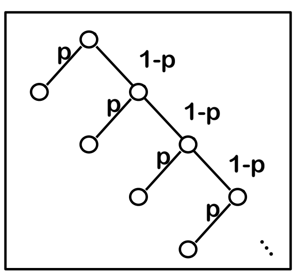
15-251

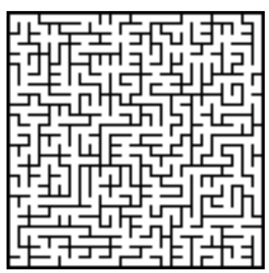
Great Theoretical Ideas in Computer Science

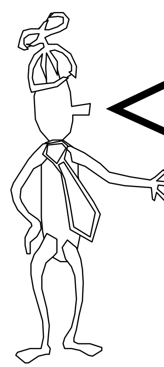
See handout for probability review and some new stuff

Infinite Sample Spaces and Random Walks

Lecture 11, September 30, 2008





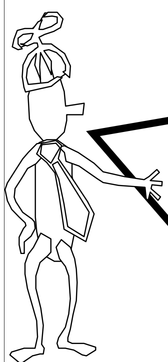


Use linearity of expectation

Suppose we have m people each with a uniformly chosen birthday from 1 to 366

X = number of pairs of people with the same birthday

E[X] = ?



X = number of pairs of people with the same birthday

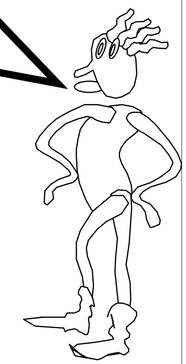
E[X] = ?

Use m(m-1)/2 indicator variables, one for each pair of people

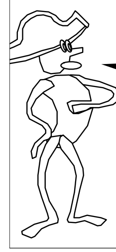
X_{jk} = 1 if person j and person k have the same birthday; else 0

$E[X_{jk}] = (1/366)\ 1 + (1 - 1/366)\ 0$
$= 1/366$

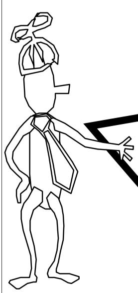
Step Right Up…



You pick a number $n \in [1..6]$. You roll 3 dice. If any match n, you win \$1. Else you pay me \$1. Want to play?



Hmm… let's see



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X_{jk} = 1 if person j and person k have the same birthday; else 0

$E[X_{jk}] = (1/366)\ 1 + (1 - 1/366)\ 0$
$= 1/366$

$$E[X] = E[\Sigma_{j \le k \le m} X_{jk}]$$
$$= \Sigma_{j \le k \le m} E[X_{jk}]$$
$$= m(m-1)/2 \times 1/366$$

Analysis

A_i = event that i-th die matches

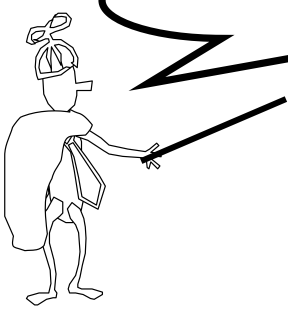
X_i = indicator RV for A_i

Expected number of dice that match:
$E[X_1+X_2+X_3] = 1/6+1/6+1/6 = ½$

But this is not the same as Pr(at least one die matches)

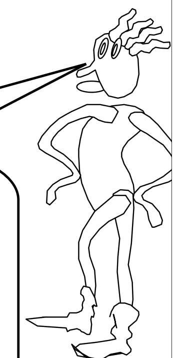
Analysis

Pr(at least one die matches)
= 1 – Pr(none match)
= 1 – $(5/6)^3$ = 0.416



But it never actually gets to 2. Is that a problem?

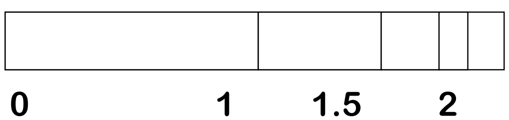
No, we really mean the limit of the partial sums.
In this case, the partial sum is $2-(\frac{1}{2})^n$ which goes to 2.

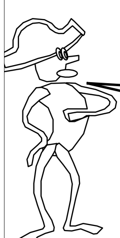


An easy question

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = ?$$

Answer: 2





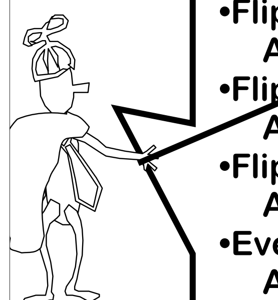
But it never actually gets to 2. Is that a problem?

