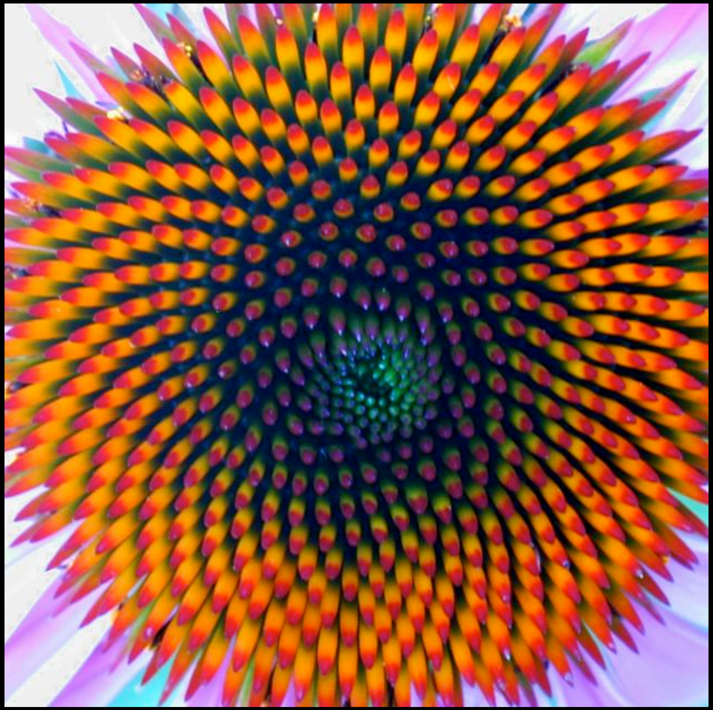
15-251

Great Theoretical Ideas in Computer Science

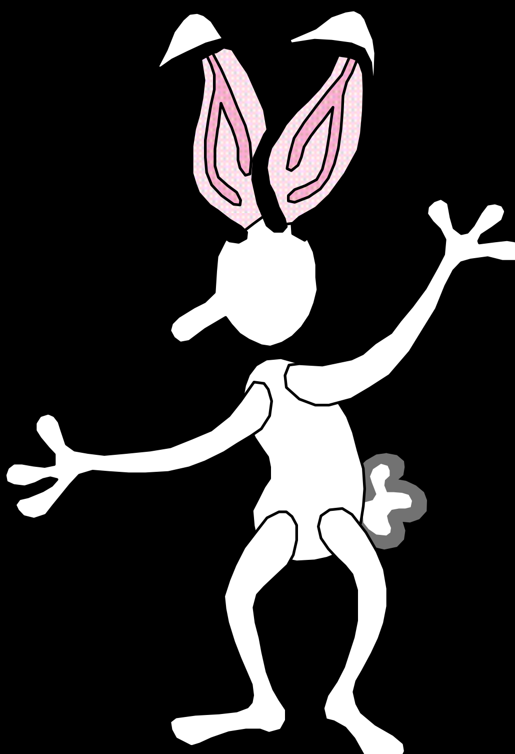
Recurrences, Fibonacci Numbers and Continued Fractions

Lecture 9, September 23, 2008



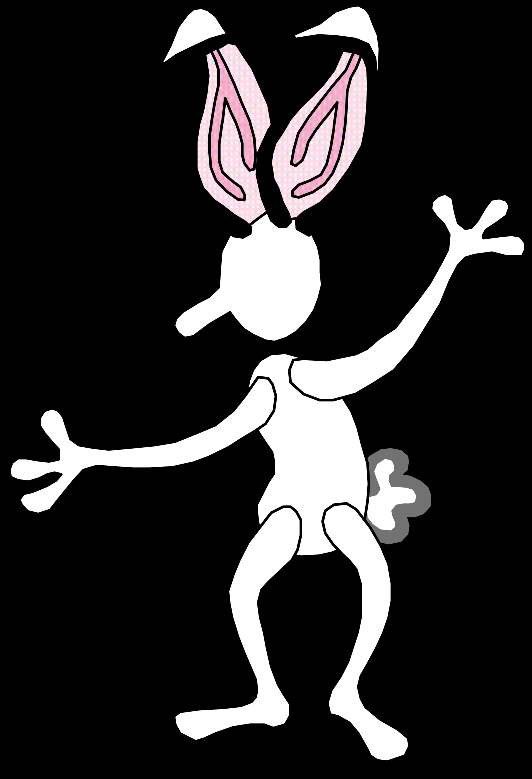

Happy Autumnal Equinox
http://apod.nasa.gov/apod/ap080922.html

Leonardo Fibonacci



Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations



Rabbit Reproduction

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month	1	2	3	4	5	6	7
rabbits							

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month	1	2	3	4	5	6	7
rabbits	1						

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rabbits	1	1	2				

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rabbits	1	1	2	3			

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rabbits	1	1	2	3	5		

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rabbits	1	1	2	3	5	8	

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rabbits	1	1	2	3	5	8	13

Fibonacci Numbers

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Stage 0, Initial Condition, or Base Case:
Fib(1) = 1; Fib (2) = 1

Fibonacci Numbers

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Inductive Rule:
For n>3, Fib(n) =

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Inductive Rule:
For n>3, Fib(n) = Fib(n-1) + Fib(n-2)

Sequences That Sum To n

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Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

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$f_1 = 1$ 0 = the empty sum

$f_2 = 1$ 1 = 1

$f_3 = 2$ 2 = 1 + 1

2

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

4 =

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

4 = 2 + 2

2 + 1 + 1

1 + 2 + 1

1 + 1 + 2

1 + 1 + 1 + 1

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# of sequences beginning with a 1

# of sequences beginning with a 2

Fibonacci Numbers Again

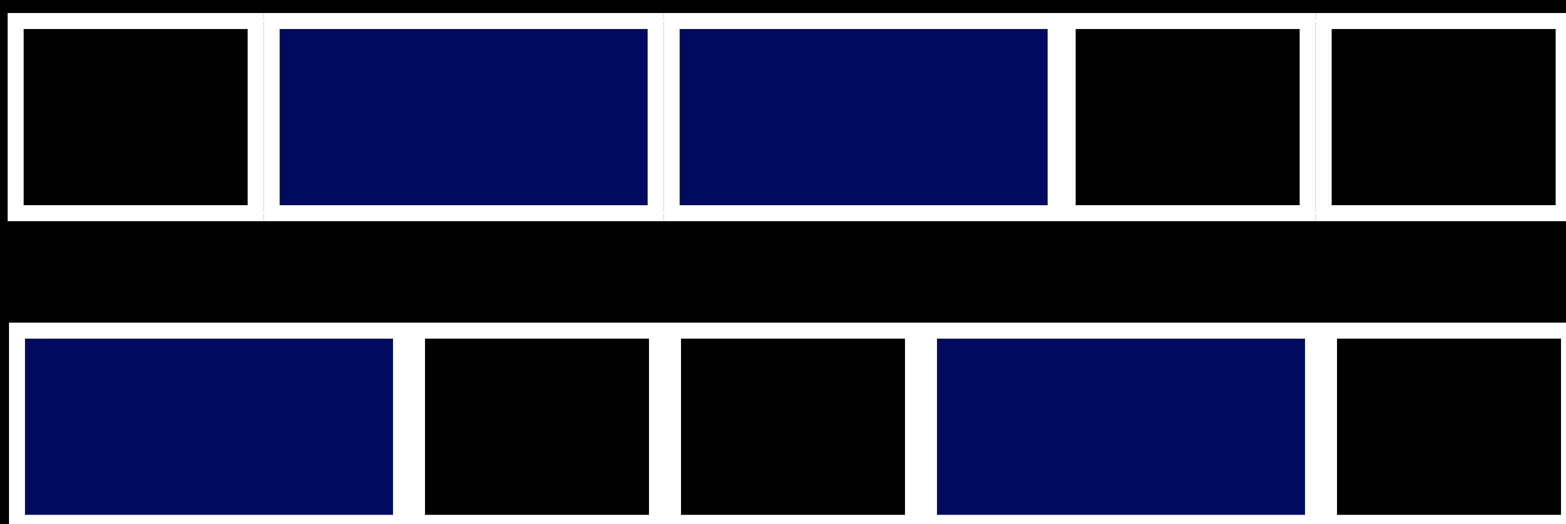
Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

$$f_1 = 1 \qquad f_2 = 1$$

Visual Representation: Tiling

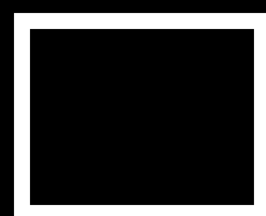
Let f_{n+1} be the number of different ways to tile a 1 × n strip with squares and dominoes.



