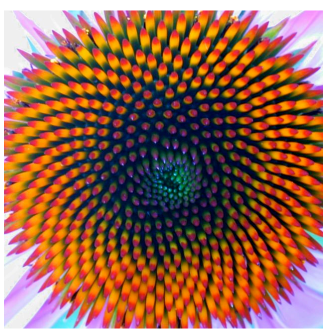
15-251

Great Theoretical Ideas in Computer Science

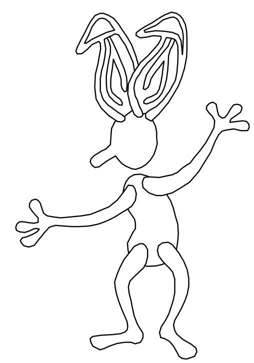
Recurrences, Fibonacci Numbers and Continued Fractions

Lecture 9, September 23, 2008



Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations



Rabbit Reproduction

A rabbit lives forever

The population starts as single newborn pair

Every month, each productive pair begets a new pair which will become productive after 2 months old

F_n= # of rabbit pairs at the beginning of the n^{th} month

month	1	2	3	4	5	6	7
rabbits	1	1	2	3	5	8	13

Fibonacci Numbers

month	1	2	3	4	5	6	7
rabbits	1	1	2	3	5	8	13

Stage 0, Initial Condition, or Base Case:
Fib(1) = 1; Fib (2) = 1

Inductive Rule:
For n>3, Fib(n) = Fib(n-1) + Fib(n-2)

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$f_1 = 1$ 0 = the empty sum

$f_2 = 1$ 1 = 1

$f_3 = 2$ 2 = 1 + 1

2

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

4 = 2 + 2
2 + 1 + 1
1 + 2 + 1
1 + 1 + 2
1 + 1 + 1 + 1

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

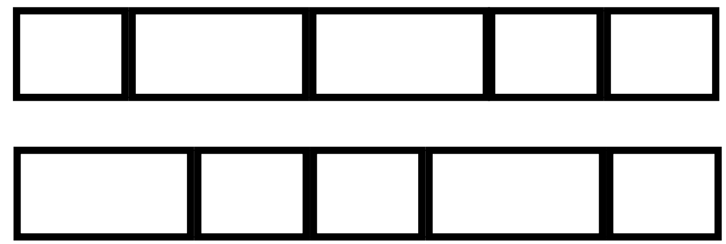
$$f_{n+1} = f_n + f_{n-1}$$

# of sequences beginning with a 1

# of sequences beginning with a 2

Visual Representation: Tiling

Let f_{n+1} be the number of different ways to tile a 1 × n strip with squares and dominoes.



Fibonacci Numbers Again

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

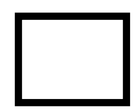
$$f_{n+1} = f_n + f_{n-1}$$

$$f_1 = 1 \qquad f_2 = 1$$

Visual Representation: Tiling

1 way to tile a strip of length 0

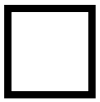
1 way to tile a strip of length 1:



2 ways to tile a strip of length 2:

$$f_{n+1} = f_n + f_{n-1}$$

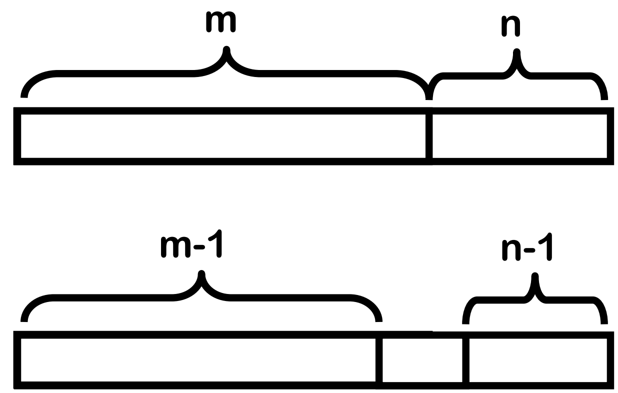
f_{n+1} is number of ways to tile length n.



f_n tilings that start with a square.

f_{n-1} tilings that start with a domino.

