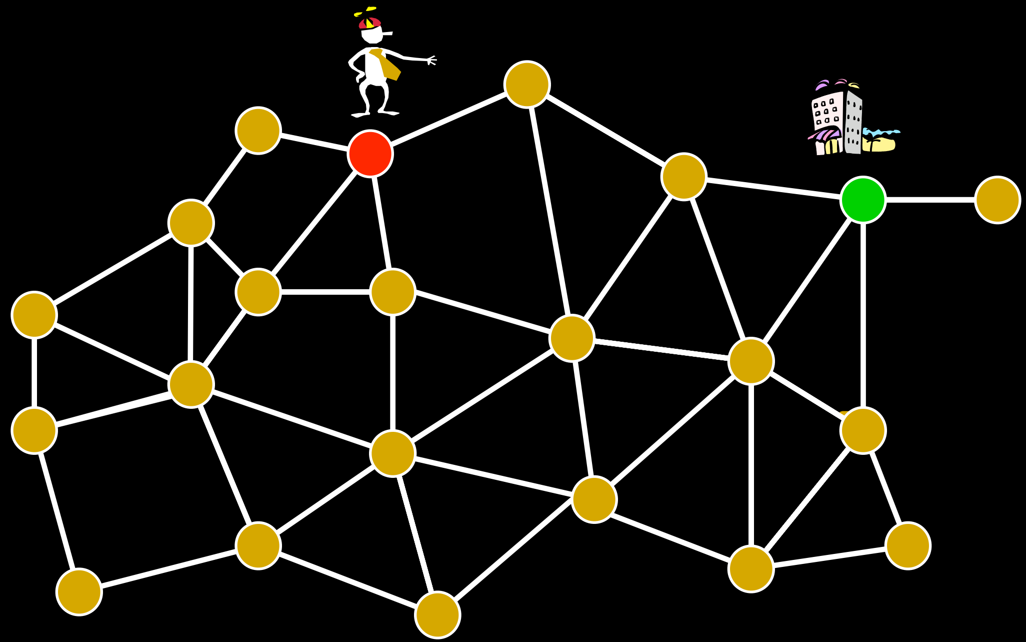
Getting Back Home

Getting Back Home



Lost in a city, you want to get back to your hotel
How should you do this?

Getting Back Home



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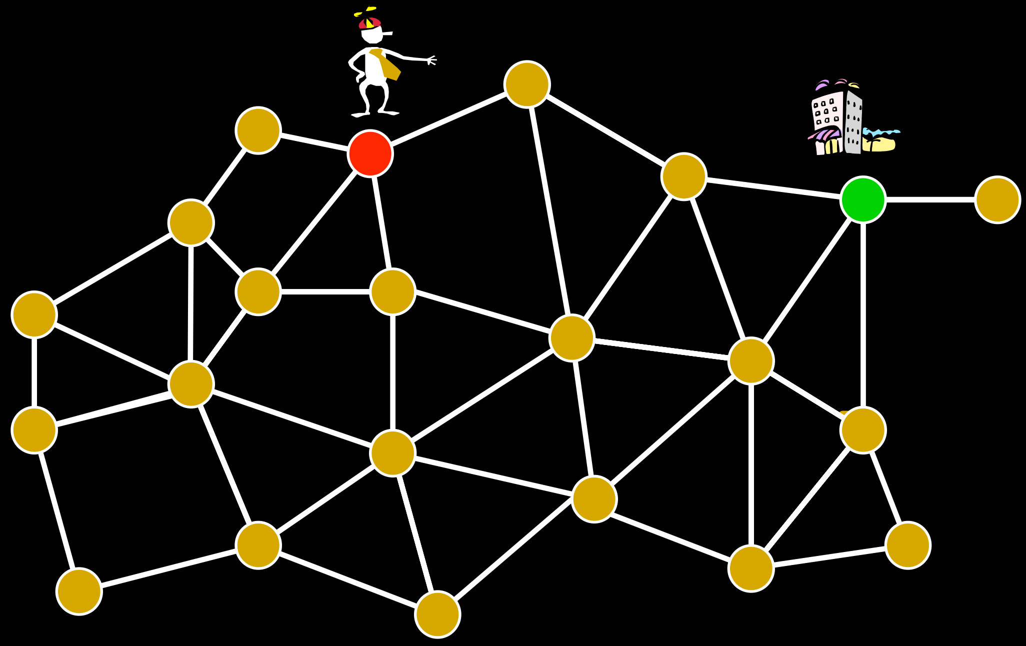
Depth First Search!

Getting Back Home

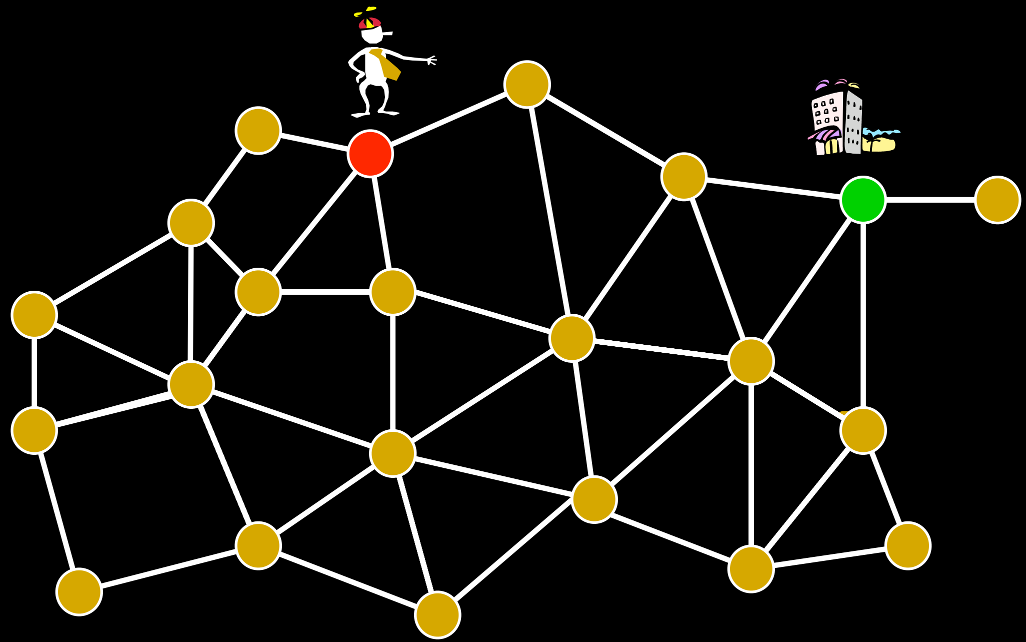


Lost in a city, you want to get back to your hotel
How should you do this?

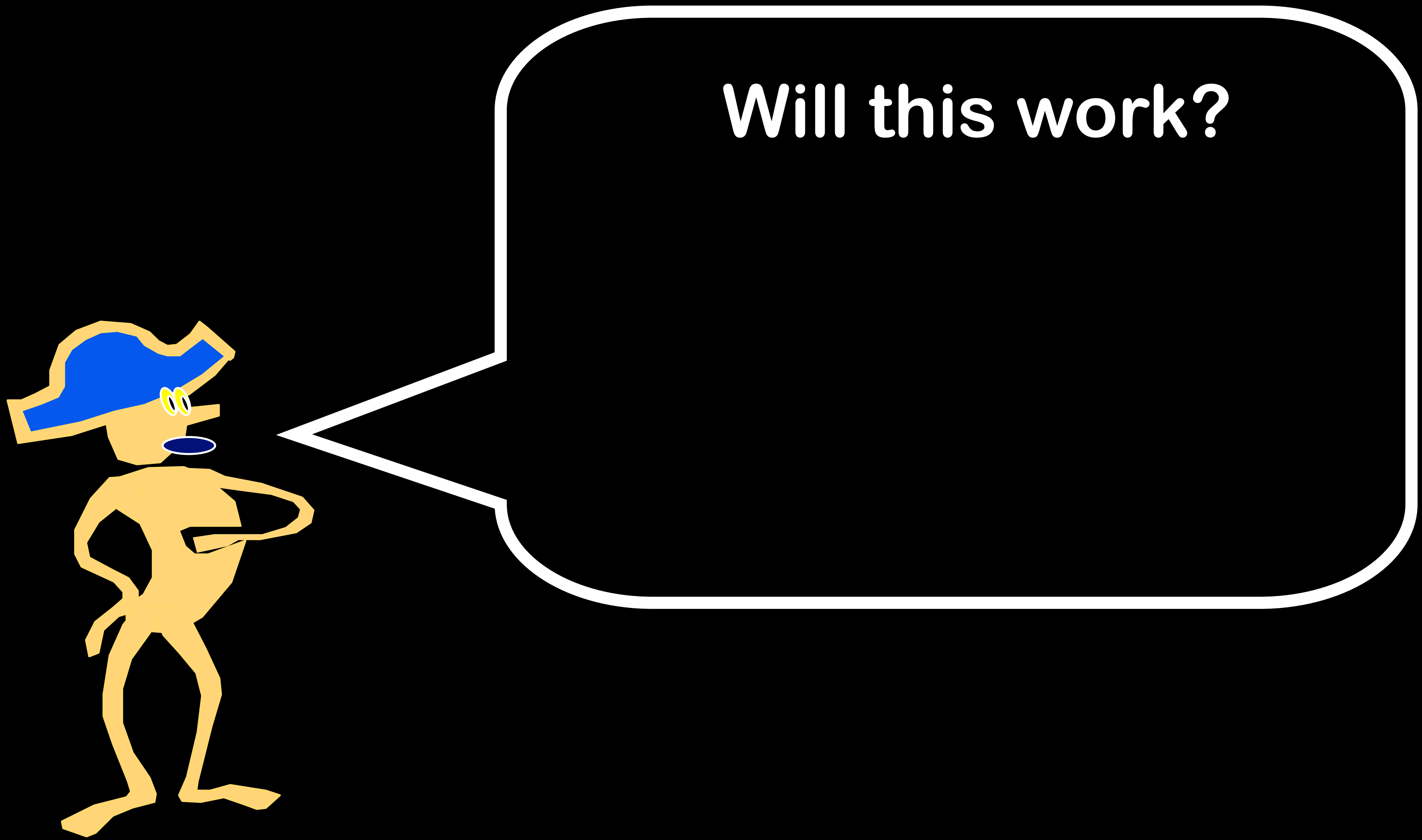
Depth First Search!