A related question

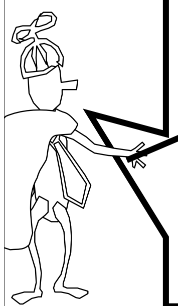
Suppose I flip a coin of bias p, stopping when I first get heads.

What's the chance that I:

- Flip exactly once?
  Ans: p
- Flip exactly two times?
  Ans: (1-p)p
- Flip exactly k times?
  Ans: $(1-p)^{k-1}p$
- Eventually stop?
  Ans: 1. (assuming p>0)



A related question



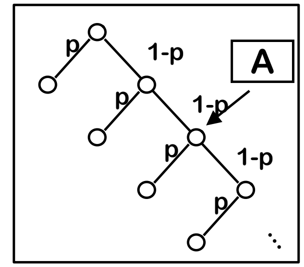
Pr(flip once) + Pr(flip 2 times) + Pr(flip 3 times) + ... = 1:

$p + (1-p)p + (1-p)^2p + (1-p)^3p + \ldots = 1.$

Or, using q = 1-p,

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q} = \frac{1}{p}$$

Reason about expectations too!



Suppose A is a node in this tree

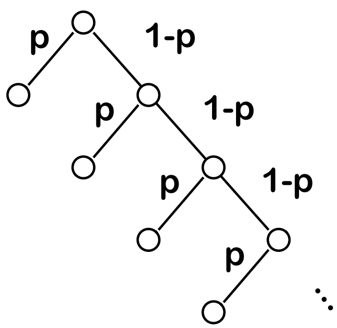
Pr(x|A)=product of edges on path from A to x.

E[X] = $\sum_x$ Pr(x)X(x).

E[X|A] = $\sum_{x \text{ in } A}$ Pr(x|A)X(x).

I.e., it is as if we started the game at A.

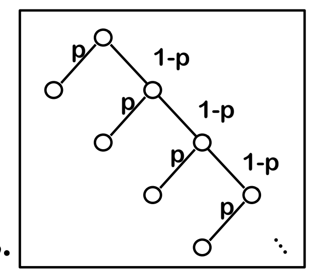
Pictorial view



Sample space S = leaves in this tree.
Pr(x) = product of edges on path to x.

If p>0, Pr(not halting by time n) goes to zero.

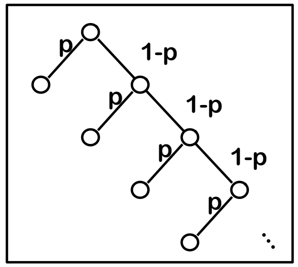
Expected number of heads



Flip bias-p coin until heads.

What is expected number of flips?

Expected number of heads



Let X = # flips.

B = event "1st flip is heads"

E[X] = E[X|B] × Pr(B) + E[X| not B] × Pr(not B)

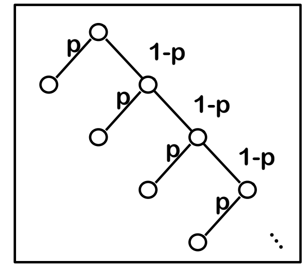
= 1 × p + (1 + E[X]) × (1-p).

Solving: p × E[X] = p + (1-p)

$\Rightarrow$ E[X] = 1/p.

General picture

Let sample space S be leaves of a choice tree.



Let S_n = {leaves at depth $\leq$ n}.

For event A, let $A_n = A \cap S_n$.

If $\lim_n Pr(S_n)=1$, can define:

$Pr(A)=\lim_n Pr(A_n)$.

Infinite Probability spaces

Notice we are using infinite probability spaces here, but we really only defined things for <u>finite</u> spaces so far.

Infinite probability spaces can sometimes be weird.

Luckily, in CS we will almost always be looking at spaces that can be viewed as choice trees where
Pr(haven't halted by time t) goes to 0 as t gets large