$$F_{m+n+1} \quad = F_{m+1}\, F_{n+1} \quad + \quad F_m\, F_n$$



Fibonacci Identities

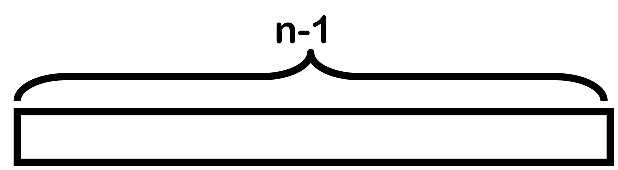
Some examples:

$F_{2n} = F_1 + F_3 + F_5 + \ldots + F_{2n-1}$

$F_{m+n+1} = F_{m+1}\, F_{n+1} + F_m\, F_n$

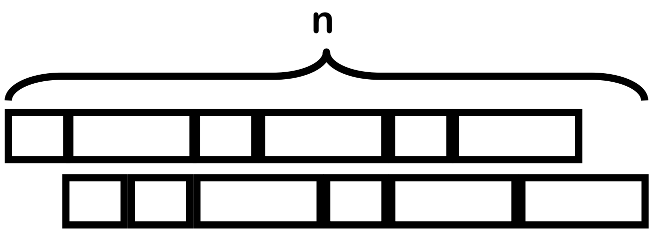
$(F_n)^2 \quad = \; F_{n-1}\, F_{n+1} + (-1)^n$

$$(F_n)^2 \quad = \; F_{n-1}\, F_{n+1} \quad + \quad (-1)^n$$



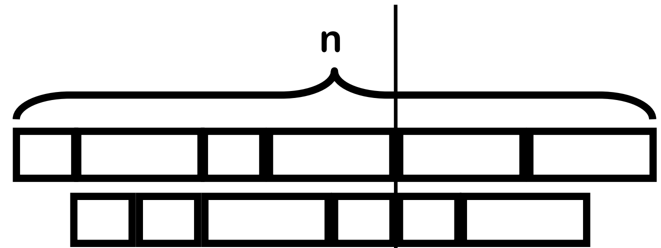
F_n tilings of a strip of length n-1

$$(F_n)^2 = F_{n-1}\,F_{n+1} + (-1)^n$$



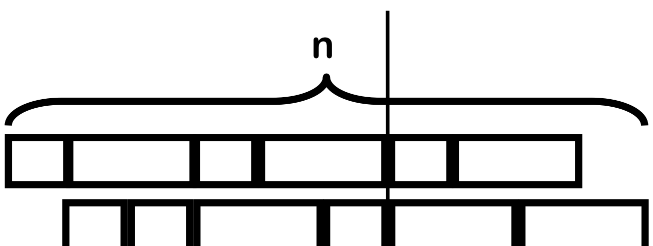
$(F_n)^2$ tilings of two strips of size n-1

$$(F_n)^2 = F_{n-1}\,F_{n+1} + (-1)^n$$



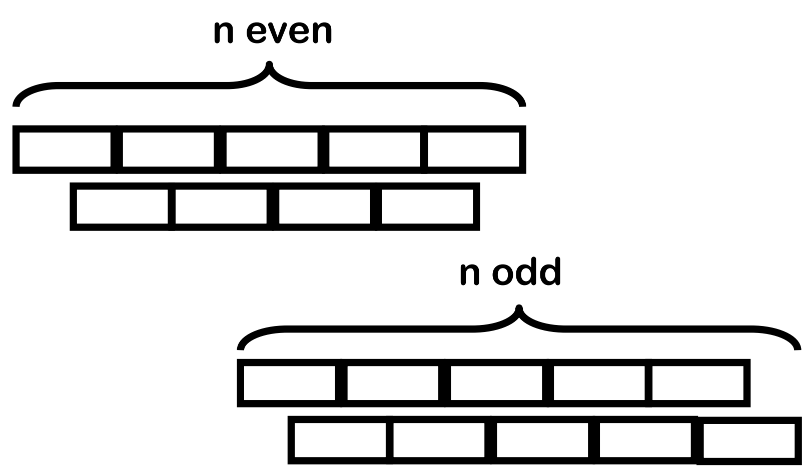
Swap the tails at the fault line to map to a tiling of 2 (n-1)'s to a tiling of an n-2 and an n.

$$(F_n)^2 = F_{n-1}\,F_{n+1} + (-1)^n$$

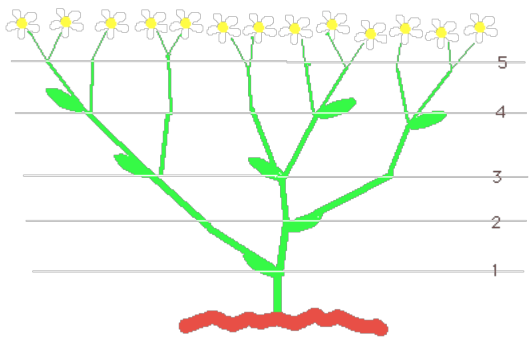


Draw a vertical "fault line" at the rightmost position (<n) possible without cutting any dominoes

$$(F_n)^2 = F_{n-1}\,F_{n+1} + (-1)^{n-1}$$



Sneezwort (Achilleaptarmica)



Each time the plant starts a new shoot it takes two months before it is strong enough to support branching.