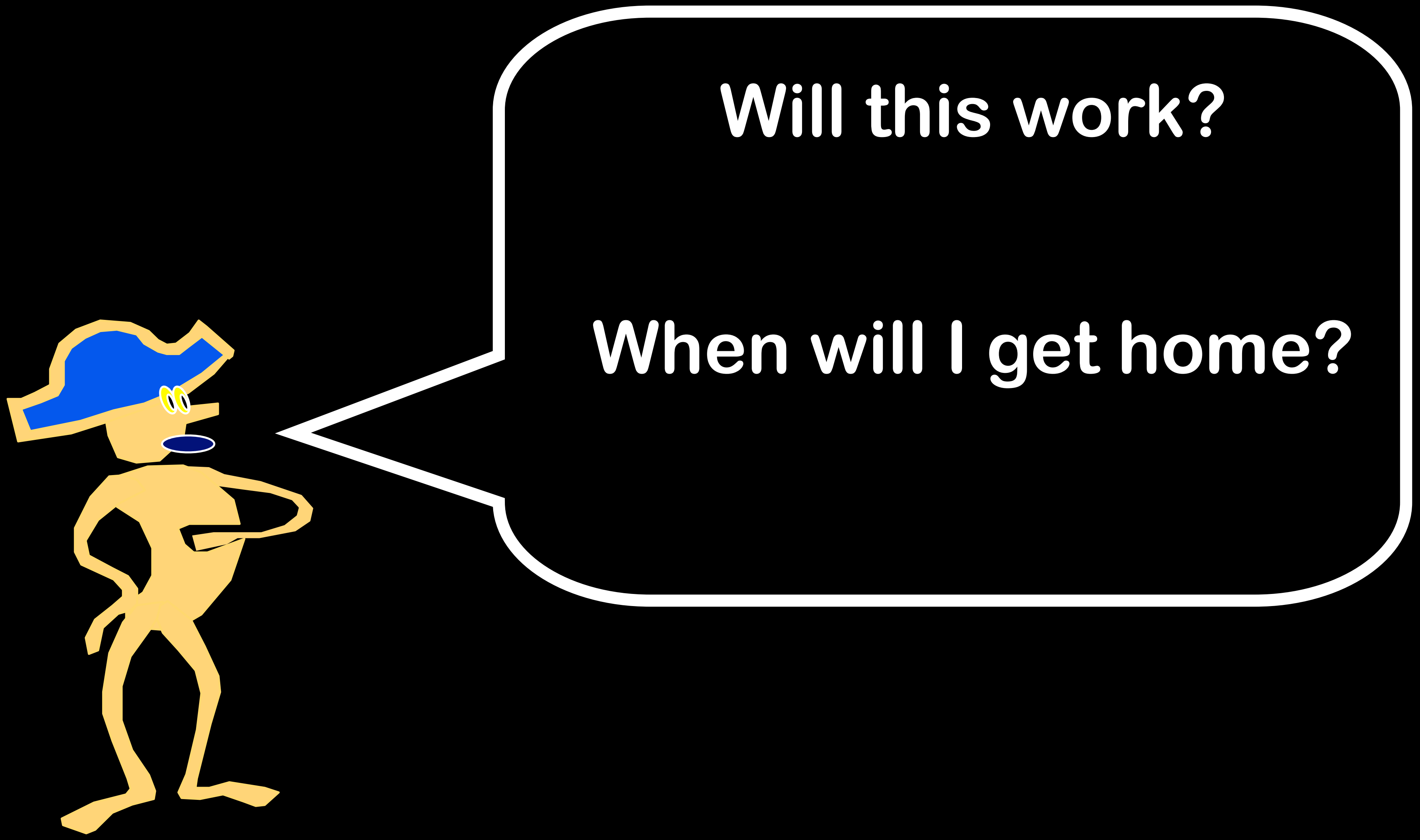
Requires a good memory and a piece of chalk

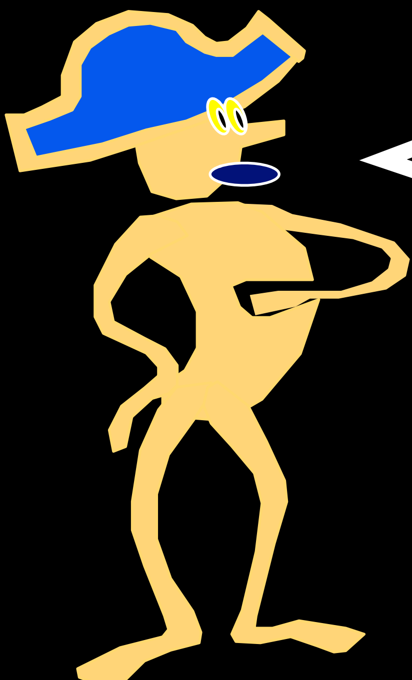
Getting Back Home



How about walking randomly?


Will this work?

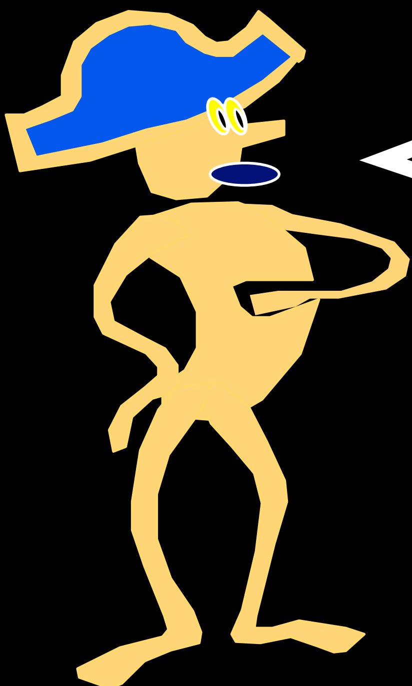

Will this work?
When will I get home?



Will this work?

Is Pr[reach home] = 1?

When will I get home?

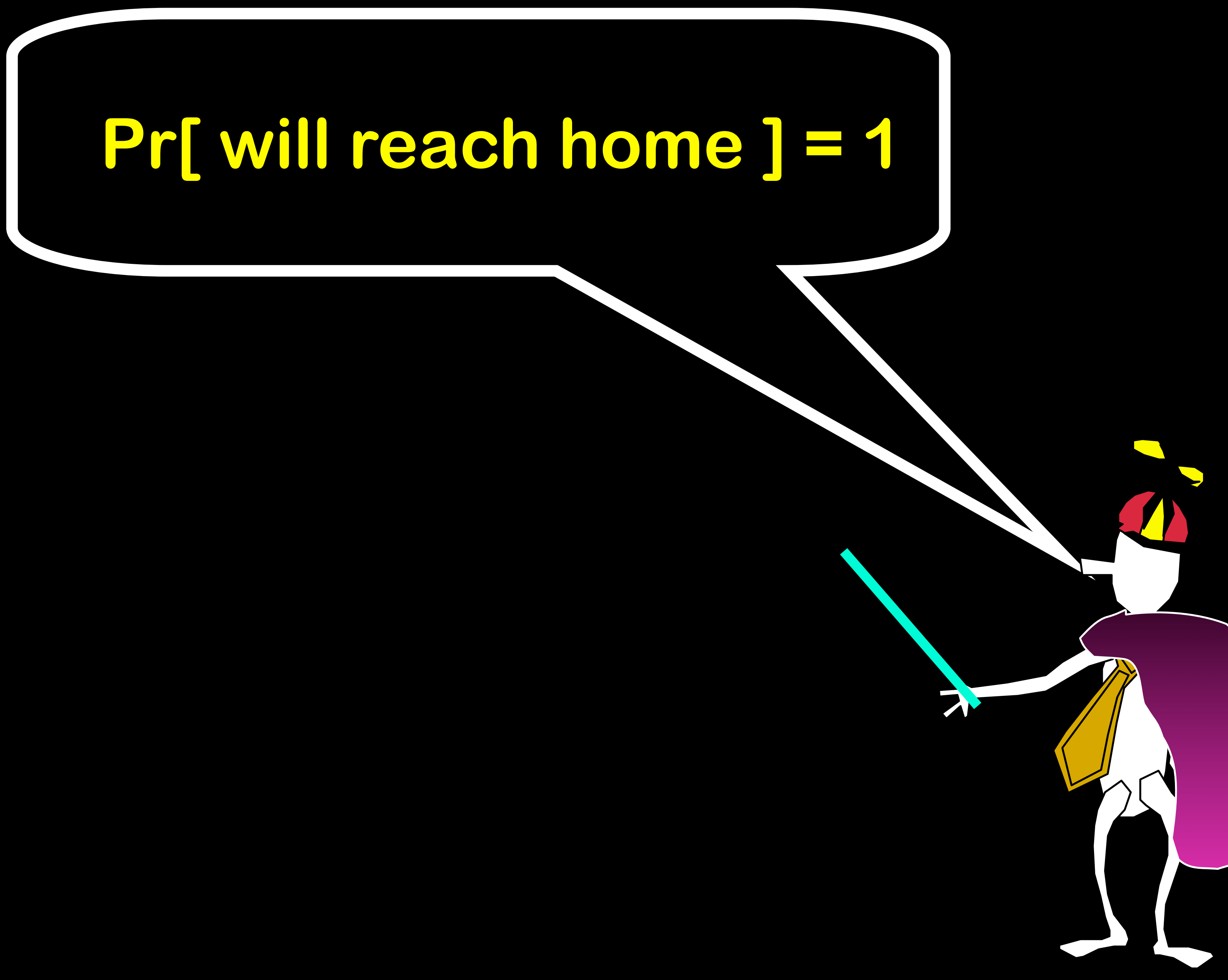


Will this work?

Is Pr[reach home] = 1?

When will I get home?

What is
E[time to reach home]?