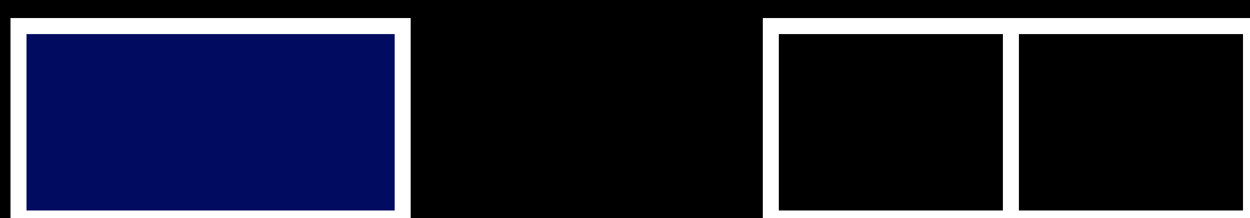
Visual Representation: Tiling

1 way to tile a strip of length 0

1 way to tile a strip of length 1:

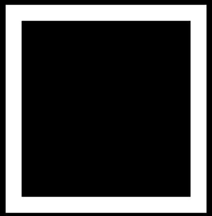


2 ways to tile a strip of length 2:

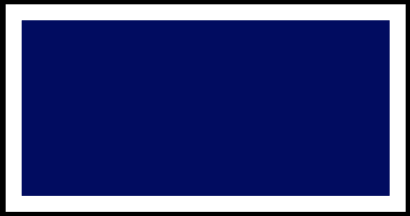


$$f_{n+1} = f_n + f_{n-1}$$

f_{n+1} is number of ways to tile length n.



f_n tilings that start with a square.



f_{n-1} tilings that start with a domino.

Fibonacci Identities

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Some examples:

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$$F_{2n} = F_1 + F_3 + F_5 + \ldots + F_{2n-1}$$

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$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$

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Fibonacci Identities

Some examples:

$$F_{2n} = F_1 + F_3 + F_5 + \ldots + F_{2n-1}$$

$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$

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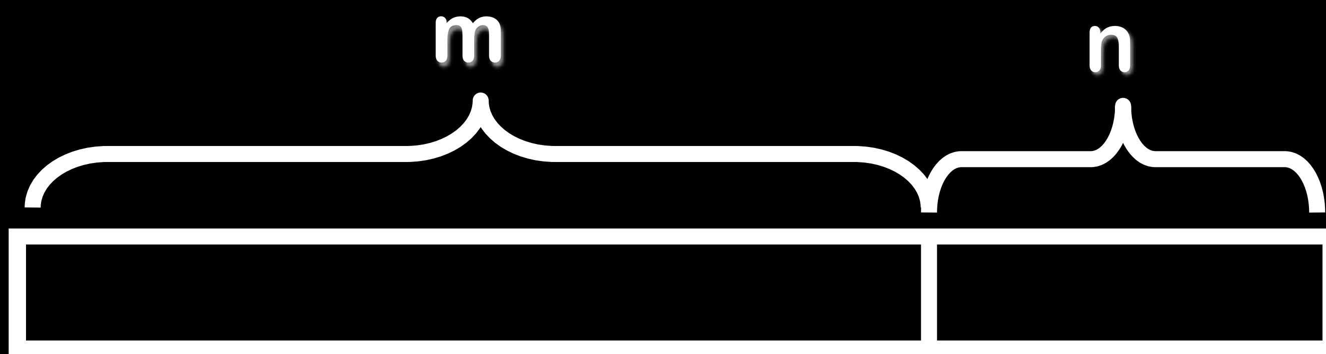

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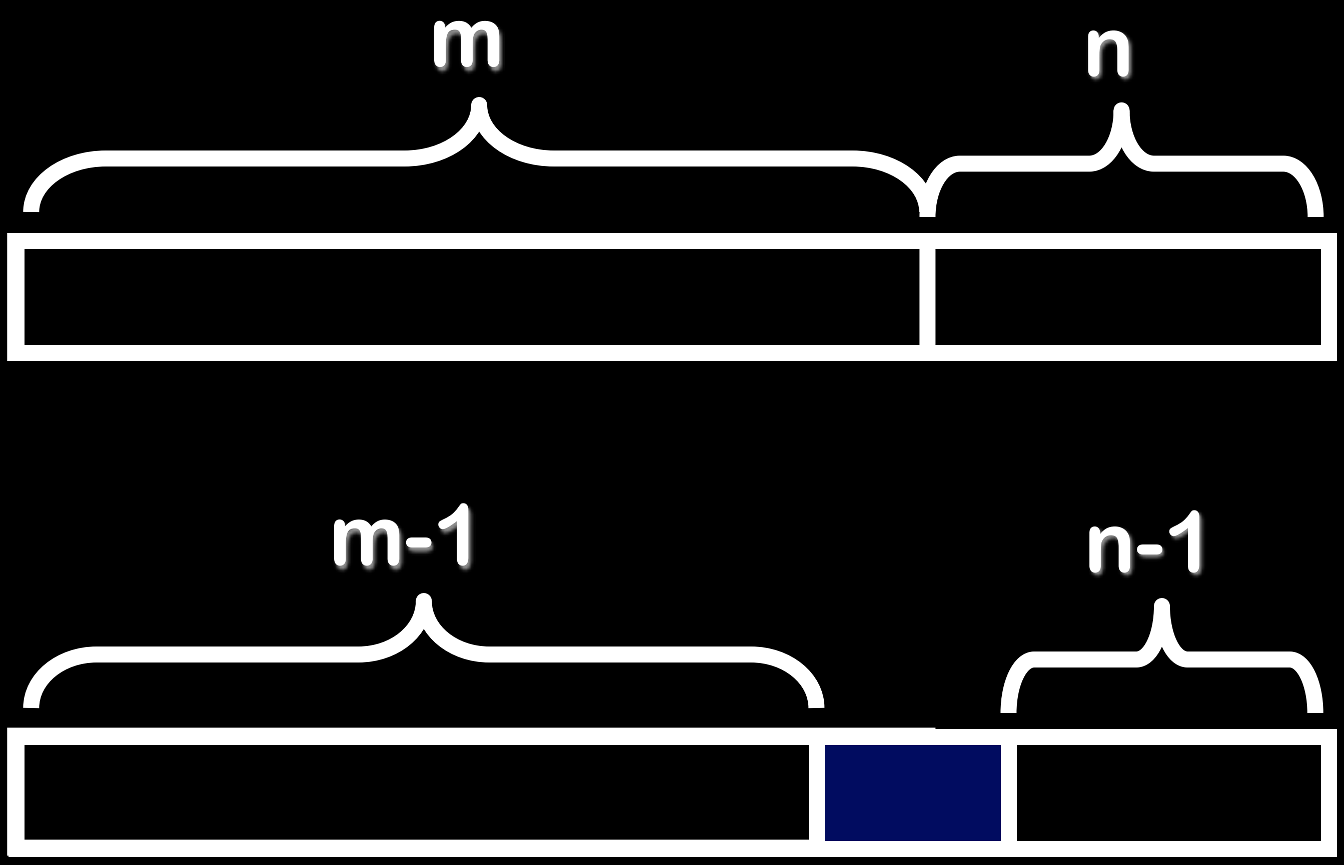