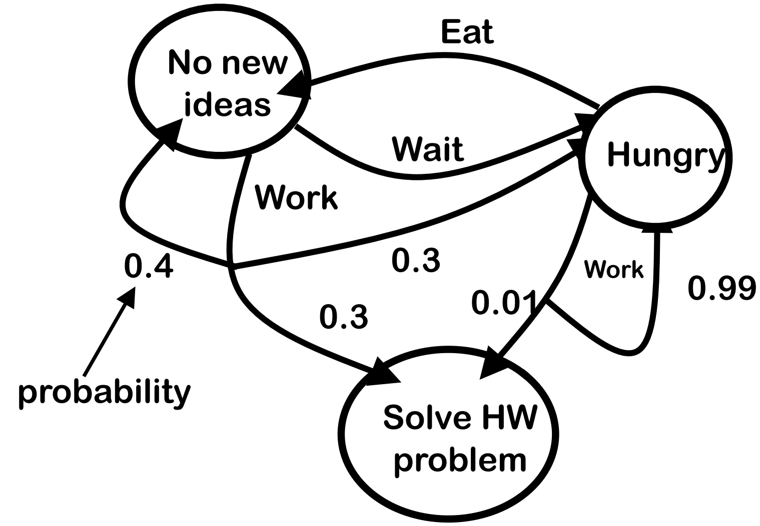
Setting that doesn't fit our model

Event: "Flip coin until #heads > 2*#tails."

There's a reasonable chance this will <u>never</u> stop...

How to walk home drunk

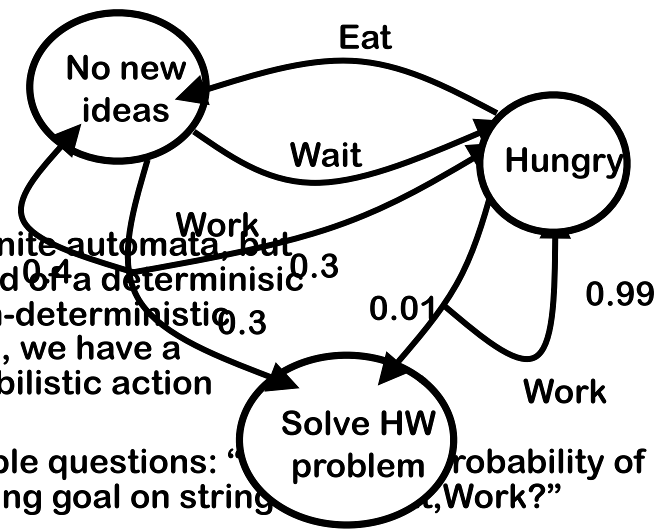
Abstraction of Student Life



Drunk man will find way home, but drunk bird may get lost forever

- Shizuo Kakutani

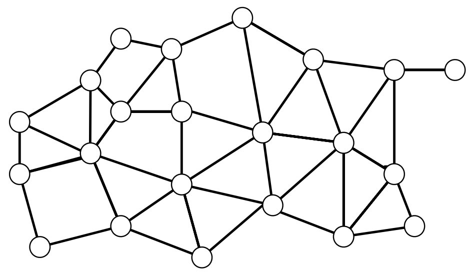
Abstraction of Student Life



Like finite automata, but instead of a determinisic or non-deterministic action, we have a probabilistic action

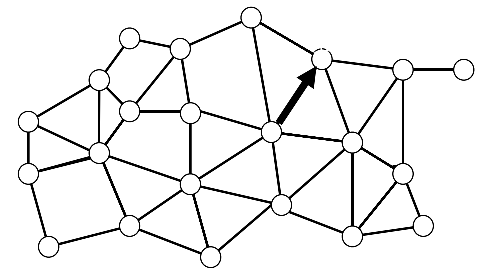
Example questions: “ robability of reaching goal on string ,Work?”

Simpler:
Random Walks on Graphs



At any node, go to one of the neighbors of the node with equal probability

Simpler:
Random Walks on Graphs



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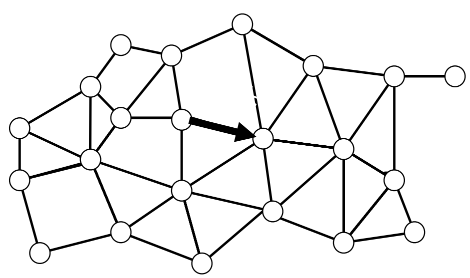

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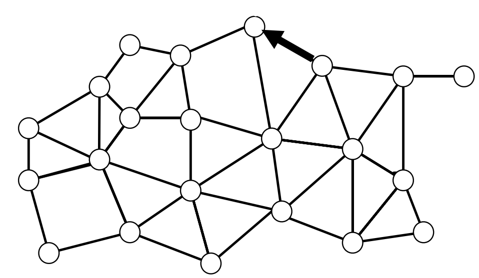