Pr[will reach home] = 1

We Will Eventually Get Home

We Will Eventually Get Home

Look at the first n steps

We Will Eventually Get Home

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There is a non-zero chance p_1 that we get home

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Also, $p_1 \geq (1/n)^n$

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Then, wherever we are, there is a chance $p_2 \geq (1/n)^n$ that we hit home in the next n steps from there

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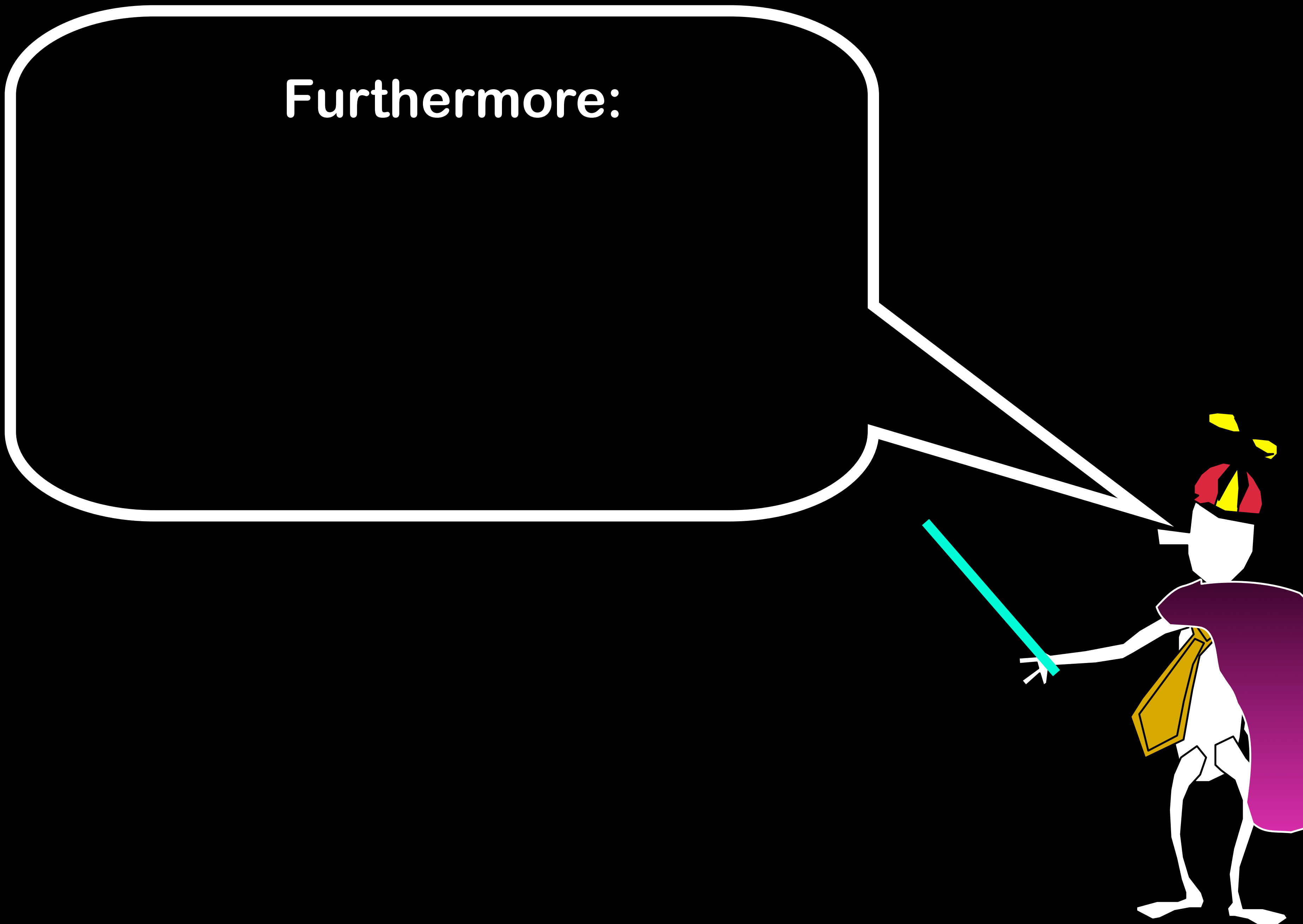
Probability of failing to reach home by time kn

$$= (1 - p_1)(1 - p_2) \ldots (1 - p_k) \to 0 \text{ as } k \to \infty$$


Furthermore:

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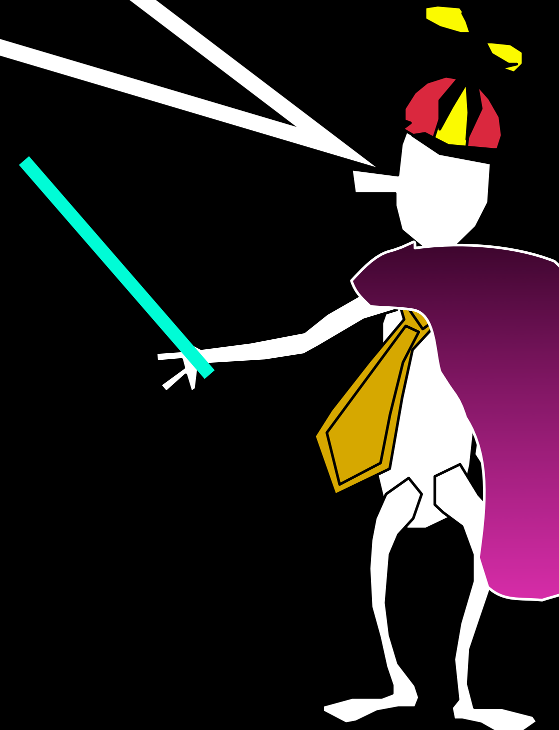
If the graph has
n nodes and m edges, then



Furthermore:

If the graph has
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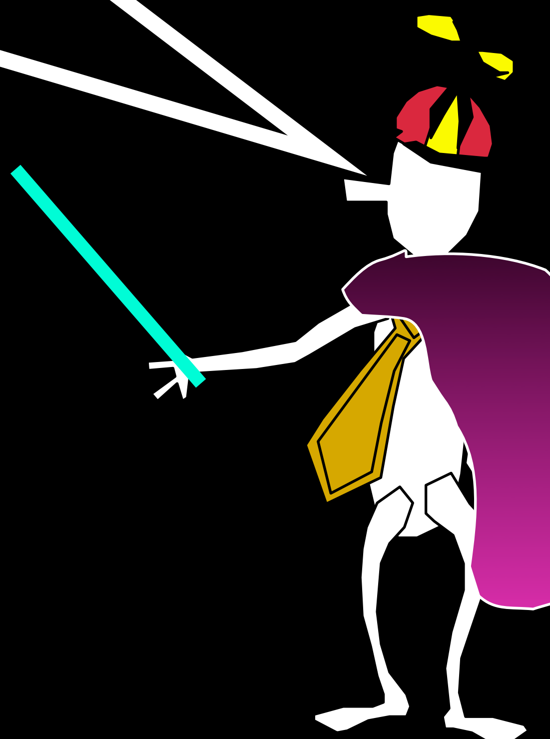
E[time to visit all nodes]
$\leq 2m \times (n-1)$

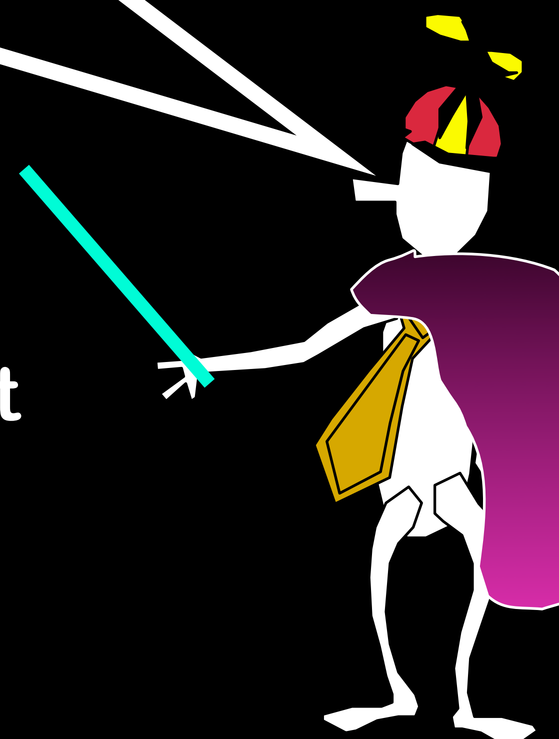


Furthermore:

If the graph has
n nodes and m edges, then

E[time to visit all nodes]
$\leq 2m \times (n-1)$

E[time to reach home] is at
most this



Cover Times

Cover Times

Cover time (from u)

C_u = E [time to visit all vertices | start at u]

Cover Times

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Cover time of the graph

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$C(G) = \max_u \{ C_u \}$

Cover Times

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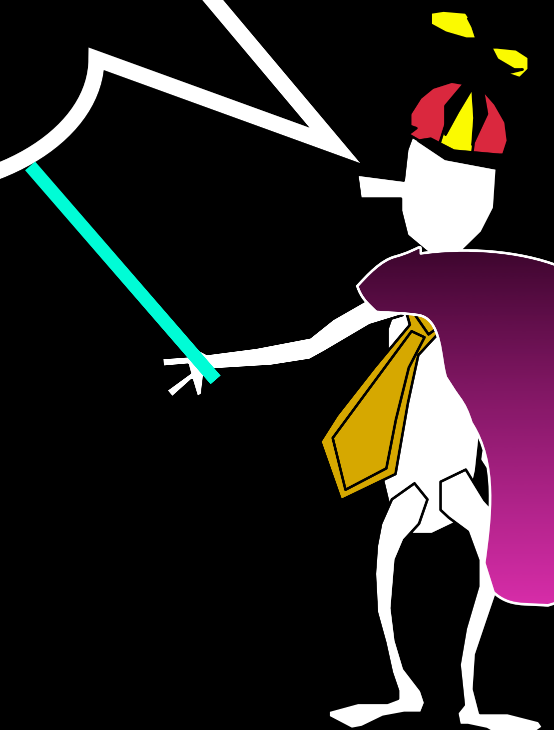
C_u = E [time to visit all vertices | start at u]