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

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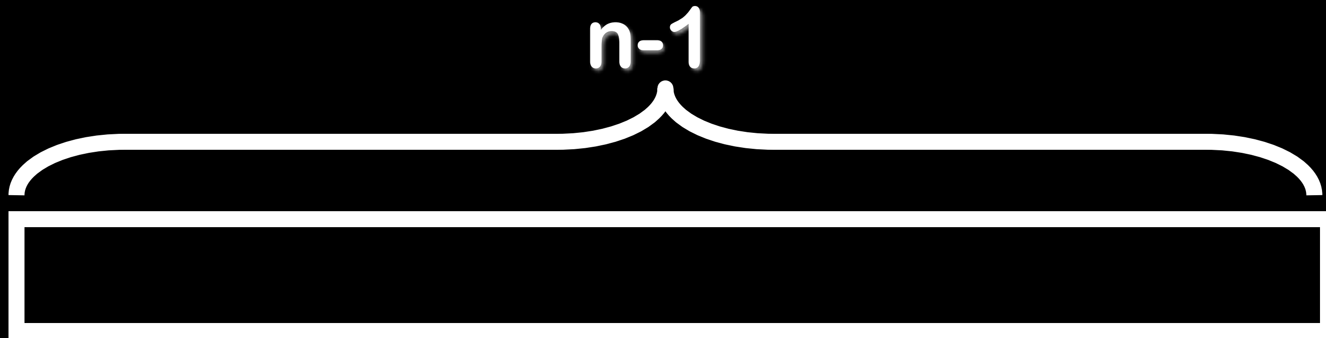
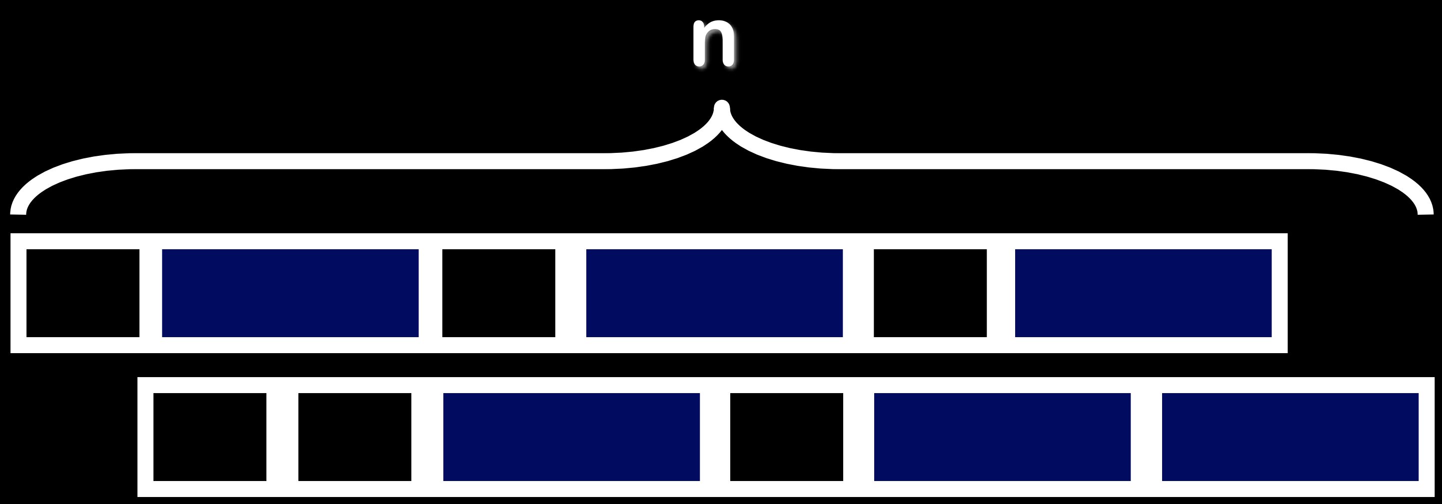


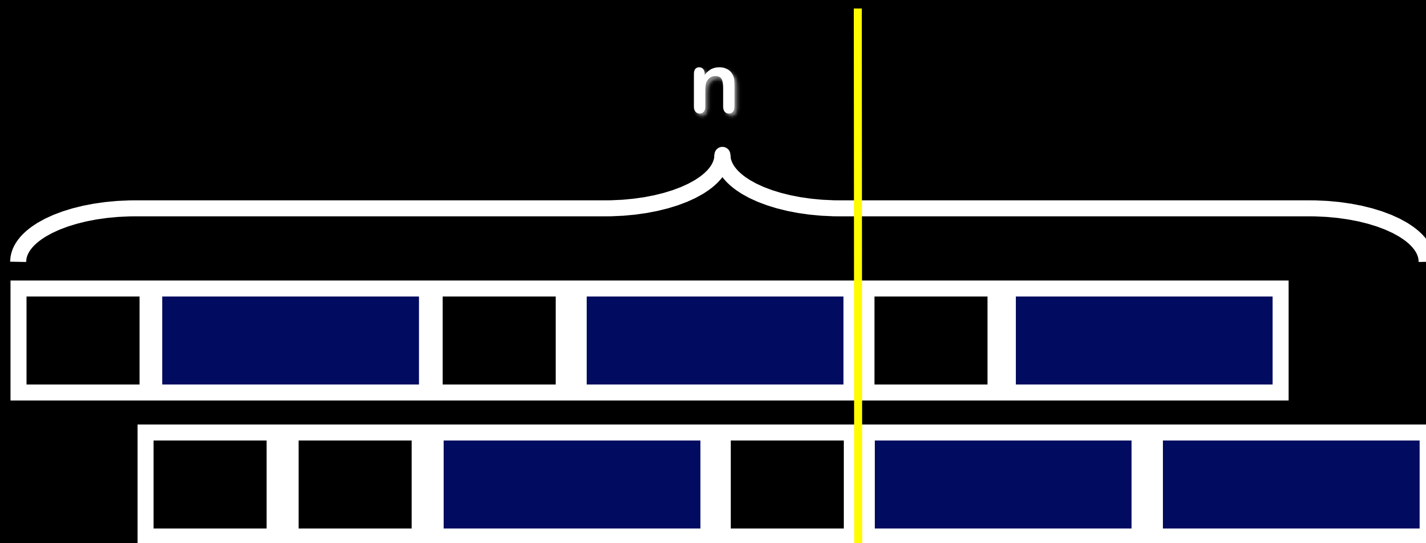
F_n tilings of a strip of length n-1

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



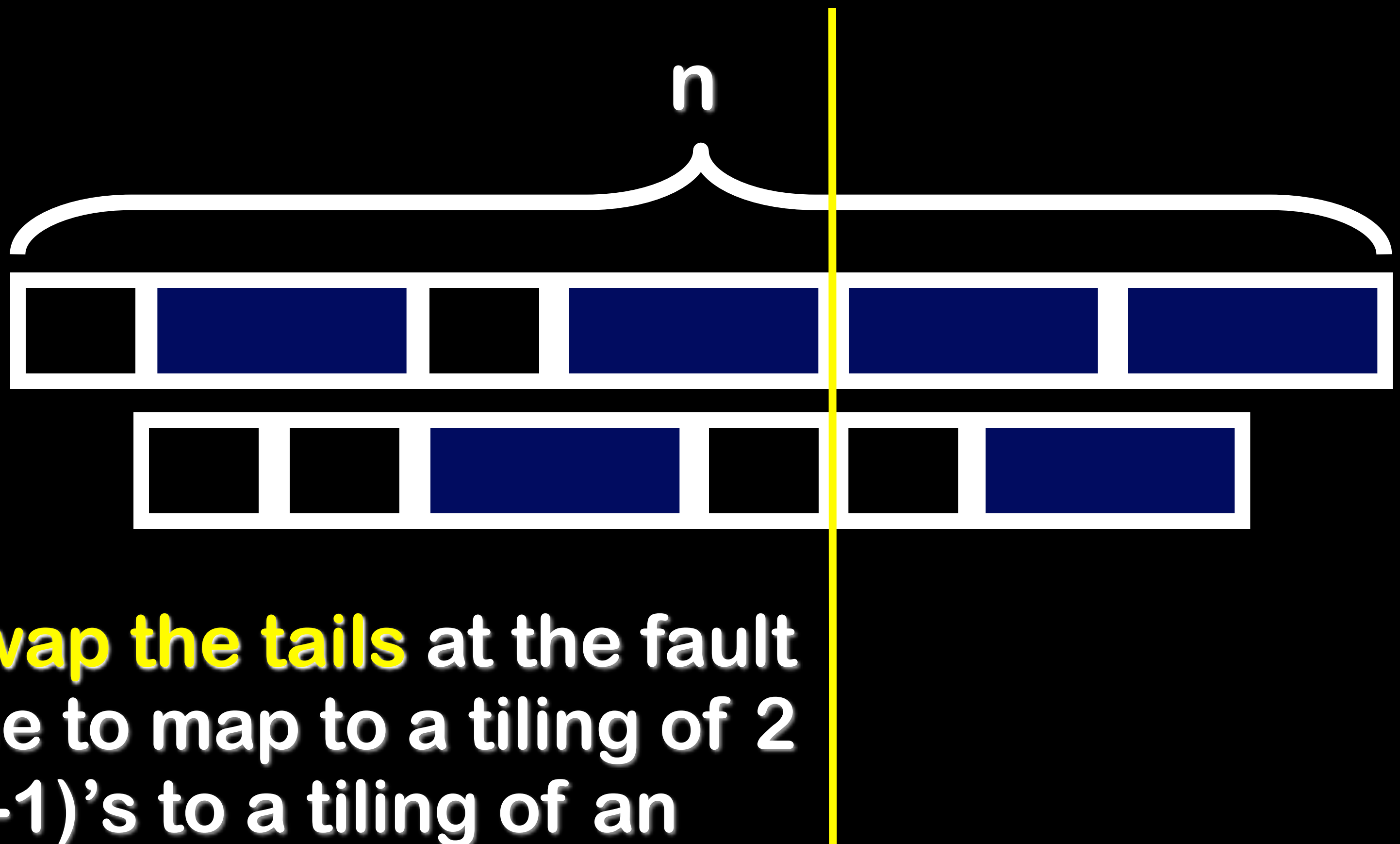
$(F_n)^2$ tilings of two strips of size n-1