The Fibonacci Quarterly

Counting Petals

5 petals: buttercup, wild rose, larkspur,
columbine (aquilegia)
8 petals: delphiniums
13 petals: ragwort, corn marigold, cineraria,
some daisies
21 petals: aster, black-eyed susan, chicory
34 petals: plantain, pyrethrum
55, 89 petals: michaelmas daisies, the
asteraceae family.

Definition of ϕ (Euclid)

Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

$$\phi = \frac{AC}{AB} = \frac{AB}{BC}$$

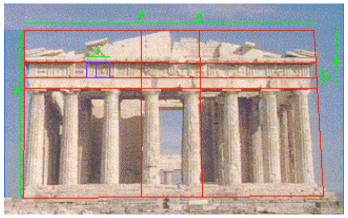
$$\phi^2 = \frac{AC}{BC}$$

$$\phi^2 - \phi = \frac{AC}{BC} - \frac{AB}{BC} = \frac{BC}{BC} = 1$$

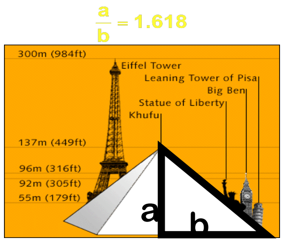
$$\phi^2 - \phi - 1 = 0$$

$$\phi = \frac{1 + \sqrt{5}}{2}$$

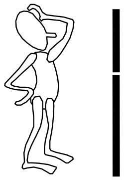
Golden ratio supposed to arise in…



Parthenon, Athens (400 B.C.)



The great pyramid at Gizeh



Ratio of a person's height to the height of his/her navel

Mostly circumstantial evidence…

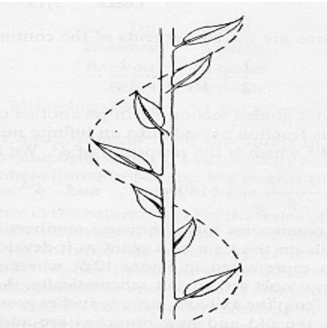


Fig. 12.4. Phyllotaxis

Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

$$= 1 + \frac{1}{1 + \frac{1}{\phi}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\phi}}}$$

A (Simple) Continued Fraction Is Any Expression Of The Form:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \cfrac{1}{g + \cfrac{1}{h + \cfrac{1}{i + \cfrac{1}{j + ...}}}}}}}}}$$

where a, b, c, ... are whole numbers.

Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + ...}}}}}}}}}$$

A Continued Fraction can have a finite or infinite number of terms.

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \cfrac{1}{g + \cfrac{1}{h + \cfrac{1}{i + \cfrac{1}{j + ...}}}}}}}}}$$

We also denote this fraction by [a,b,c,d,e,f,...]

A Finite Continued Fraction

$$2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2}}}$$

Denoted by [2,3,4,2,0,0,0,...]

Recursively Defined Form For CF

$$\begin{aligned} CF &= \text{whole number, or} \\ &= \text{whole number} + \frac{1}{CF} \end{aligned}$$

An Infinite Continued Fraction

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+....}}}}}}}}}$$

Denoted by [1,2,2,2,...]

Continued fraction representation of a standard fraction

$$\frac{67}{29}=2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2}}}$$

$$\frac{67}{29} = 2 + \frac{1}{\frac{29}{9}} = 2 + \frac{1}{3 + \frac{2}{9}} 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{2}}}$$

e.g., 67/29 = 2 with remainder 9/29
= 2 + 1/ (29/9)

Ancient Greek Representation: Continued Fraction Representation

$$\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}$$

= [1,1,1,1,0,0,0,...]