Cover time of the graph

$C(G) = \max_u \{ C_u \}$

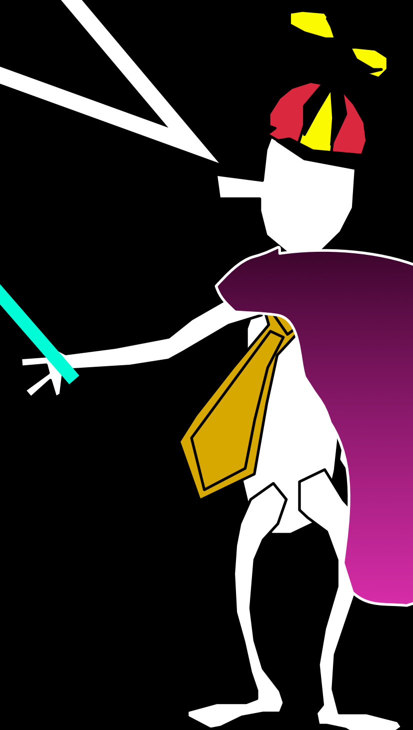
(worst case expected time to see all vertices)

Cover Time Theorem



Cover Time Theorem

If the graph **G** has **n** nodes and **m** edges, then the cover time of **G** is



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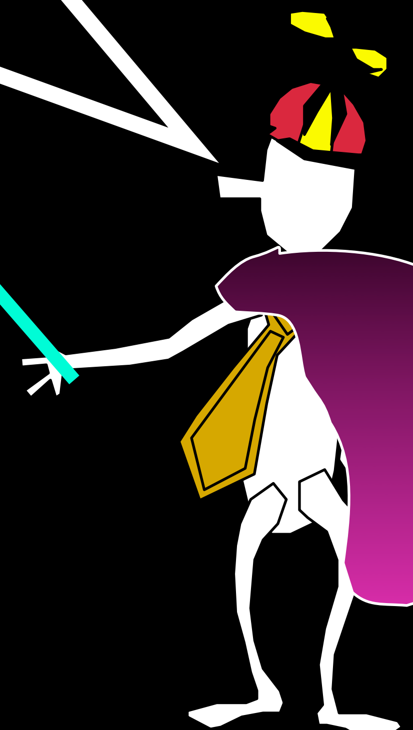

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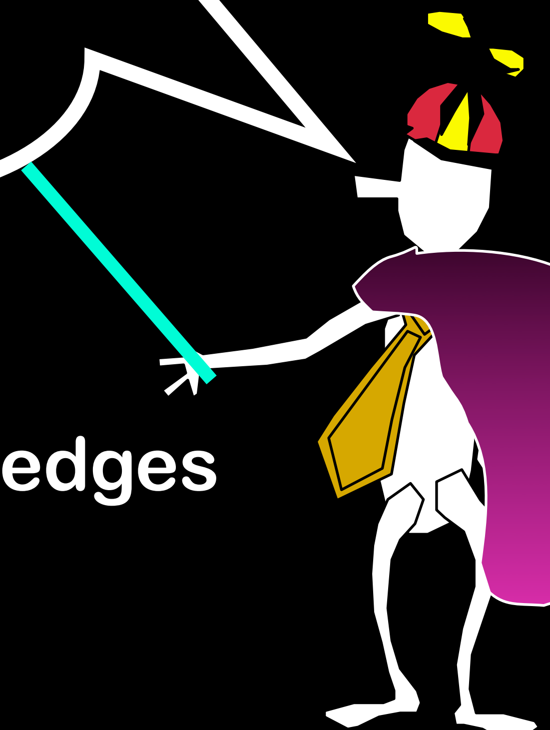


Any graph on n vertices has < $n^2/2$ edges

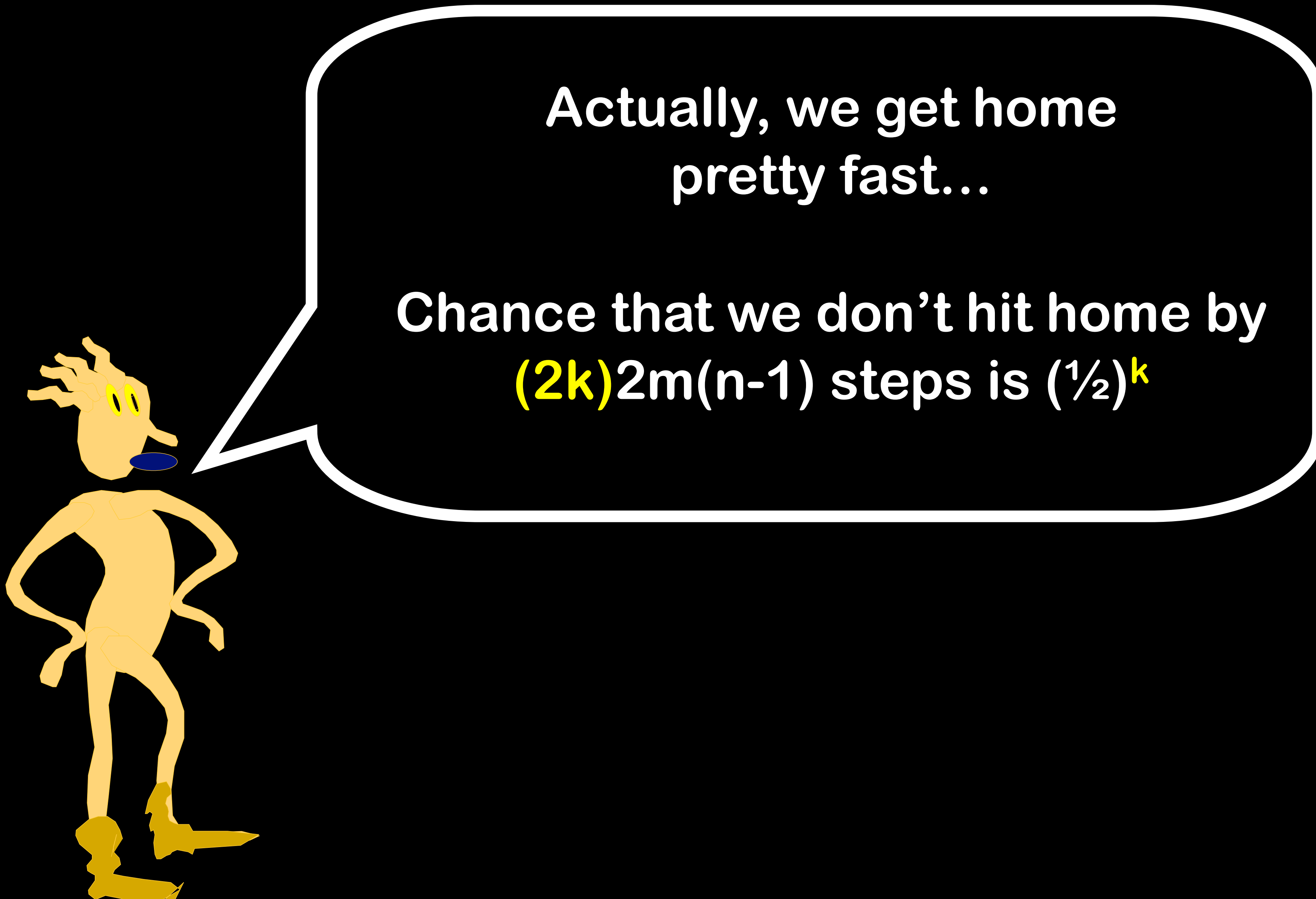
Cover Time Theorem

If the graph G has
n nodes and m edges, then
the cover time of G is

$$C(G) \leq 2m\,(n-1)$$

Any graph on n vertices has $< n^2/2$ edges

Hence $C(G) < n^3$ for all graphs G


Actually, we get home pretty fast…
Chance that we don't hit home by (2k)2m(n-1) steps is $(½)^k$

A Simple Calculation

A Simple Calculation

True or False:

A Simple Calculation

True or False:

If the average income of people is $100 then

more than 50% of the people can be earning more than $200 each

A Simple Calculation

True or False:

If the average income of people is $100 then more than 50% of the people can be earning more than $200 each

False! else the average would be higher!!!

Markov's Inequality

Andrei A. Markov

Markov's Inequality

If X is a non-negative r.v. with mean E[X], then

Andrei A. Markov

Markov's Inequality

If X is a non-negative r.v. with mean E[X], then

$$\Pr[\, X > 2\,E[X]\,] \leq \tfrac{1}{2}$$

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If X is a non-negative r.v. with mean E[X], then

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$$\Pr[\, X > k\, E[X]\,] \leq 1/k$$

Andrei A. Markov

Markov’s Inequality

Markov's Inequality

Non-neg random variable X has expectation
A = E[X]

Markov's Inequality

Non-neg random variable X has expectation A = E[X]

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$\Rightarrow A \geq 2A \times Pr[X > 2A]$

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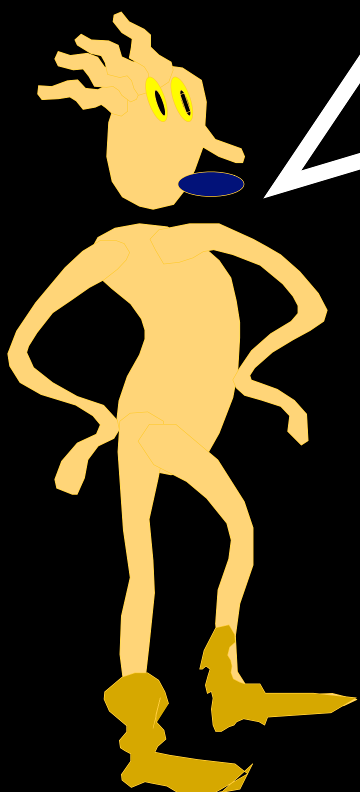
$\Pr[\,X > k \times \text{expectation}\,] \le 1/k$



Actually, we get home pretty fast…

Chance that we don't hit home by $(2k)2m(n-1)$ steps is $(\frac{1}{2})^k$

An Averaging Argument

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Suppose I start at u

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E[time to hit all vertices | start at u] $\leq$ C(G)

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Hence, by Markov's Inequality:
Pr[time to hit all vertices > 2C(G) | start at u] $\leq$ ½

So Let's Walk Some More!