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



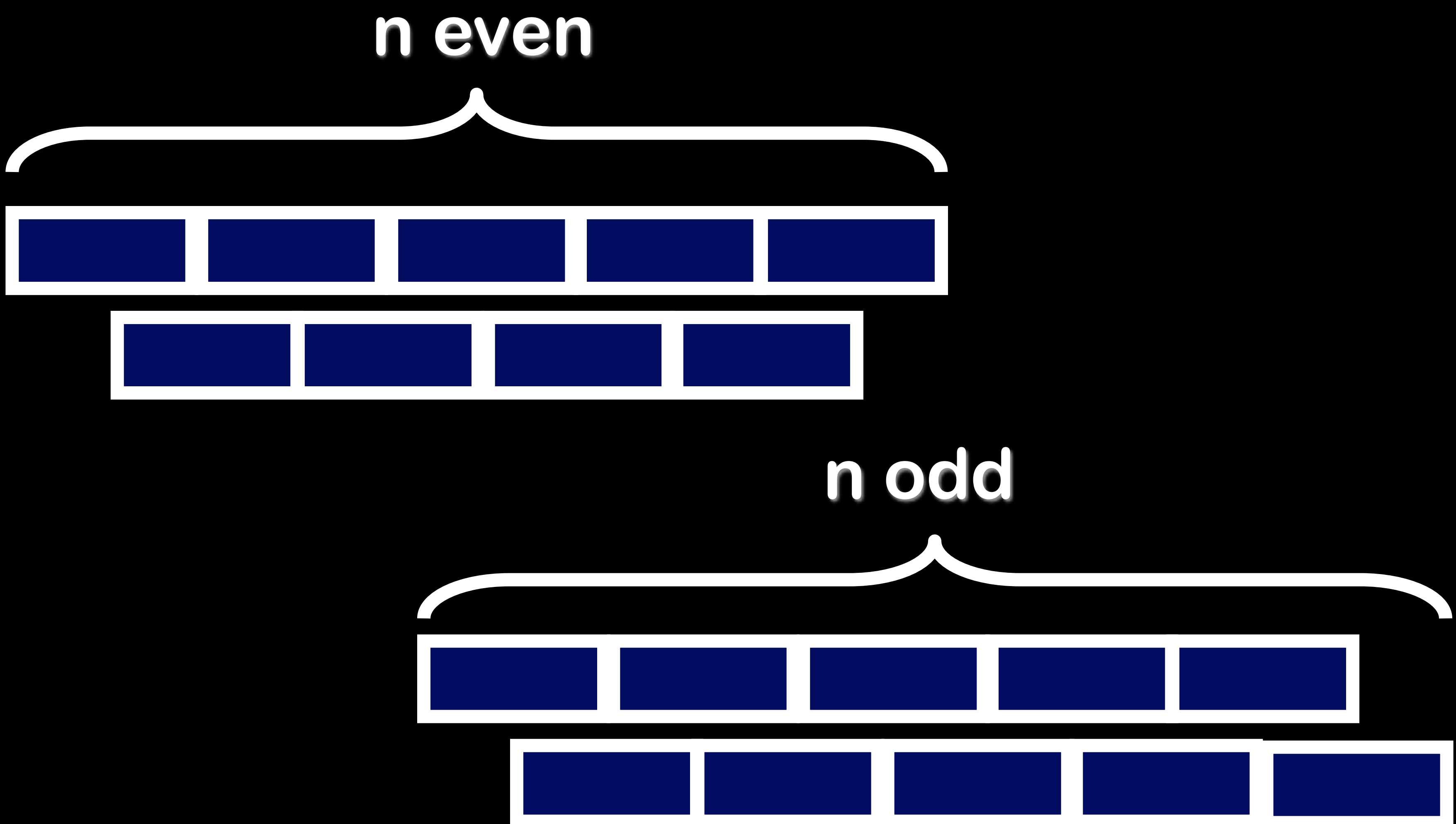
Draw a vertical "fault line" at the **rightmost** position (<n) possible without cutting any dominoes

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

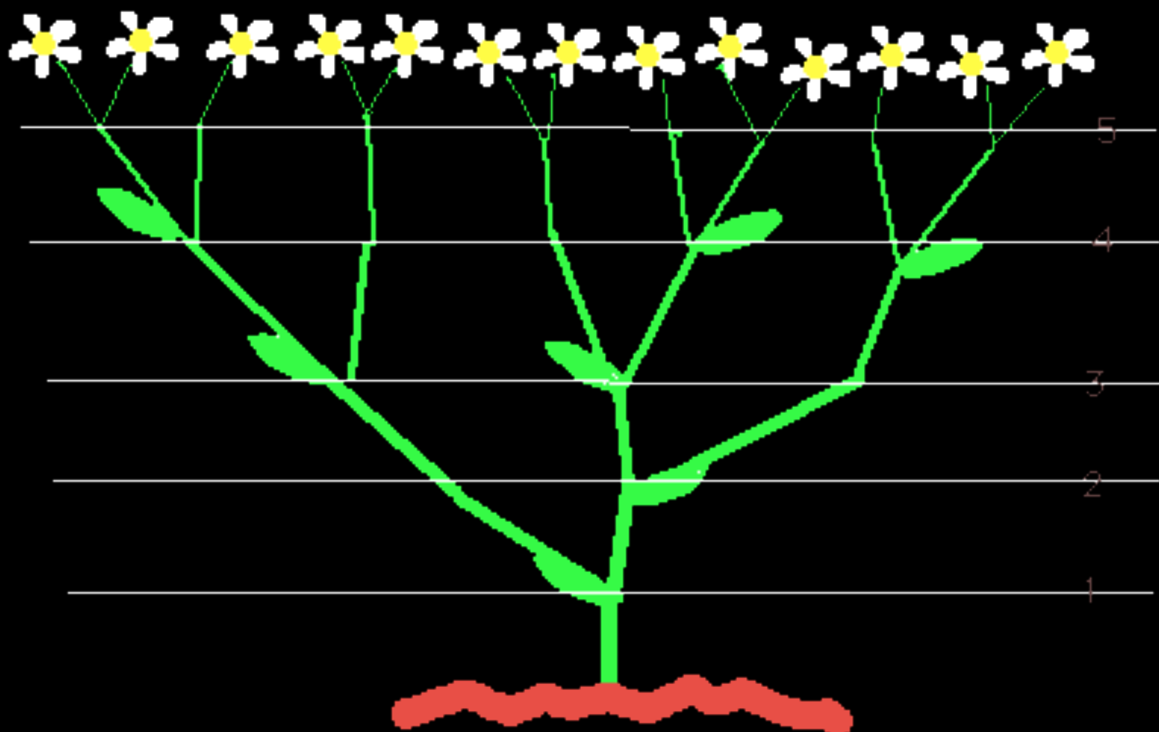


Swap the tails at the fault line to map to a tiling of 2 (n-1)'s to a tiling of an n-2 and an n.

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^{n-1}$$



Sneezwort (Achilleaptarmica)



Each time the plant starts a new shoot it takes two months before it is strong enough to support branching.

Counting Petals

Counting Petals

5 petals: buttercup, wild rose, larkspur, columbine (aquilegia)
8 petals: delphiniums
13 petals: ragwort, corn marigold, cineraria, some daisies
21 petals: aster, black-eyed susan, chicory
34 petals: plantain, pyrethrum
55, 89 petals: michaelmas daisies, the asteraceae family.