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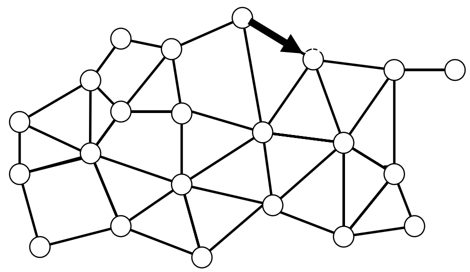
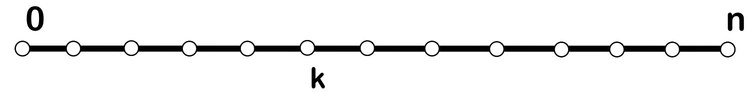


At any node, go to one of the neighbors of the node with equal probability

Random Walk on a Line

You go into a casino with $k, and at each time step, you bet $1 on a fair game

You leave when you are broke or have $n



$X_t = k + \delta_1 + \delta_2 + \ldots + \delta_t,$

(δ_i is RV for change in your money at time i)

$E[\delta_i] = 0$

So, $E[X_t] = k$

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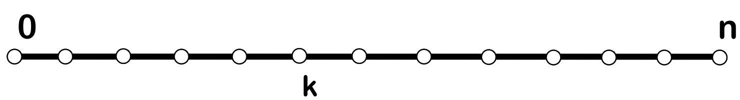

Question 1: what is your expected amount of money at time t?

Let X_t be a R.V. for the amount of $$$ at time t

Random Walk on a Line

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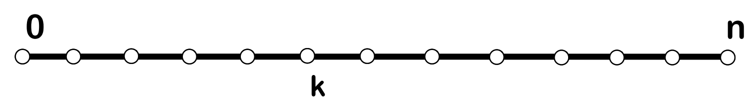


Question 2: what is the probability that you leave with $n?

Random Walk on a Line

Question 2: what is the probability that you leave with $n?

$E[X_t] = k$

$E[X_t] = E[X_t | X_t = 0] \times Pr(X_t = 0)$

$\quad + E[X_t | X_t = n] \times Pr(X_t = n)$

$\quad + E[X_t | \text{neither}] \times Pr(\text{neither})$

$k = n \times Pr(X_t = n)$

$\quad + (\text{something}_t) \times Pr(\text{neither})$

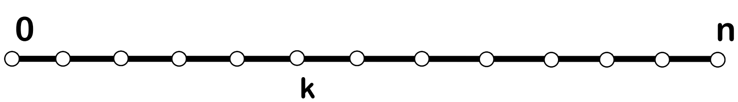
As $t \to \infty$, $Pr(\text{neither}) \to 0$, also $\text{something}_t < n$

Hence $Pr(X_t = n) \to k/n$

Another Way To Look At It

You go into a casino with $k, and at each time step, you bet $1 on a fair game

You leave when you are broke or have $n

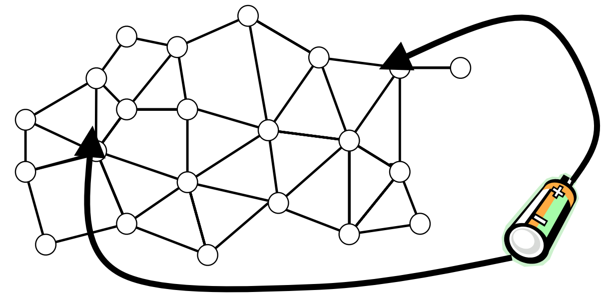


Question 2: what is the probability that you leave with $n?

= probability that I hit green before I hit red

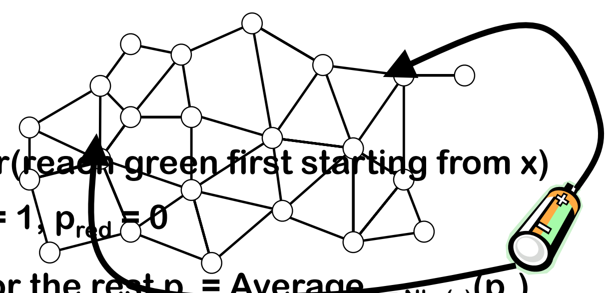
Random Walks and Electrical Networks

What is chance I reach green before red?



Same as voltage if edges are resistors and we put 1-volt battery between green and red

Random Walks and Electrical Networks



p_x = Pr(reach green first starting from x)

$p_{green} = 1$, $p_{red} = 0$

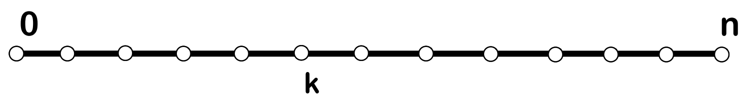
And for the rest $p_x = \text{Average}_{y \in Nbr(x)}(p_y)$

Same as equations for voltage if edges all have same resistance!