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Suppose at time 2C(G), I'm at some node
with more nodes still to visit

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Pr [time to hit all vertices > 2C(G) | start at u] $\leq$ ½

Suppose at time 2C(G), I'm at some node with more nodes still to visit

Pr [haven't hit all vertices in 2C(G) <u>more</u> time | start at v] $\leq$ ½

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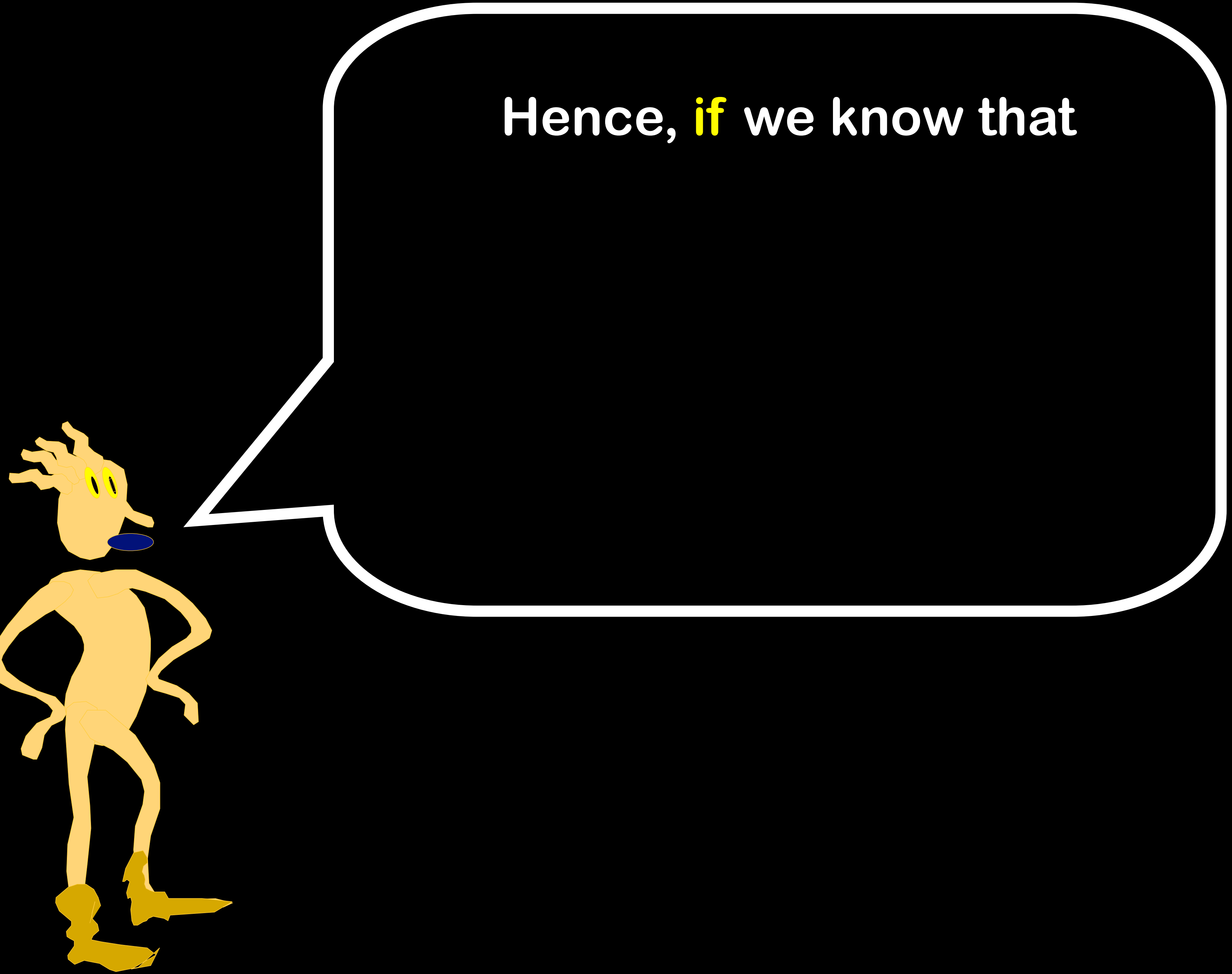
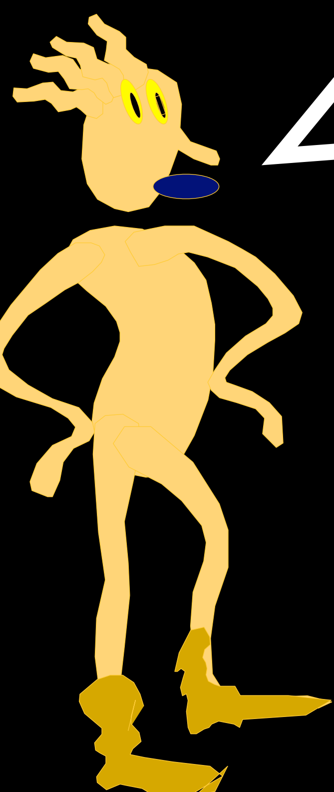
Pr [haven't hit all vertices in 2C(G) more time | start at v] $\leq$ ½

Chance that you failed both times $\leq$ ¼ = $(½)^2$

Hence,
Pr[havent hit everyone in time k × 2C(G)] $\leq$ $(½)^k$


Hence, if we know that



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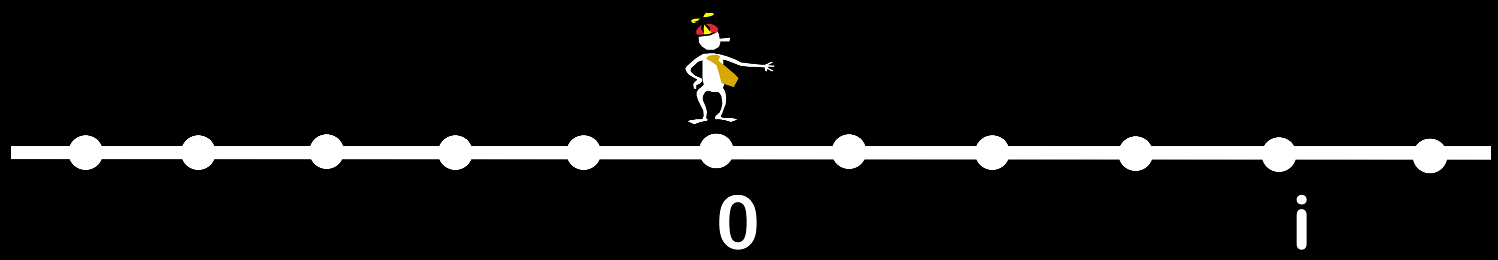
Pr[home by time $4k\, m(n-1)$]
$\geq 1 - (½)^k$

Random walks on infinite graphs

Drunk man will find way home, but drunk bird may get lost forever

- Shizuo Kakutani

Random Walk On a Line



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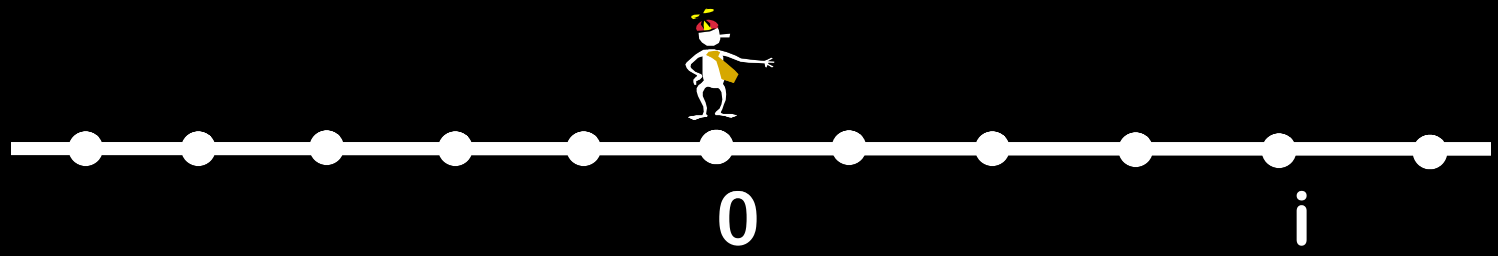
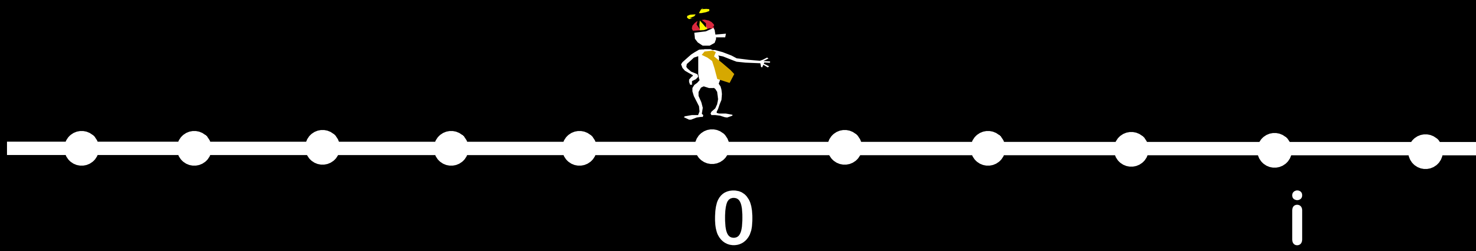


Flip an unbiased coin and go left/right

Random Walk On a Line



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Let X_t be the position at time t

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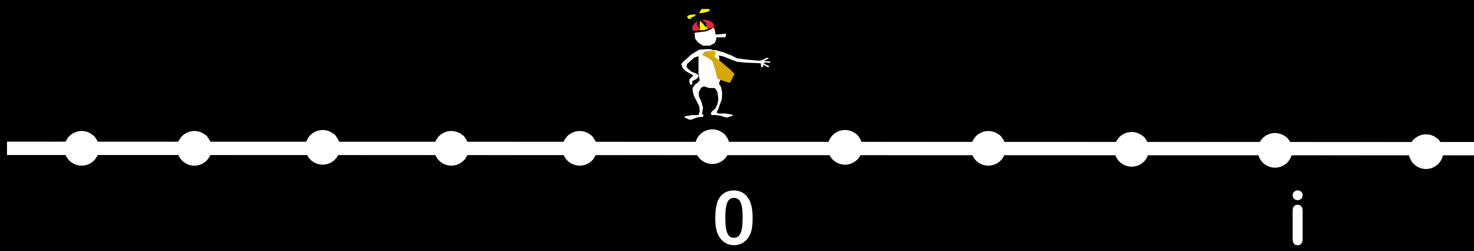