The Fibonacci Quarterly

Definition of ϕ (Euclid)

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Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

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$$\phi = \frac{AC}{AB} = \frac{AB}{BC}$$

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$$\phi^2 - \phi =$$

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$$\phi^2 - \phi = \frac{AC}{BC} - \frac{AB}{BC} =$$

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$$\phi^2 = \frac{AC}{BC}$$

$$\phi^2 - \phi = \frac{AC}{BC} - \frac{AB}{BC} = \frac{BC}{BC} = 1$$

$$\phi^2 - \phi - 1 = 0$$

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$$\phi = \frac{1 + \sqrt{5}}{2}$$

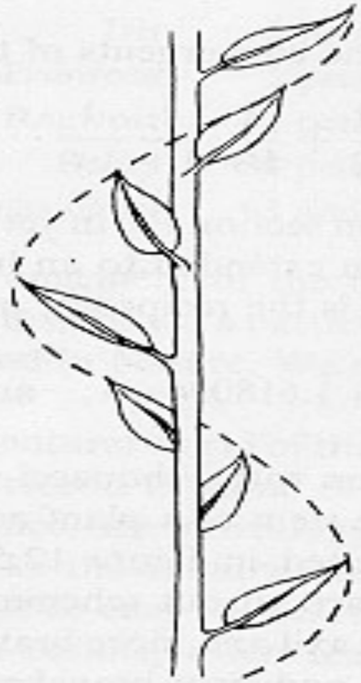
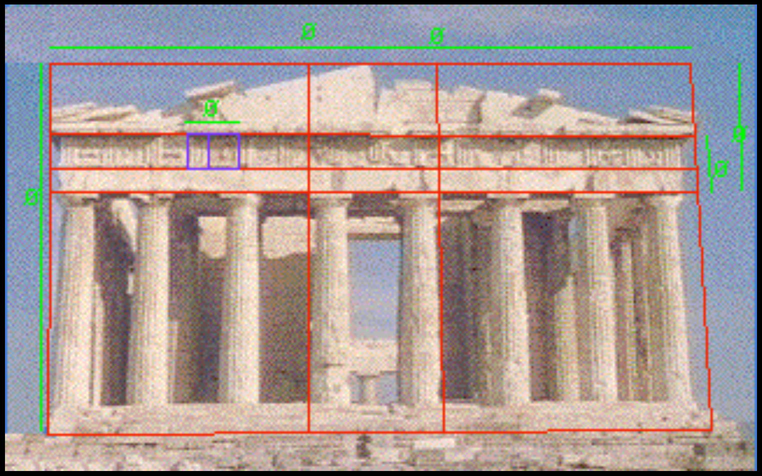


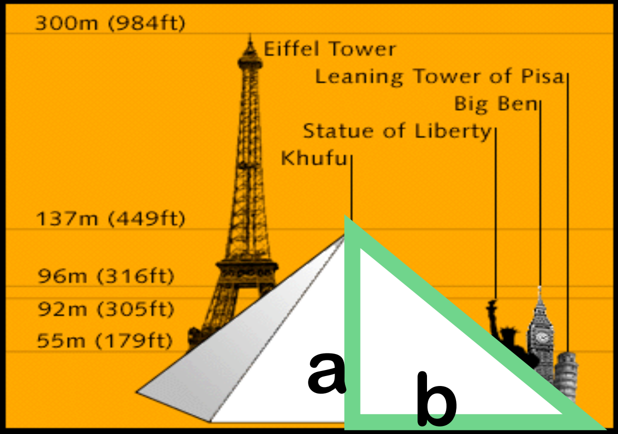
Fig. 12.4. Phyllotaxis

Golden ratio supposed to arise in…

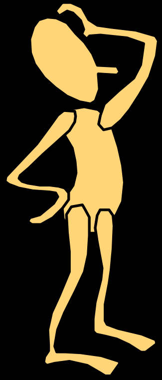


Parthenon, Athens (400 B.C.)

$\frac{a}{b} = 1.618$



The great pyramid at Gizeh



Ratio of a person's height
to the height of his/her navel

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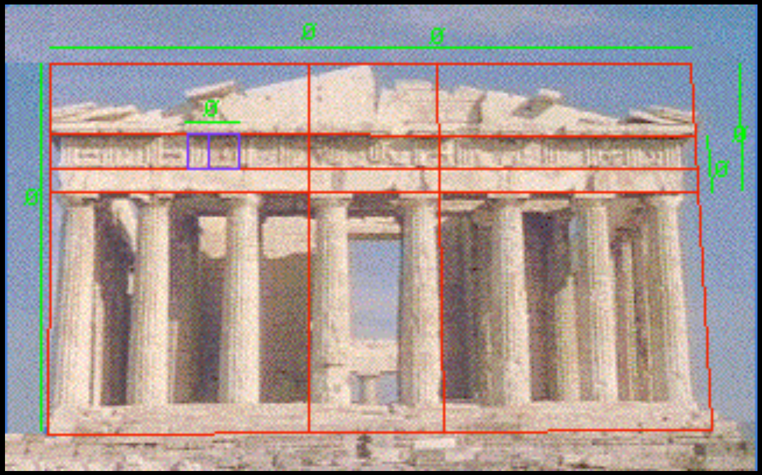

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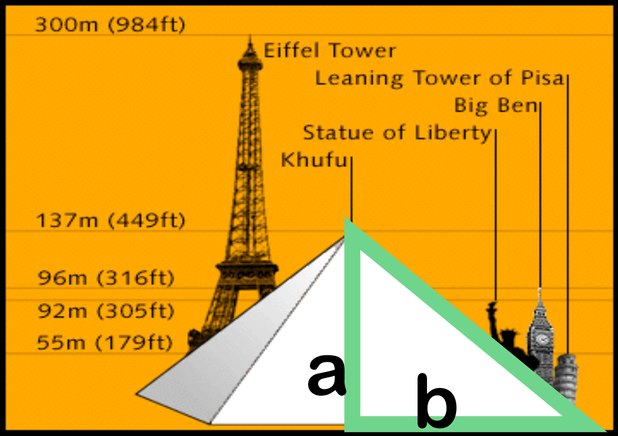

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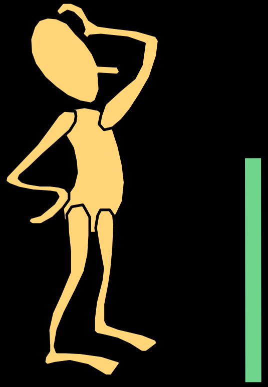

Ratio of a person's height
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Mostly circumstantial evidence…

Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

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$$\phi = 1 + \frac{1}{\phi}$$

$$= 1 + \frac{1}{1 + \frac{1}{\phi}}$$

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Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

A (Simple) Continued Fraction Is Any Expression Of The Form:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \cfrac{1}{g + \cfrac{1}{h + \cfrac{1}{i + \cfrac{1}{j + \dots}}}}}}}}}$$

where a, b, c, … are whole numbers.

A Continued Fraction can have a finite or infinite number of terms.

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \cfrac{1}{g + \cfrac{1}{h + \cfrac{1}{i + \cfrac{1}{j + \ldots}}}}}}}}}$$

We also denote this fraction by [a,b,c,d,e,f,...]

A Finite Continued Fraction

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

Denoted by [2,3,4,2,0,0,0,...]

An Infinite Continued Fraction

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}}}}}}}$$

Denoted by [1,2,2,2,...]

Recursively Defined Form For CF

$$\text{CF} = \text{whole number, or}$$

$$= \text{whole number} + \frac{1}{\text{CF}}$$

Continued fraction representation of a standard fraction

$$\frac{67}{29} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

$$\frac{67}{29} = 2 + \frac{1}{\frac{29}{9}} = 2 + \frac{1}{3 + \frac{2}{9}} 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{2}}}$$

e.g., 67/29 = 2 with remainder 9/29
= 2 + 1/ (29/9)

Ancient Greek Representation: Continued Fraction Representation

$$\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{2}}$$

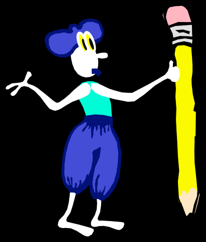
Ancient Greek Representation: Continued Fraction Representation

$$\frac{5}{3} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$$

= [1,1,1,1,0,0,0,...]

Ancient Greek Representation: Continued Fraction Representation

$$? = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$



Ancient Greek Representation: Continued Fraction Representation

$$\frac{8}{5} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

= [1,1,1,1,1,0,0,0,...]

Ancient Greek Representation: Continued Fraction Representation

$$\frac{13}{8} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}$$

= [1,1,1,1,1,1,0,0,0,...]

A Pattern?

Let $r_1 = [1,0,0,0,\ldots] = 1$

$r_2 = [1,1,0,0,0,\ldots] = 2/1$

$r_3 = [1,1,1,0,0,0\ldots] = 3/2$

$r_4 = [1,1,1,1,0,0,0\ldots] = 5/3$

and so on.