Another Way To Look At It

You go into a casino with \$k, and at each time step, you bet \$1 on a fair game

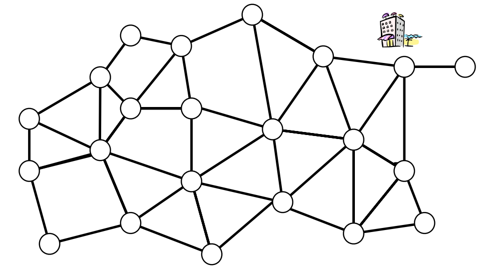
You leave when you are broke or have \$n



Question 2: what is the probability that you leave with \$n?

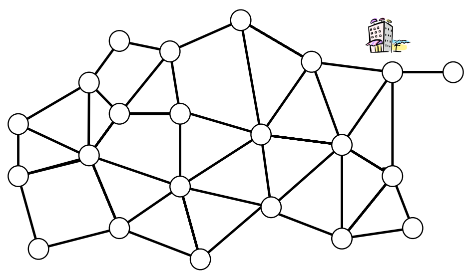
voltage(k) = k/n
= Pr[hitting n before 0 starting at k] !!!

Getting Back Home



How about walking randomly?

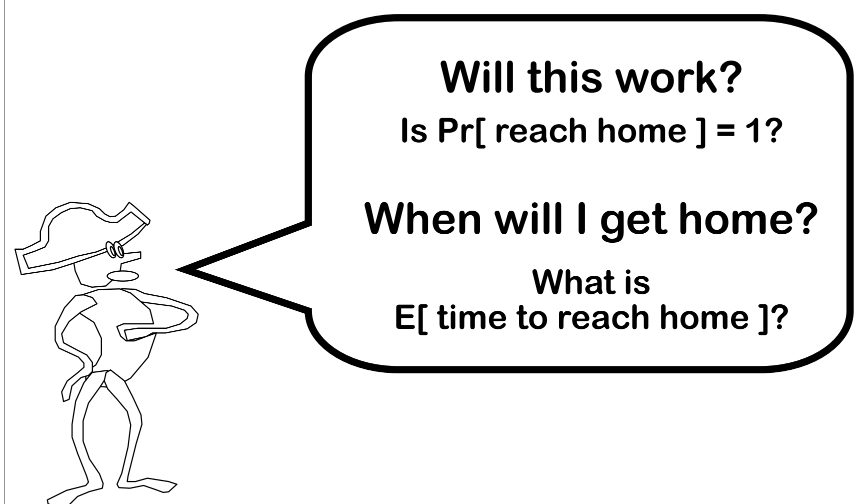
Getting Back Home

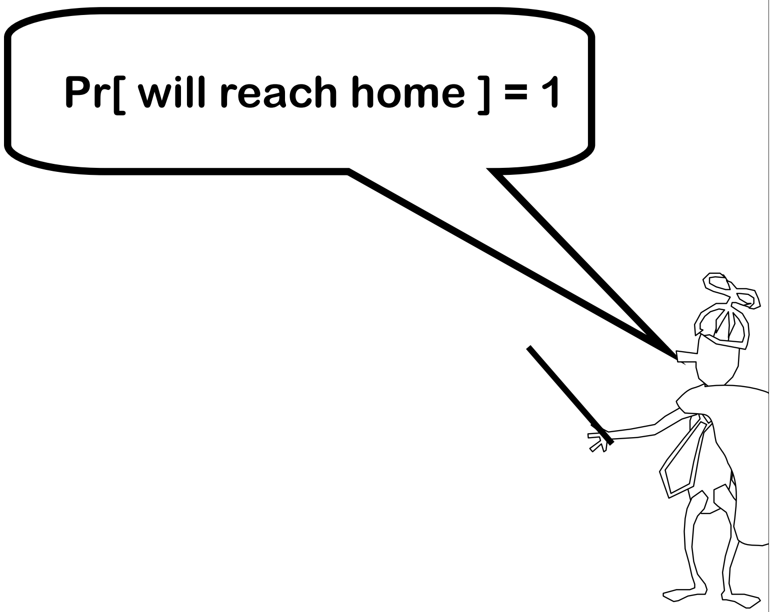


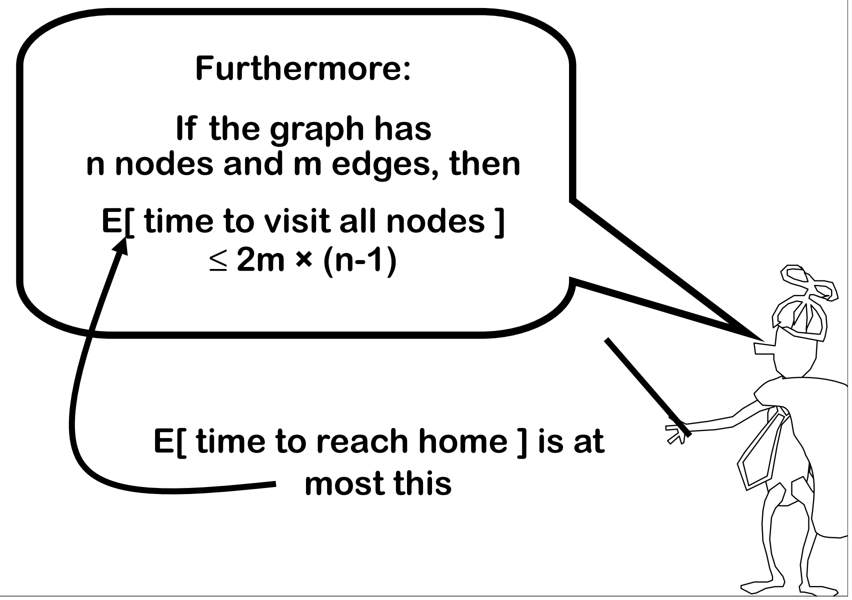
Lost in a city, you want to get back to your hotel
How should you do this?

Depth First Search!

Requires a good memory and a piece of chalk







We Will Eventually Get Home

Look at the first n steps

There is a non-zero chance p_1 that we get home

Also, $p_1 \geq (1/n)^n$

Suppose we fail

Then, wherever we are, there is a chance $p_2 \geq (1/n)^n$ that we hit home in the next n steps from there

Probability of failing to reach home by time kn

$= (1 – p_1)(1 – p_2) \ldots (1 – p_k) \to 0$ as $k \to \infty$

Cover Times

Cover time (from u)

C_u = E [time to visit all vertices | start at u]