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$\Pr[\, X_t = i \,]$

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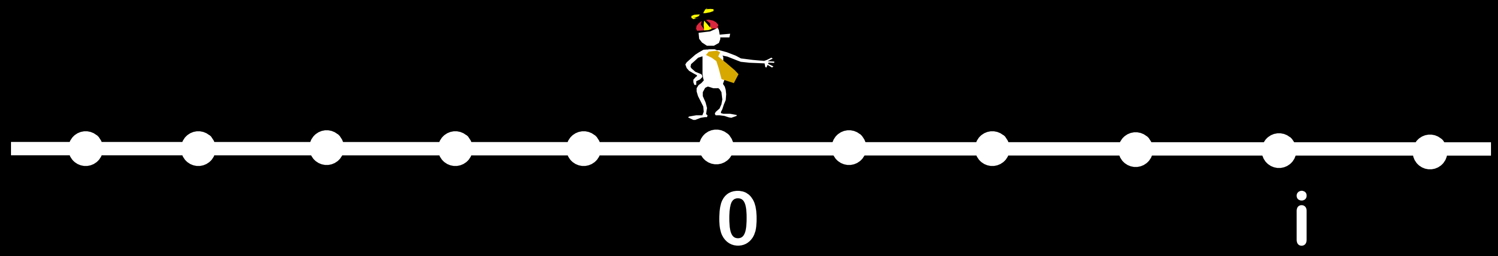

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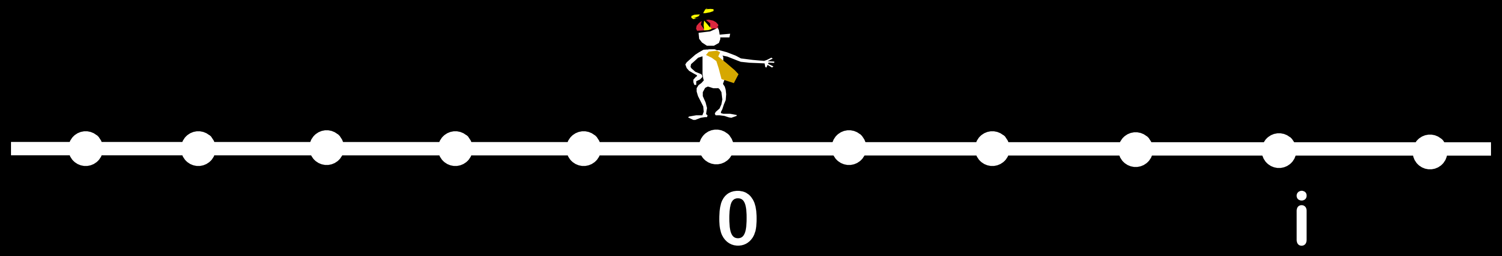

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$Pr[\ X_t = i\ ]\quad = Pr[\ \#heads - \#tails = i]$

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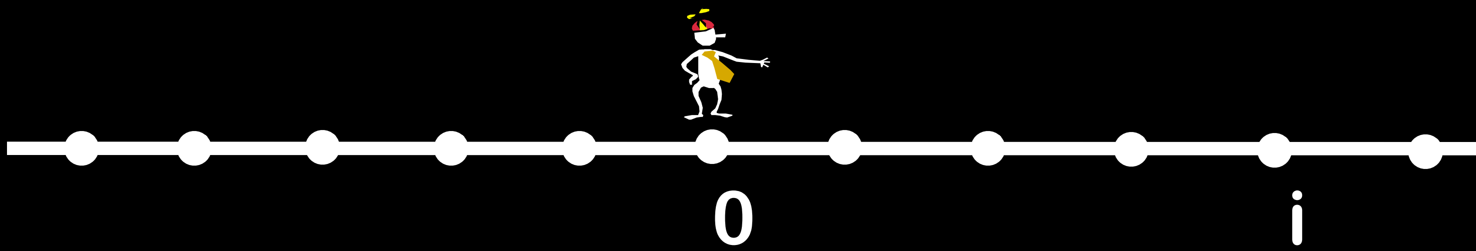

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$$= \binom{t}{(t+i)/2} / 2^t$$

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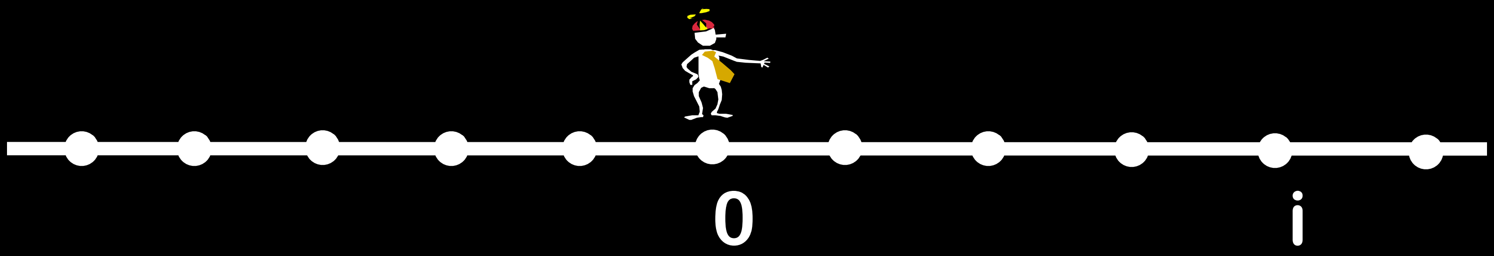

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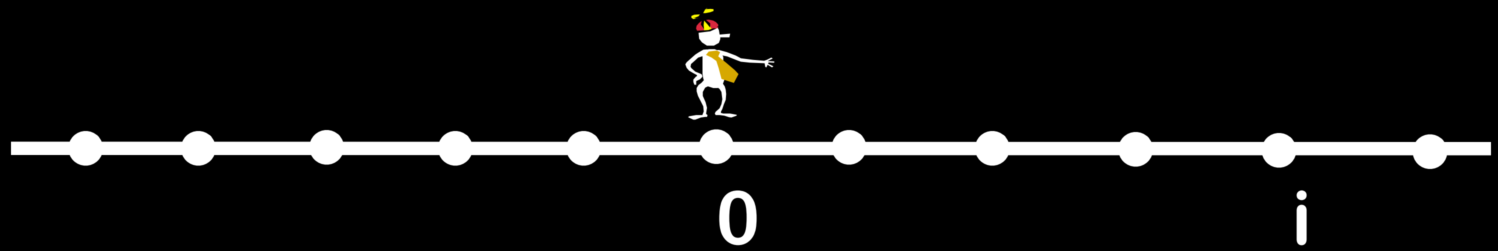

$$\Pr[\, X_{2t} = 0 \,] = \binom{2t}{t} / 2^{2t}$$

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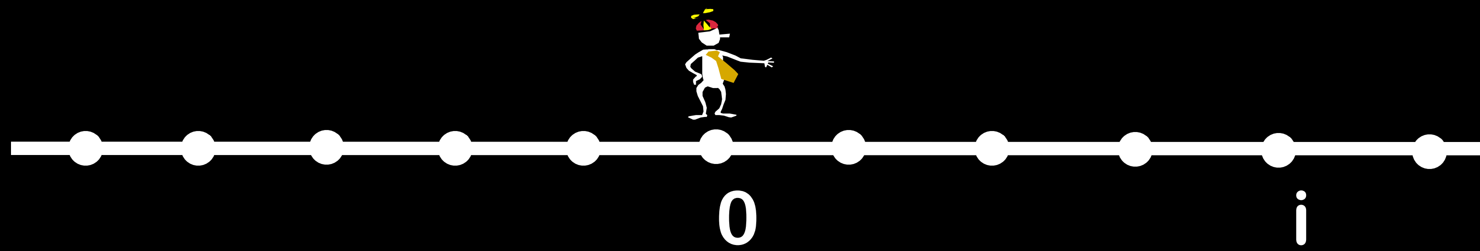

$$\Pr[\, X_{2t} = 0 \,] = \binom{2t}{t} / 2^{2t} \leq \Theta(1/\sqrt{t})$$

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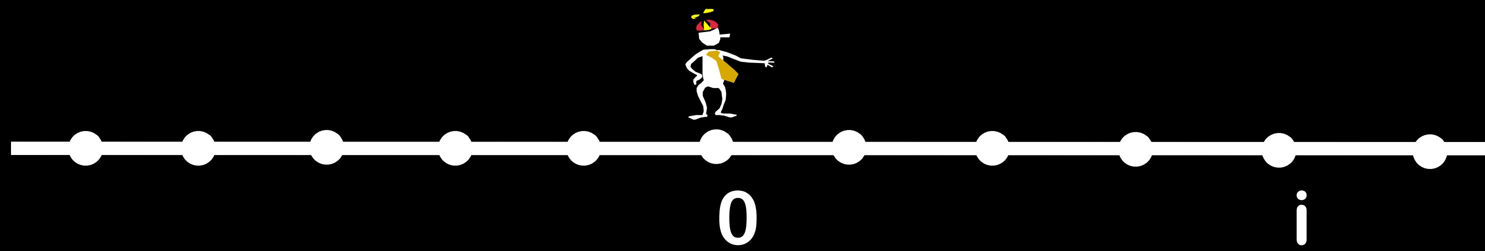

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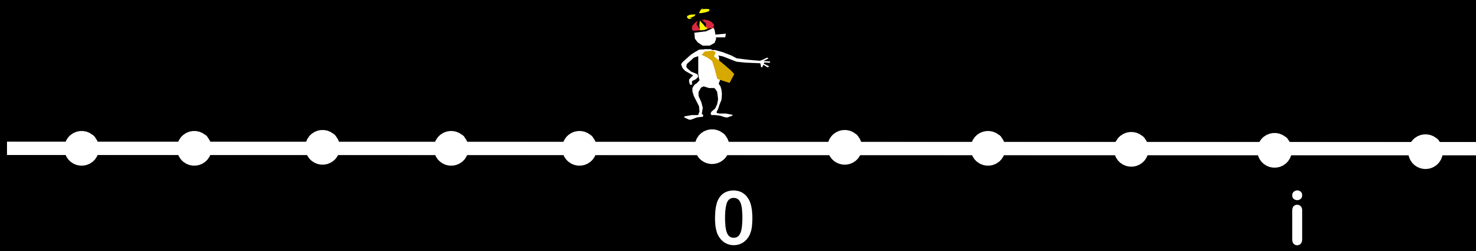