Ancient Greek Representation: Continued Fraction Representation

$$\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{2}}$$

Ancient Greek Representation: Continued Fraction Representation

$$? = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}$$

Ancient Greek Representation: Continued Fraction Representation

$$\frac{8}{5} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

= [1,1,1,1,1,0,0,0,...]

A Pattern?

Let r_1 = [1,0,0,0,...] = 1

r_2 = [1,1,0,0,0,...] = 2/1

r_3 = [1,1,1,0,0,0...] = 3/2

r_4 = [1,1,1,1,0,0,0...] = 5/3

and so on.

Theorem:

r_n = Fib(n+1)/Fib(n)

Ancient Greek Representation: Continued Fraction Representation

$$\frac{13}{8} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}$$

= [1,1,1,1,1,1,0,0,0,...]

1,1,2,3,5,8,13,21,34,55,....

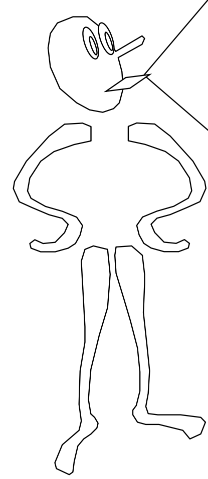
2/1	=	2
3/2	=	1.5
5/3	=	1.666...
8/5	=	1.6
13/8	=	1.625
21/13	=	1.6153846...
34/21	=	1.61904...

ϕ = 1.6180339887498948482045

Pineapple whorls

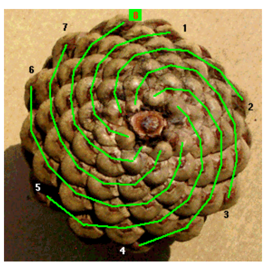
Church and Turing were both interested in the number of whorls in each ring of the spiral.

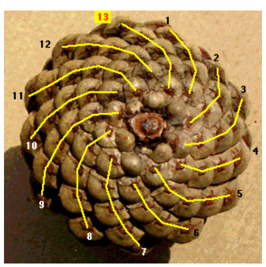
The ratio of consecutive ring lengths approaches the Golden Ratio.

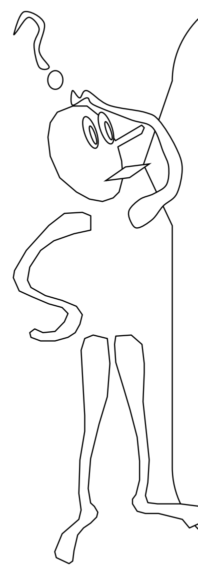


Proposition:
Any finite continued fraction evaluates to a rational.

Theorem
Any rational has a finite continued fraction representation.







Hmm.
Finite CFs = Rationals.

Then what do <u>infinite</u> continued fractions represent?

An infinite continued fraction

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}}}}}}}$$

A Periodic CF

$$\frac{3+\sqrt{13}}{2} = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \ldots}}}}}}}}}$$