Cover time of the graph

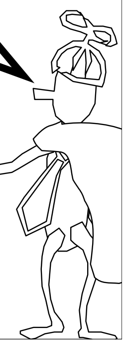
$C(G) = \max_u \{ C_u \}$

(worst case expected time to see all vertices)

Cover Time Theorem

If the graph G has n nodes and m edges, then the cover time of G is

$C(G) \leq 2m(n-1)$



Any graph on n vertices has < $n^2/2$ edges

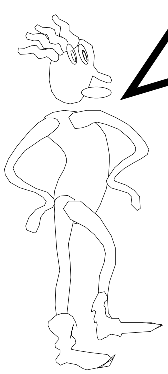
Hence $C(G) < n^3$ for all graphs G

A Simple Calculation

True or False:

If the average income of people is $100 then more than 50% of the people can be earning more than $200 each

False! else the average would be higher!!!



Actually, we get home pretty fast…

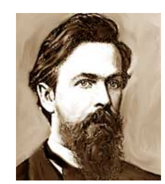
Chance that we don't hit home by $(2k)2m(n-1)$ steps is $(\frac{1}{2})^k$

Markov's Inequality

If X is a non-negative r.v. with mean E[X], then

$\Pr[X > 2E[X]] \leq \frac{1}{2}$

$\Pr[X > k E[X]] \leq 1/k$



Andrei A. Markov

Markov's Inequality

Non-neg random variable X has expectation
A = E[X]

A = E[X] = E[X | X > 2A] Pr[X > 2A]
+ E[X | X ≤ 2A] Pr[X ≤ 2A]

≥ E[X | X > 2A] Pr[X > 2A] (since X is non-neg)

Also, E[X | X > 2A] > 2A

⇒ A ≥ 2A × Pr[X > 2A]

⇒ ½ ≥ Pr[X > 2A]