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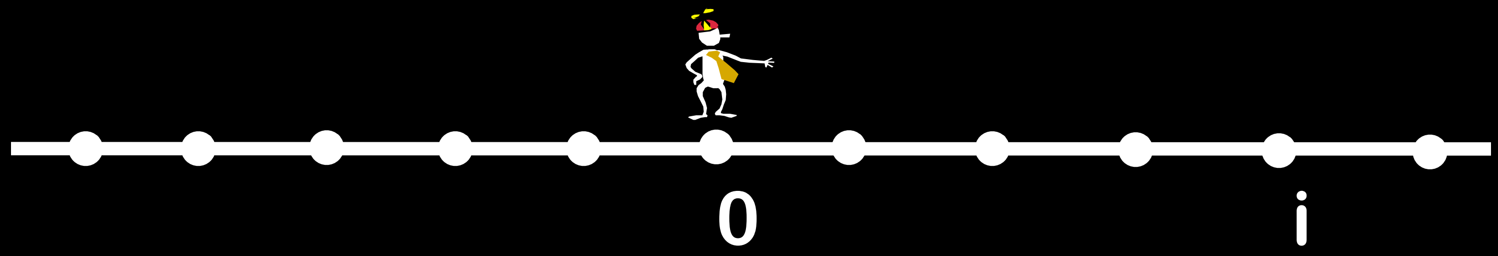

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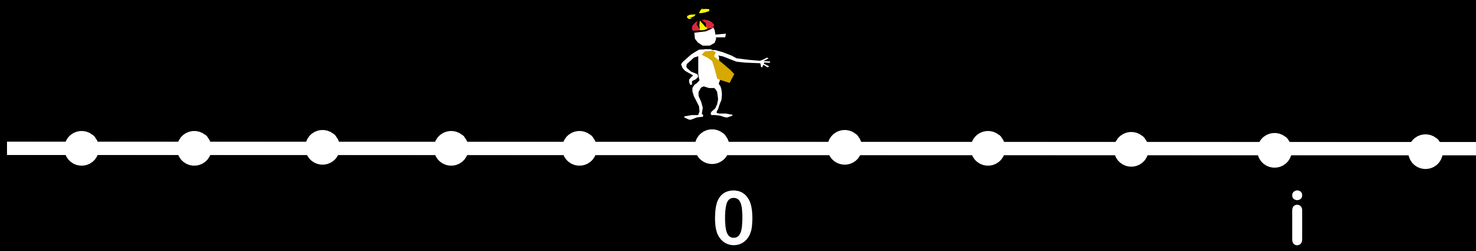

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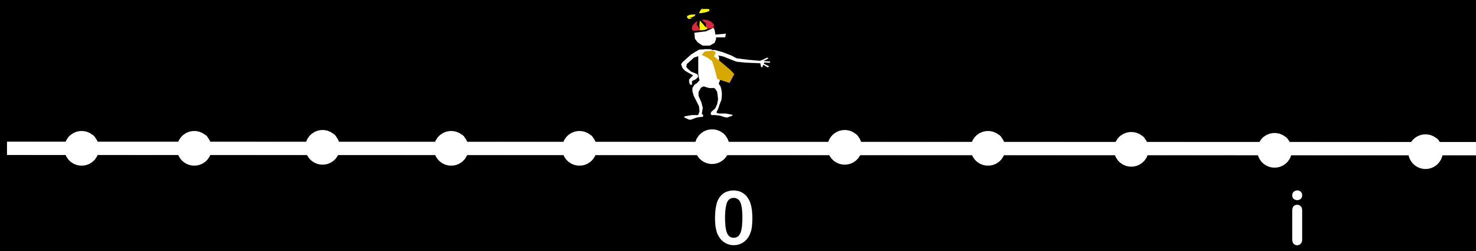

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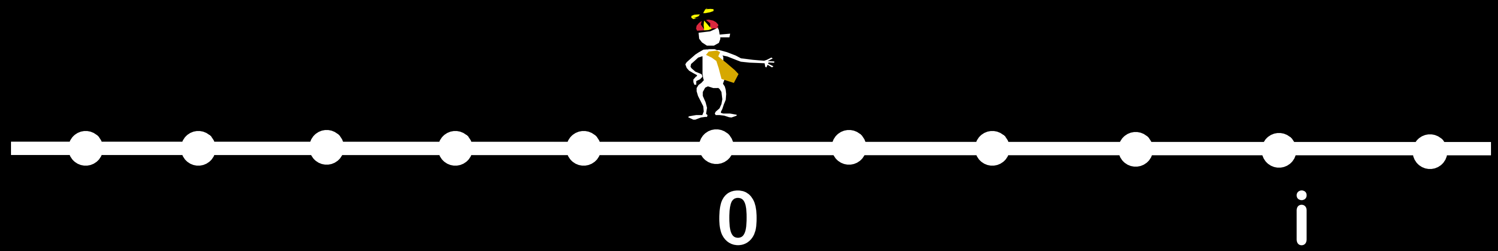

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$$E[\, Z_{2n} \,] = E[\, \textstyle\sum_{t = 1 \ldots n} Y_{2t} \,]$$

$$\le \Theta(1/\sqrt{1} + 1/\sqrt{2} + \ldots + 1/\sqrt{n})$$

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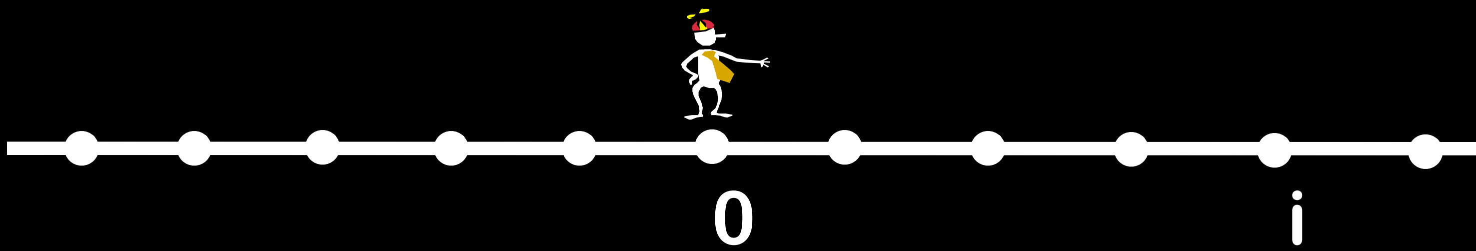

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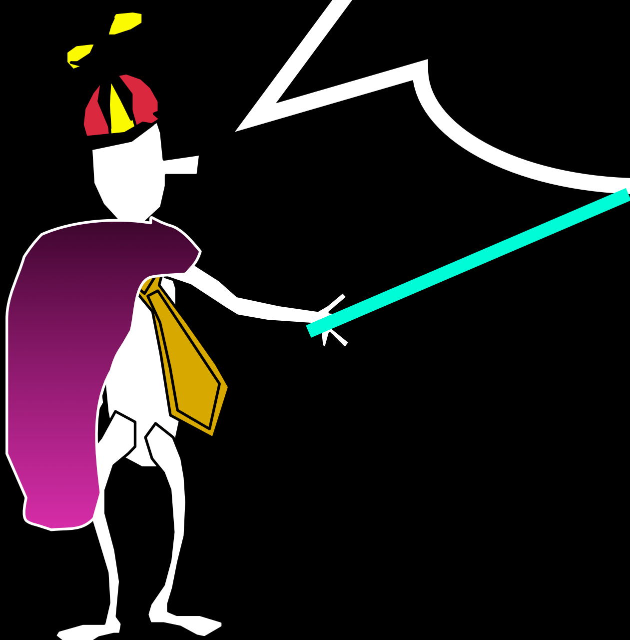
$\leq \Theta(1/\sqrt{1} + 1/\sqrt{2} + \ldots + 1/\sqrt{n}) = \Theta(\sqrt{n})$

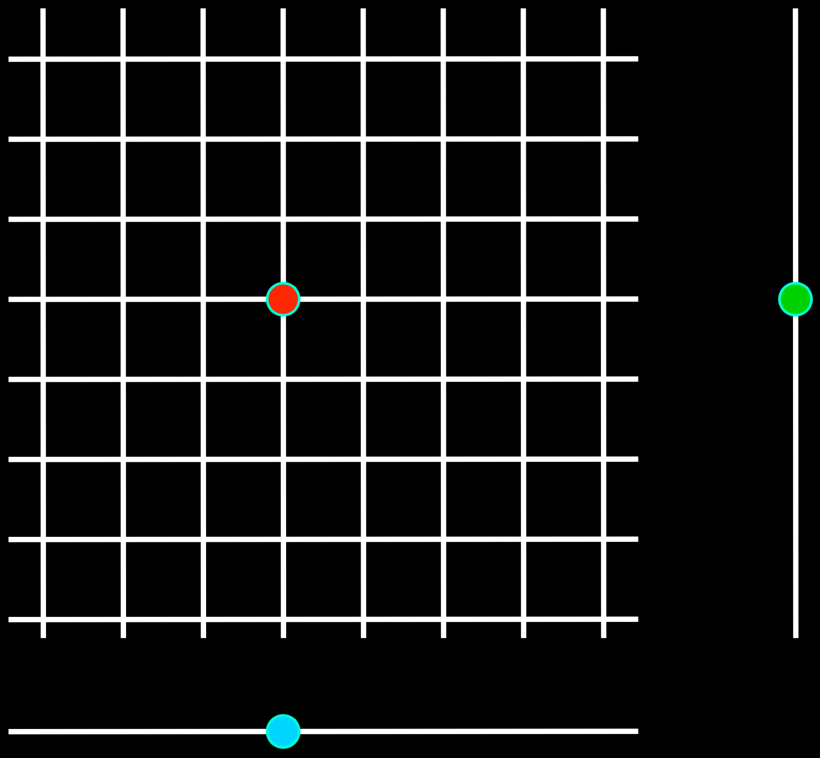


In n steps, you expect to return to the origin $\Theta(\sqrt{n})$ times!