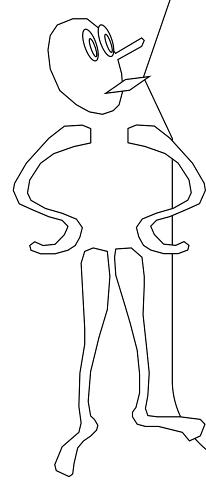
Quadratic Equations

- $X^2 - 3x - 1 = 0$

$$X = \frac{3+\sqrt{13}}{2}$$

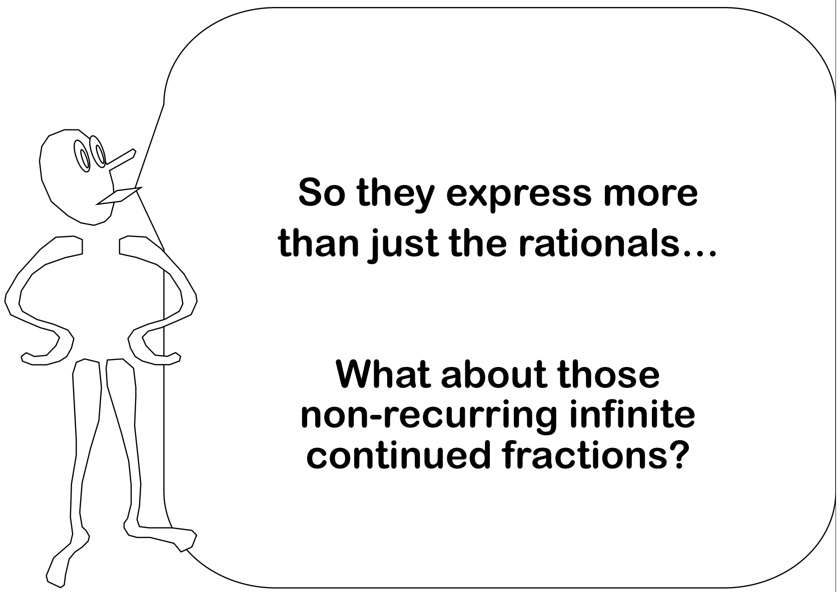
- $X^2 = 3X + 1$
- $X = 3 + 1/X$

- $X = 3 + 1/X = 3 + 1/[3 + 1/X] = \ldots$



Theorem:
Any solution to a <u>quadratic</u> equation has a periodic continued fraction.

Converse:
Any periodic continued fraction is the solution of a quadratic equation.
(try to prove this!)



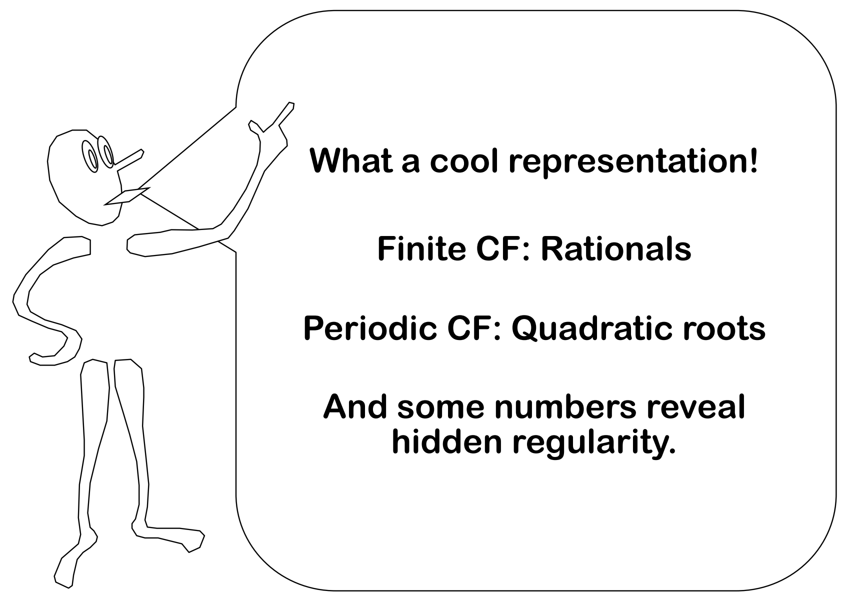
What is the pattern?

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ldots.}}}}}}}}}$$

No one knows!

Non-periodic CFs

$$e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ldots.}}}}}}}}}$$



More good news: Convergents

Let $\alpha = [a_1, a_2, a_3, \ldots]$ be a CF.

Define: $C_1 = [a_1,0,0,0,0..]$

$C_2 = [a_1,a_2,0,0,0,\ldots]$

$C_3 = [a_1,a_2,a_3,0,0,\ldots]$ and so on.

C_k is called the k-th convergent of α

α is the limit of the sequence $C_1, C_2, C_3, \ldots$

Best Approximators of π

$C_1 = 3$

$C_2 = 22/7$

$C_3 = 333/106$

$C_4 = 355/113$

$C_5 = 103993/33102$

$C_6 = 104348/33215$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Best Approximator Theorem

- A rational p/q is the <u>best approximator</u> to a real α if no rational number of denominator smaller than q comes closer to α.

BEST APPROXIMATOR THEOREM:

Given any CF representation of α, each convergent of the CF is a best approximator for α !

Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Continued Fraction Representation

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}}}}$$

1,1,2,3,5,8,13,21,34,55,….

- 2/1 = 2
- 3/2 = 1.5
- 5/3 = 1.666…
- 8/5 = 1.6
- 13/8 = 1.625
- 21/13 = 1.6153846…
- 34/21 = 1.61904…

- ϕ = 1.6180339887498948482045…

Remember?