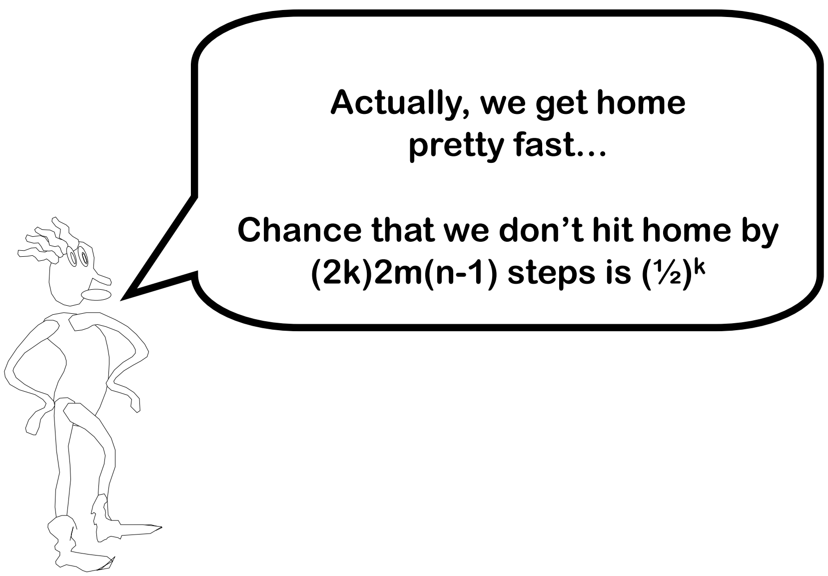
Pr[X > k × expectation] ≤ 1/k

An Averaging Argument

Suppose I start at u

E[time to hit all vertices | start at u] ≤ C(G)

Hence, by Markov's Inequality:
Pr[time to hit all vertices > 2C(G) | start at u] ≤ ½



So Let's Walk Some More!

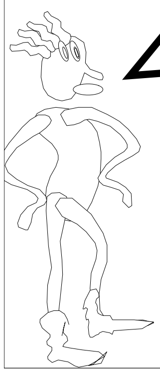
Pr [time to hit all vertices > 2C(G) | start at u] ≤ ½

Suppose at time 2C(G), I'm at some node
with more nodes still to visit

Pr [haven't hit all vertices in 2C(G) more time
| start at v] ≤ ½

Chance that you failed both times ≤ ¼ = $(½)^2$

Hence,
Pr[havent hit everyone in time k × 2C(G)] ≤ $(½)^k$



Hence, if we know that

Expected Cover Time
$C(G) < 2m(n-1)$

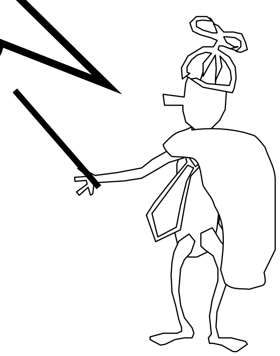
then

Pr[home by time $4k\ m(n-1)$]
$\geq 1 - (½)^k$

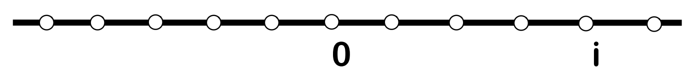
Drunk man will find way home, but drunk bird may get lost forever

- Shizuo Kakutani

Random walks
on infinite graphs



Random Walk On a Line



Flip an unbiased coin and go left/right

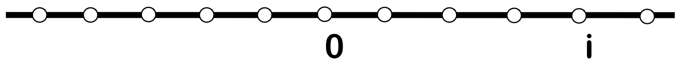
Let X_t be the position at time t

$$\Pr[X_t = i] = \Pr[\#\text{heads} - \#\text{tails} = i]$$

$$= \Pr[\#\text{heads} - (t - \#\text{heads}) = i]$$

$$= \binom{t}{(t+i)/2} / 2^t$$

Random Walk On a Line



$\Pr[\, X_{2t} = 0 \,] = \binom{2t}{t} / 2^{2t} \le \Theta(1/\sqrt{t})$ Stirling's approx

Y_{2t} = indicator for $(X_{2t} = 0) \Rightarrow E[\, Y_{2t} \,] = \Theta(1/\sqrt{t})$

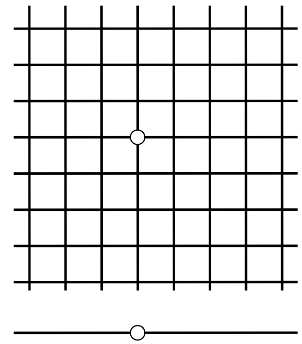
Z_{2n} = number of visits to origin in 2n steps