How About a 2-d Grid?

Let us simplify our 2-d random walk:

move in both the x-direction and y-direction…



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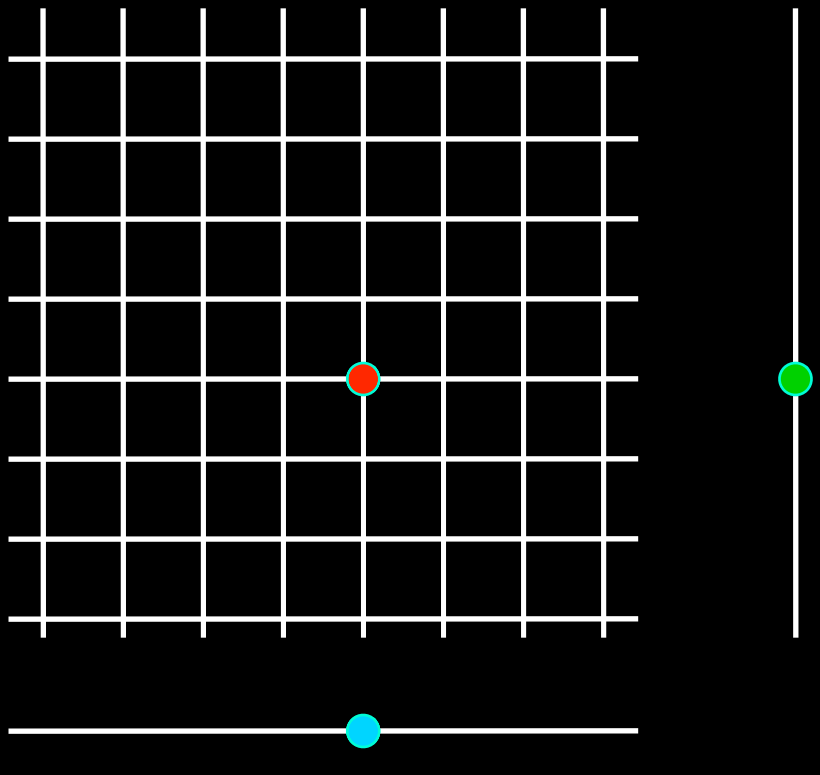

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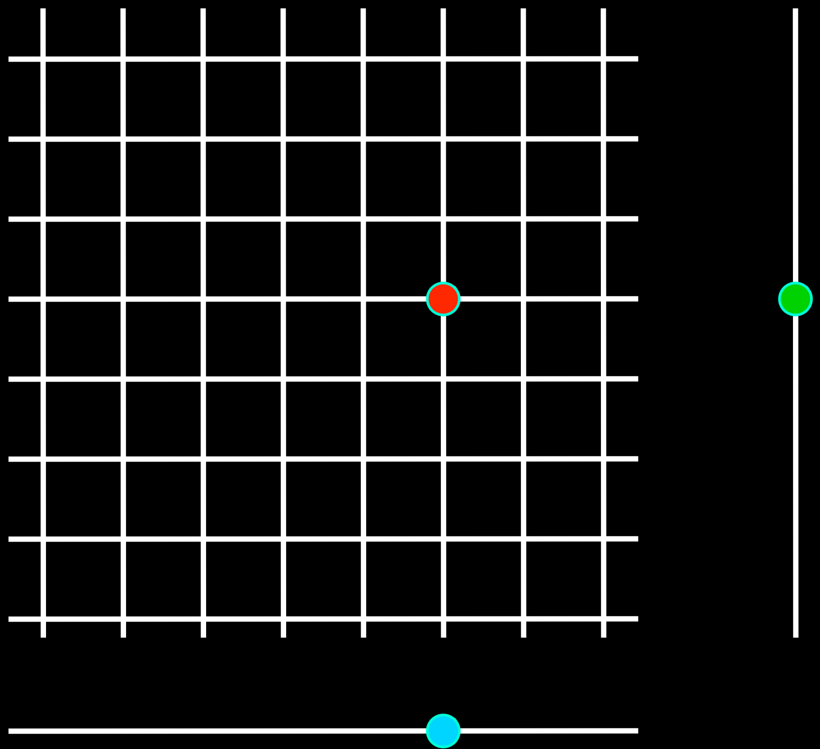

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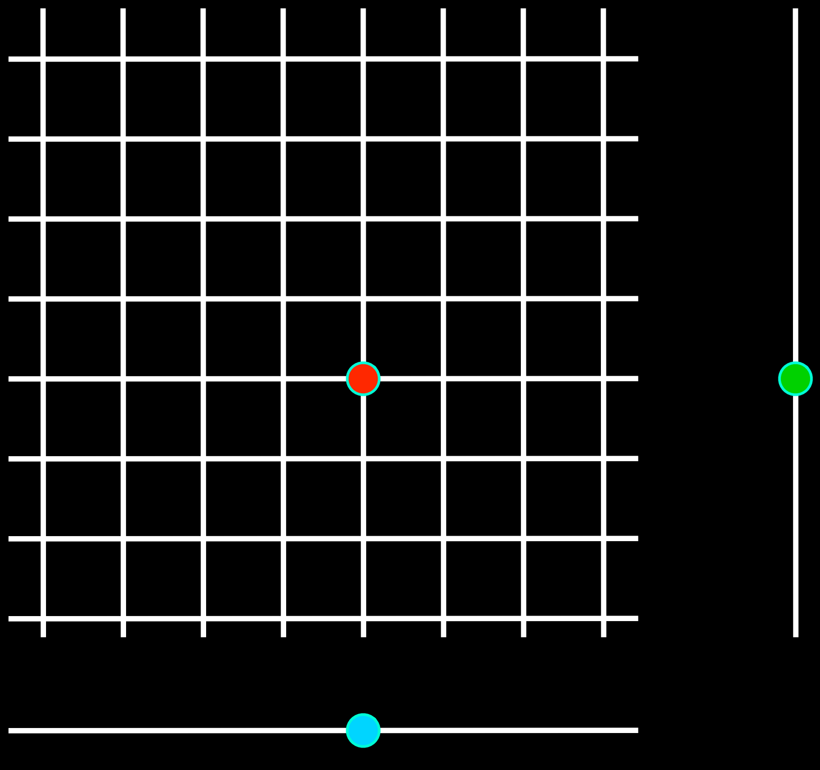

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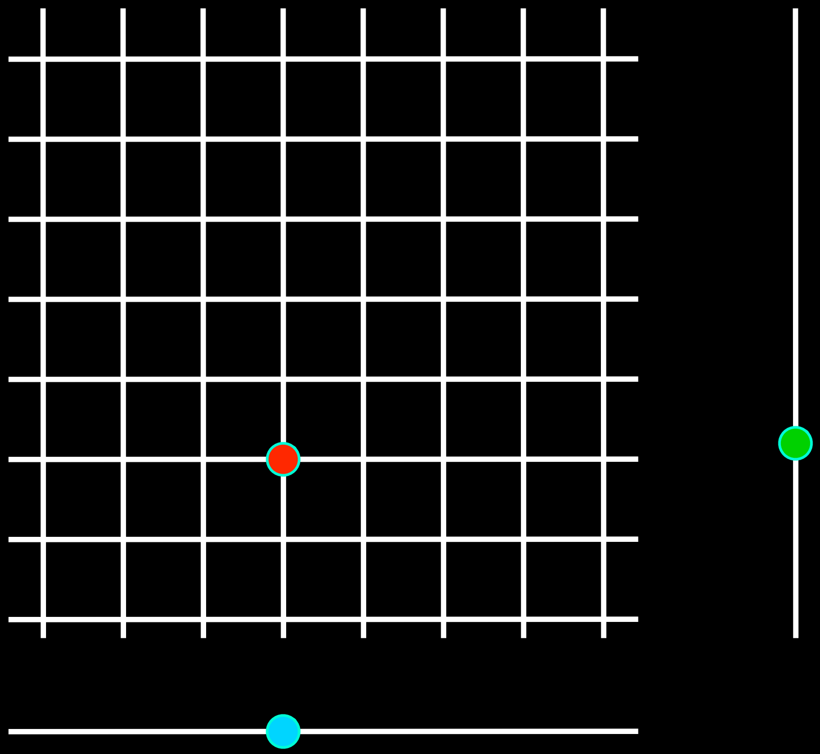

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$= \Theta(1/t)$

E[# of visits to origin by time n]
$= \Theta(1/1 + 1/2 + 1/3 + \ldots + 1/n) = \Theta(\log n)$

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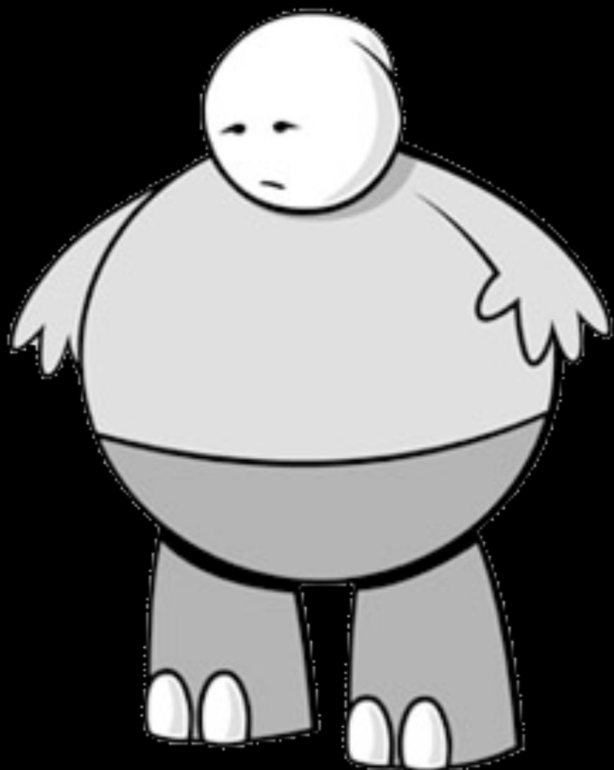
Hence Pr[never return to origin] > 1/K



Here's What You Need to Know…

Conditional expectation

Flipping coins with bias p

Expected number of flips before a heads

Random Walk on a Line

Cover Time of a Graph

Markov's Inequality