We already saw the convergents of this CF
[1,1,1,1,1,1,1,1,1,1,1,…]
are of the form Fib(n+1)/Fib(n)

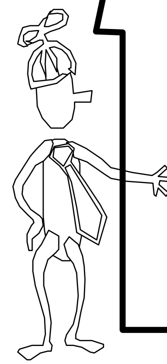
Hence: $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi = \frac{1+\sqrt{5}}{2}$

As we've seen…

$$\begin{aligned}\frac{z}{1-z-z^2} &= 0 \times 1 + z + z^2 + 2z^3 + 3z^4 + 5z^5 + \cdots \\ &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4 + F_5 z^5 + \cdots\end{aligned}$$

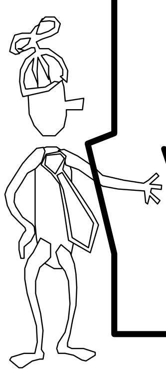
Going the Other Way

$$\begin{aligned}
&(1 - z - z^2)(F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots) \\
&= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots \\
&\qquad - F_0 z - F_1 z^2 - F_2 z^3 - \cdots \\
&\qquad\qquad - F_0 z^2 - F_1 z^3 - \cdots \\
&= F_0 + (F_1 - F_0) z \\
&= z
\end{aligned}$$



Since the bottom is quadratic we can factor it:

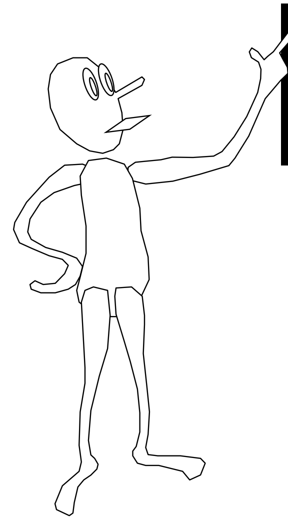
$$\begin{aligned}
\frac{z}{1 - z - z^2} &= \frac{z}{(1 - \phi z)\left(1 - \widehat{\phi} z\right)} \\
\widehat{\phi} &= -1/\phi
\end{aligned}$$



What is the Power Series Expansion of z/(1-z-z^2) ?

What does this look like when we expand it as an infinite sum?

Write as a sum of two terms



$$F(z) = \frac{a}{1 - \phi z} + \frac{b}{1 - \widehat{\phi} z}$$

where

$$a(1 - \widehat{\phi} z) + b(1 - \phi z) = z.$$

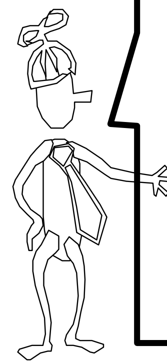
Solving:

$$a = \frac{1}{\phi - \widehat{\phi}} = \frac{1}{\sqrt{5}} = -b$$

Now use the geometric series

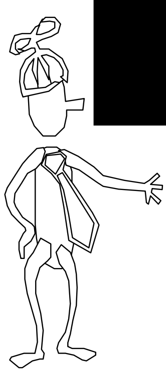
$$
\begin{aligned}
F(z) &= \frac{1}{\sqrt{5}}\sum_{n\geq 0}\phi^n z^n - \frac{1}{\sqrt{5}}\sum_{n\geq 0}\widehat{\phi}^n z^n \\
&= \sum_{n\geq 0}\frac{1}{\sqrt{5}}\left(\phi^n - \widehat{\phi}^n\right) z^n.
\end{aligned}
$$

Leonhard Euler (1765)
J. P. M. Binet (1843)
A de Moivre (1730)



The n^{th} Fibonacci number is:

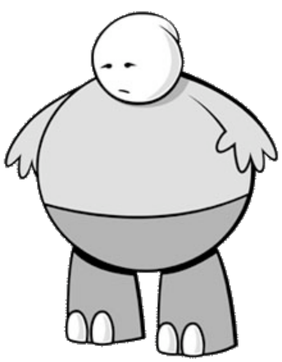
$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}}$$



$$
\begin{aligned}
F(z) &= F_0 + F_1 z + F_2 z^2 + \cdots = \frac{z}{1-z-z^2} \\
\frac{z}{1-z-z^2} &= \sum_{n\geq 0}\frac{1}{\sqrt{5}}\left(\phi^n - \widehat{\phi}^n\right) z^n.
\end{aligned}
$$

$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}} \approx \frac{\phi^n}{\sqrt{5}}$$

$$\frac{F_n}{F_{n-1}} = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} \longrightarrow \phi$$



Here's What You Need to Know…

- Recurrences and generating functions
- Golden ratio
- Continued fractions
- Convergents
- Closed form for Fibonaccis