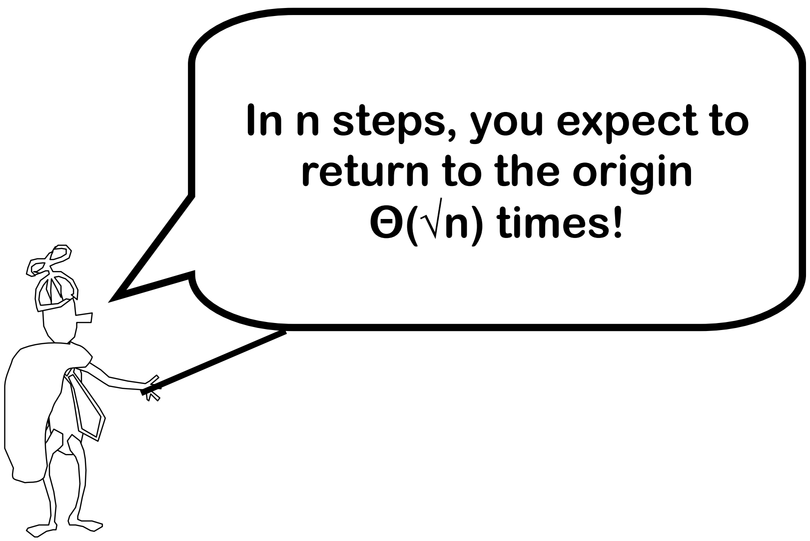
$E[\, Z_{2n} \,] = E[\, \sum_{t = 1\ldots n} Y_{2t} \,]$

$\le \Theta(1/\sqrt{1} + 1/\sqrt{2} + \ldots + 1/\sqrt{n}) = \Theta(\sqrt{n})$

How About a 2-d Grid?

Let us simplify our 2-d random walk:
move in both the x-direction and y-direction…





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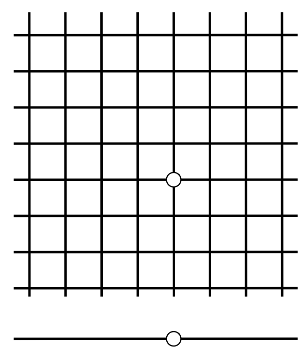

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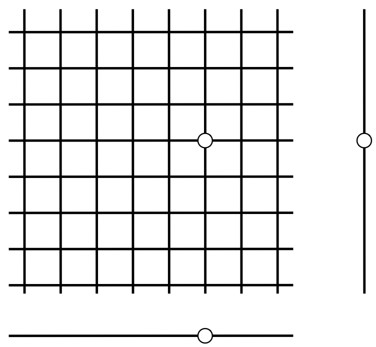

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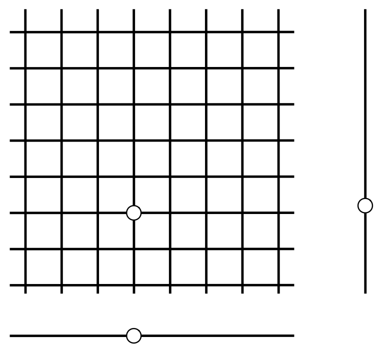

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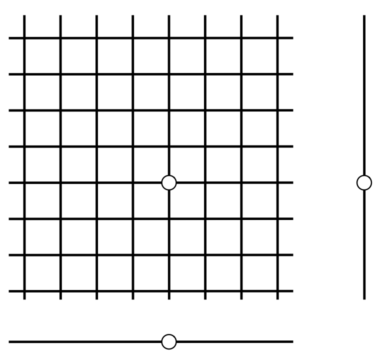


In The 2-d Walk

Returning to the origin in the grid
$\Leftrightarrow$ both "line" random walks return to their origins

Pr[visit origin at time t] $= \Theta(1/\sqrt{t}) \times \Theta(1/\sqrt{t})$
$= \Theta(1/t)$

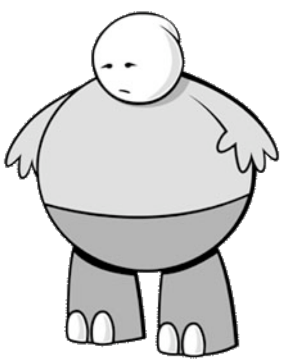
E[# of visits to origin by time n]
$= \Theta(1/1 + 1/2 + 1/3 + \ldots + 1/n) = \Theta(\log n)$

But In 3D

Pr[visit origin at time t] = $\Theta(1/\sqrt{t})^3 = \Theta(1/t^{3/2})$

$\lim_{n \to \infty}$ E[# of visits by time n] < K (constant)

Hence Pr[never return to origin] > 1/K



Here's What You Need to Know…

Conditional expectation

Flipping coins with bias p
Expected number of flips before a heads

Random Walk on a Line

Cover Time of a Graph

Markov's Inequality