Theorem:

$$r_n = Fib(n+1)/Fib(n)$$

1,1,2,3,5,8,13,21,34,55,….

1,1,2,3,5,8,13,21,34,55,….

2/1 = 2

1,1,2,3,5,8,13,21,34,55,….

2/1	=	2
3/2	=	1.5

1,1,2,3,5,8,13,21,34,55,….

2/1	=	2
3/2	=	1.5
5/3	=	1.666…

1,1,2,3,5,8,13,21,34,55,….

2/1	=	2
3/2	=	1.5
5/3	=	1.666…
8/5	=	1.6

1,1,2,3,5,8,13,21,34,55,….

2/1	=	2
3/2	=	1.5
5/3	=	1.666…
8/5	=	1.6
13/8	=	1.625

1,1,2,3,5,8,13,21,34,55,….

2/1	=	2
3/2	=	1.5
5/3	=	1.666…
8/5	=	1.6
13/8	=	1.625
21/13	=	1.6153846…

1,1,2,3,5,8,13,21,34,55,….

2/1	=	2
3/2	=	1.5
5/3	=	1.666…
8/5	=	1.6
13/8	=	1.625
21/13	=	1.6153846…
34/21	=	1.61904…

1,1,2,3,5,8,13,21,34,55,....

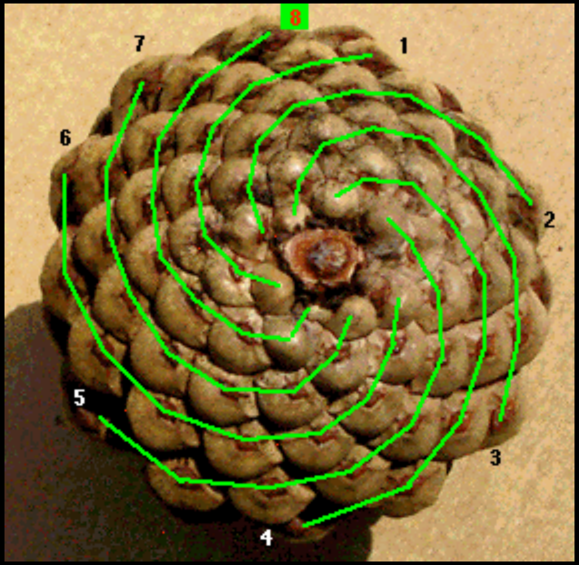
2/1	=	2
3/2	=	1.5
5/3	=	1.666...
8/5	=	1.6
13/8	=	1.625
21/13	=	1.6153846...
34/21	=	1.61904...

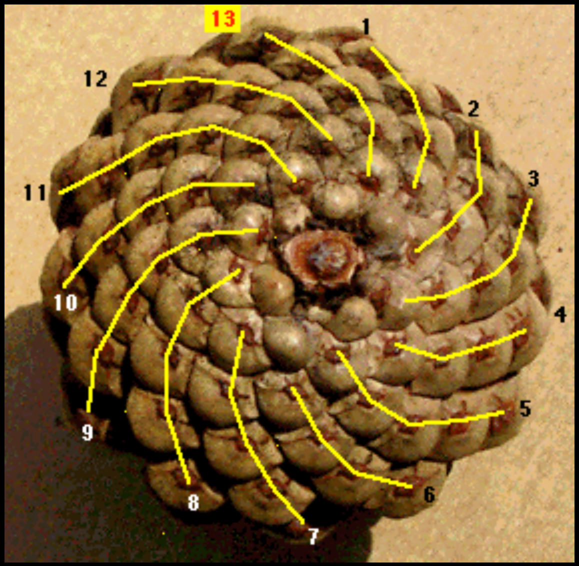
ϕ = 1.6180339887498948482045

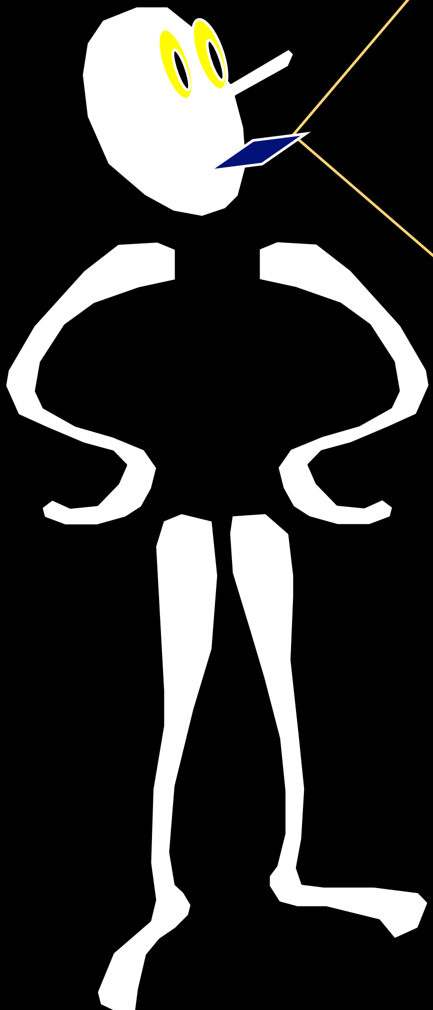
Pineapple whorls

Church and Turing were both interested in the number of whorls in each ring of the spiral.

The ratio of consecutive ring lengths approaches the Golden Ratio.

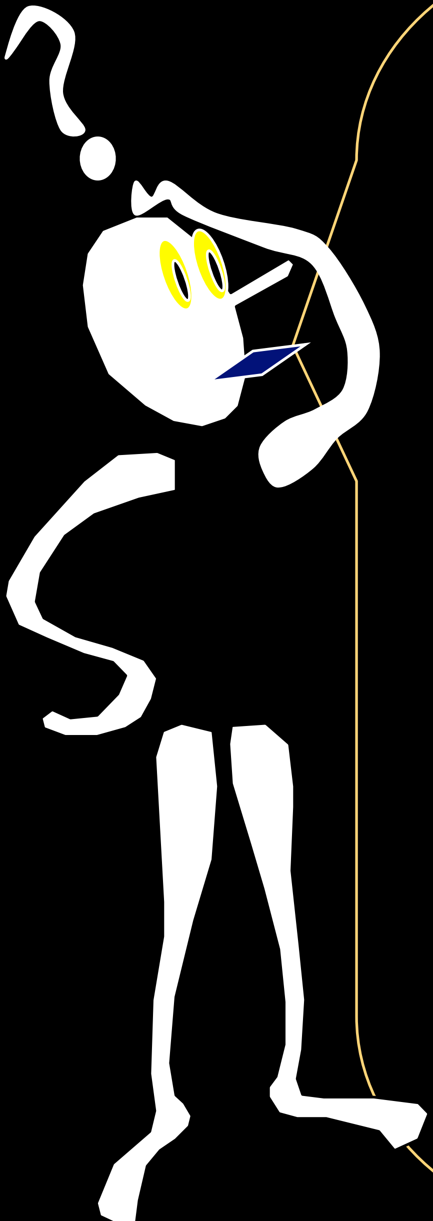

8
1
2
3
4
5
6
7


13
1
2
3
4
5
6
7
8
9
10
11
12



Proposition:
Any finite continued fraction evaluates to a rational.

Theorem
Any rational has a finite continued fraction representation.



Hmm.

Finite CFs = Rationals.

Then what do
infinite continued fractions
represent?

An infinite continued fraction

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}}}}}}}$$

Quadratic Equations

- $X^2 - 3x - 1 = 0$

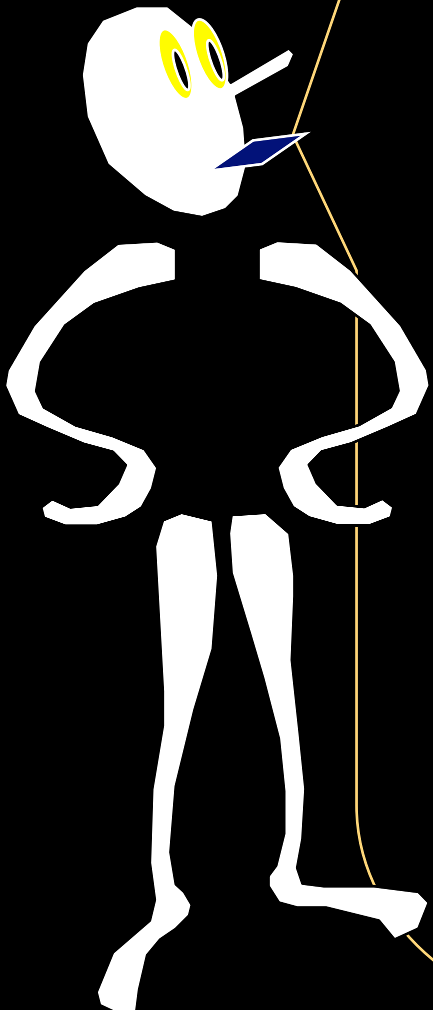
$$X = \frac{3 + \sqrt{13}}{2}$$

- $X^2 = 3X + 1$
- $X = 3 + 1/X$

- $X = 3 + 1/X = 3 + 1/[3 + 1/X] = \ldots$

A Periodic CF

$$\frac{3+\sqrt{13}}{2} = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \ldots}}}}}}}}}$$



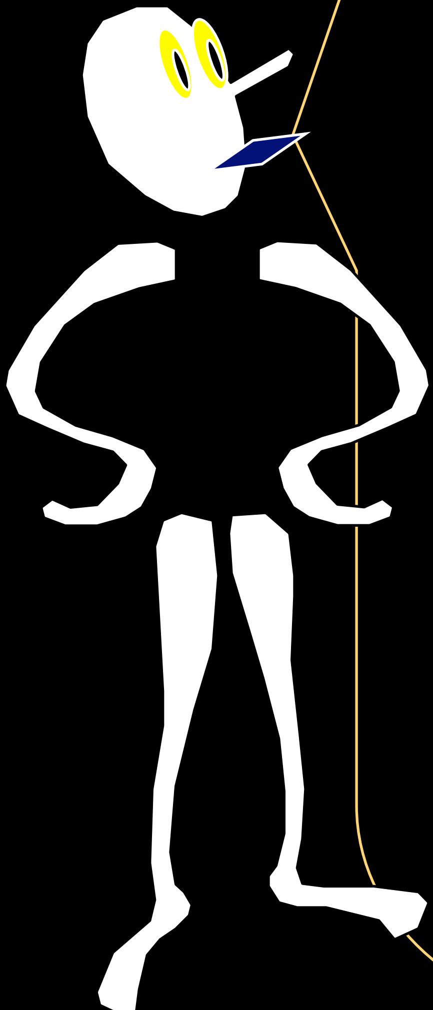
Theorem:

Any solution to a quadratic equation has a periodic continued fraction.

Converse:

Any periodic continued fraction is the solution of a quadratic equation.

(try to prove this!)



So they express more than just the rationals…

What about those non-recurring infinite continued fractions?

Non-periodic CFs

$$e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

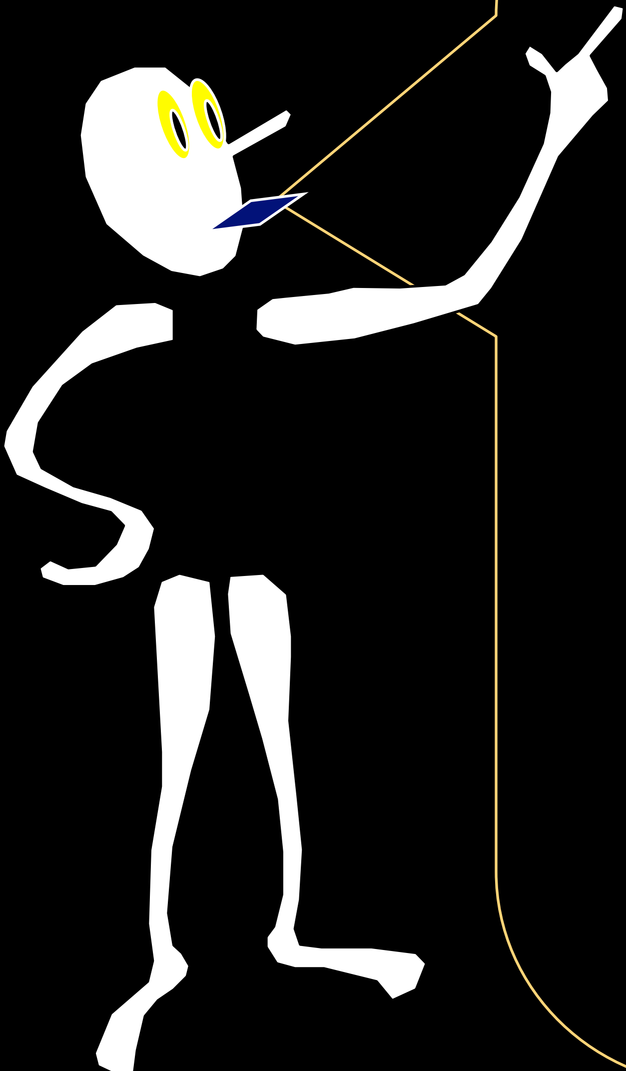
What is the pattern?

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \dots}}}}}}}}}$$

What is the pattern?

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \dots}}}}}}}}}$$

No one knows!



What a cool representation!

Finite CF: Rationals

Periodic CF: Quadratic roots

And some numbers reveal hidden regularity.

More good news: Convergents

Let $\alpha = [a_1, a_2, a_3, \ldots]$ be a CF.

Define: $C_1 = [a_1,0,0,0,0..]$

$C_2 = [a_1,a_2,0,0,0,\ldots]$

$C_3 = [a_1,a_2,a_3,0,0,\ldots]$ and so on.

C_k is called the **k-th convergent** of α

α is the limit of the sequence $C_1, C_2, C_3, \ldots$

Best Approximator Theorem

- A rational p/q is the best approximator to a real α if no rational number of denominator smaller than q comes closer to α.

Best Approximator Theorem

- A rational p/q is the best approximator to a real α if no rational number of denominator smaller than q comes closer to α.

BEST APPROXIMATOR THEOREM:

Given any CF representation of α, each convergent of the CF is a best approximator for α !

Best Approximators of π

$C_1 = 3$

$C_2 = 22/7$

$C_3 = 333/106$

$C_4 = 355/113$

$C_5 = 103993/33102$

$C_6 = 104348/33215$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Continued Fraction Representation

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Remember?

We already saw the convergents of this CF

[1,1,1,1,1,1,1,1,1,1,1,...]

are of the form Fib(n+1)/Fib(n)

Hence: $$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi = \frac{1+\sqrt{5}}{2}$$

1,1,2,3,5,8,13,21,34,55,….

- 2/1 = 2
- 3/2 = 1.5
- 5/3 = 1.666…
- 8/5 = 1.6
- 13/8 = 1.625
- 21/13 = 1.6153846…
- 34/21 = 1.61904…

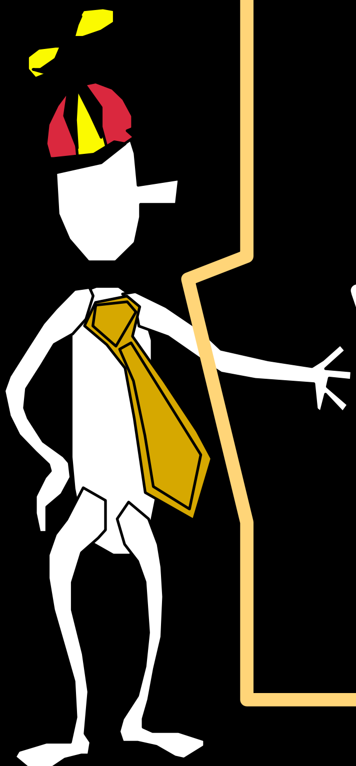
- ϕ = 1.6180339887498948482045…

As we've seen...

$$\frac{z}{1-z-z^2} = 0\times 1 + z + z^2 + 2z^3 + 3z^4 + 5z^5 + \cdots$$
$$= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4 + F_5 z^5 + \cdots$$

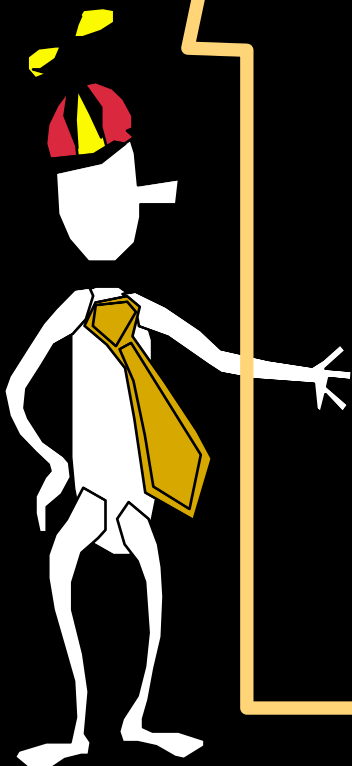
Going the Other Way

$$\begin{aligned}
(1 - z - z^2)&(F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots) \\
= \; & F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots \\
& \qquad - F_0 z - F_1 z^2 - F_2 z^3 - \cdots \\
& \qquad\qquad - F_0 z^2 - F_1 z^3 - \cdots \\
= \; & F_0 + (F_1 - F_0) z \\
= \; & z
\end{aligned}$$



What is the Power Series Expansion of $z/(1-z-z^2)$?

What does this look like when we expand it as an infinite sum?

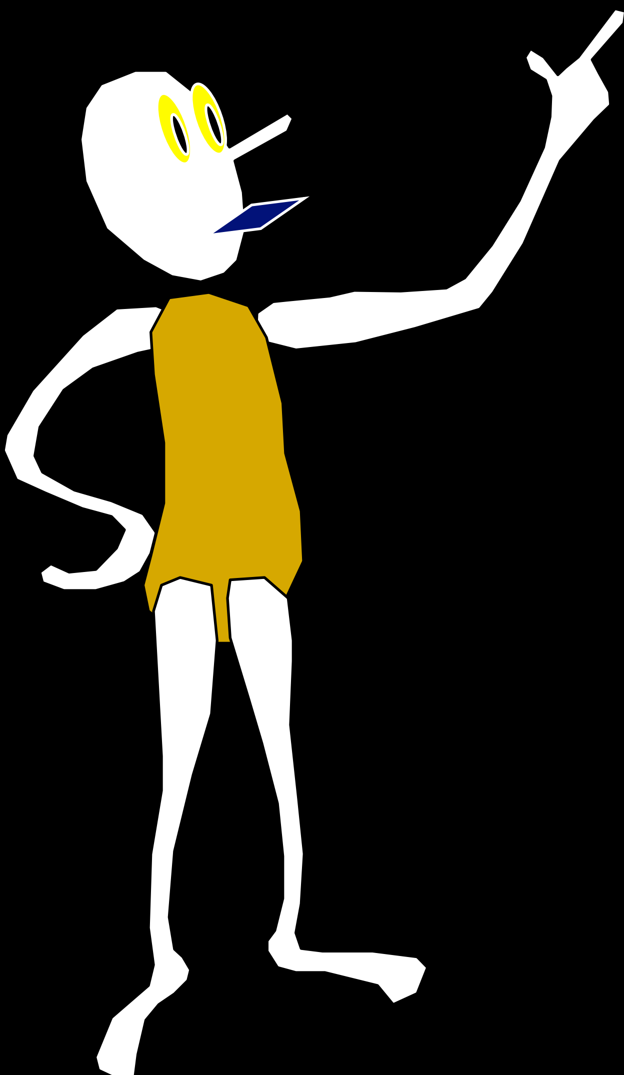


Since the bottom is quadratic we can factor it:

$$\frac{z}{1-z-z^2} = \frac{z}{(1-\phi z)\left(1-\hat{\phi} z\right)}$$

$$\hat{\phi} = -1/\phi$$

Write as a sum of two terms



$$F(z) = \frac{a}{1-\phi z} + \frac{b}{1-\widehat{\phi} z}$$

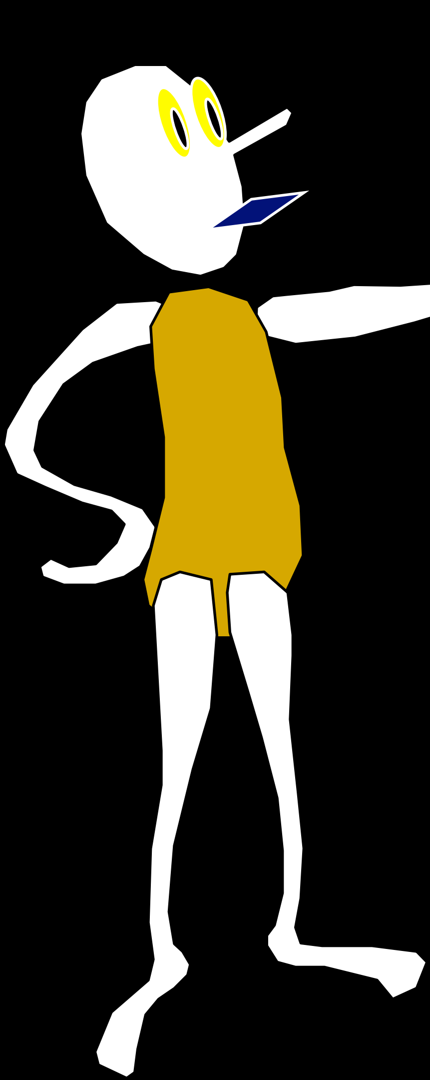
where

$$a(1-\widehat{\phi}z) + b(1-\phi z) = z.$$

Solving:

$$a = \frac{1}{\phi - \widehat{\phi}} = \frac{1}{\sqrt{5}} = -b$$

Now use the geometric series

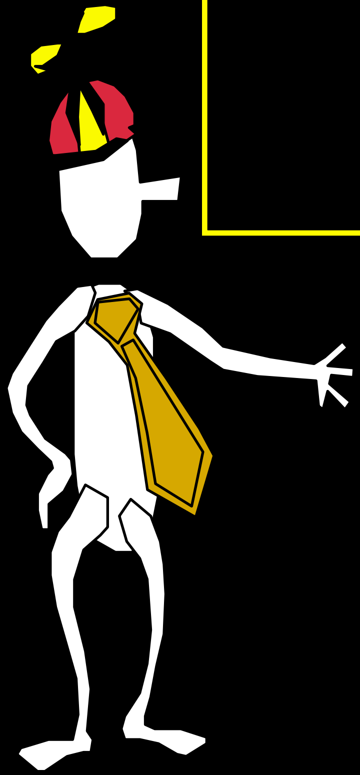


$$F(z) = \frac{1}{\sqrt{5}} \sum_{n \ge 0} \phi^n z^n - \frac{1}{\sqrt{5}} \sum_{n \ge 0} \widehat{\phi}^n z^n$$

$$= \sum_{n \ge 0} \frac{1}{\sqrt{5}} \left(\phi^n - \widehat{\phi}^n \right) z^n.$$

$$F(z) = F_0 + F_1 z + F_2 z^2 + \cdots = \frac{z}{1 - z - z^2}$$

$$\frac{z}{1 - z - z^2} = \sum_{n \geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \widehat{\phi}^n \right) z^n.$$



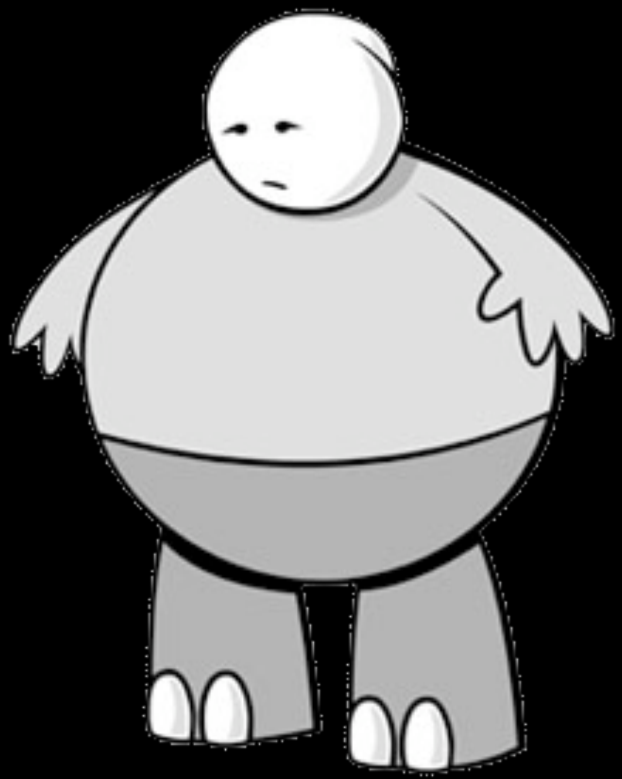
Leonhard Euler (1765)
J. P. M. Binet (1843)
A de Moivre (1730)

The n^{th} Fibonacci number is:

$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}}$$

$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}} \approx \frac{\phi^n}{\sqrt{5}}$$

$$\frac{F_n}{F_{n-1}} = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} \longrightarrow \phi$$



Here's What You Need to Know…

Recurrences and generating functions

Golden ratio

Continued fractions

Convergents

Closed form for Fibonaccis