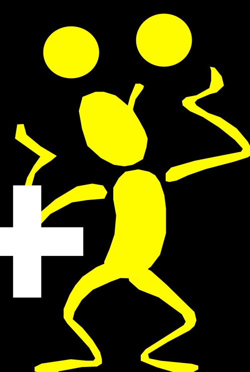
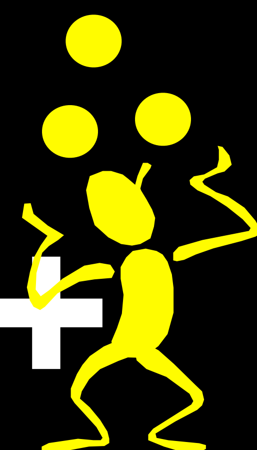
15-251

Great Theoretical Ideas in Computer Science

Counting III

Lecture 8, September 18, 2008

 X^1 +  X^2 +  X^3

Review from last time…

Arrange n symbols: r_1 of type 1, r_2 of type 2, ..., r_k of type k

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$$\binom{n}{r_1}$$

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$$\binom{n}{r_1}\binom{n-r_1}{r_2}$$

Arrange n symbols: r_1 of type 1, r_2 of type 2, ..., r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\ldots-r_{k-1}}{r_k}$$

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$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n - r_1 - r_2 - \ldots - r_{k-1}}{r_k}$$

$$= \frac{n!}{(n-r_1)!r_1!}$$

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$$= \frac{n!}{(n-r_1)!r_1!}\frac{(n-r_1)!}{(n-r_1-r_2)!r_2!}\cdots$$

$$= \frac{n!}{r_1!r_2!\ldots r_k!}$$

CARNEGIEMELLON

CARNEGIEMELLON

$$\frac{14!}{2!3!2!} = 3{,}632{,}428{,}800$$


5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?

How many different ways to divide up the loot?

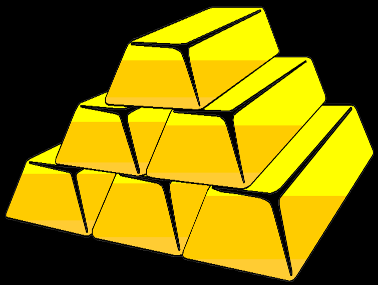
Sequences with 20 G’s and 4 /’s

How many different ways to divide up the loot?

Sequences with 20 G’s and 4 /’s

$$\binom{24}{4}$$

How many different ways can n distinct pirates divide k identical, indivisible bars of gold?



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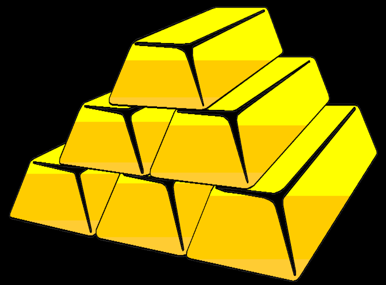

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + \ldots + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

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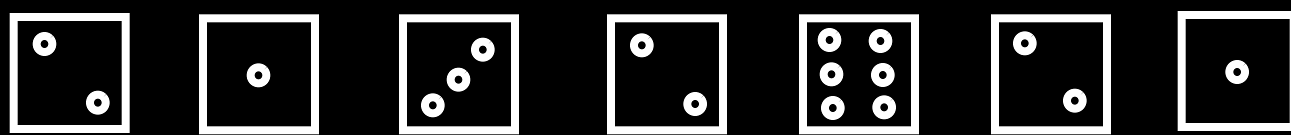
Identical/Distinct Dice

Suppose that we roll seven dice



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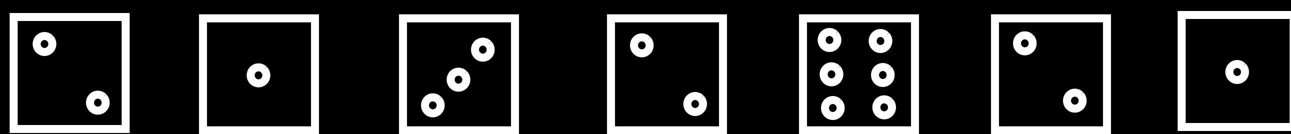

How many different outcomes are there, if order matters?

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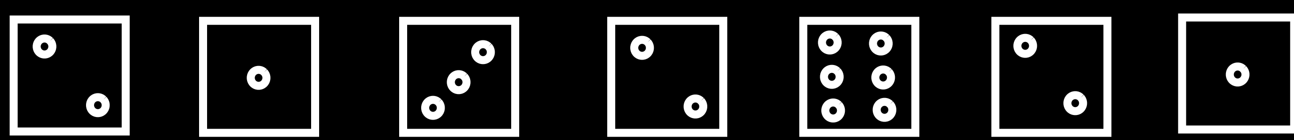

How many different outcomes are there, if order matters?

6^7

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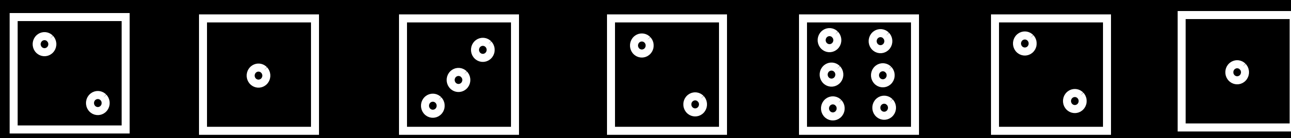

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What if order doesn't matter?
(E.g., Yahtzee)

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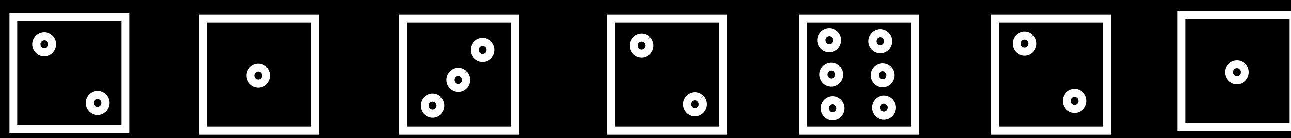

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$\binom{12}{7}$

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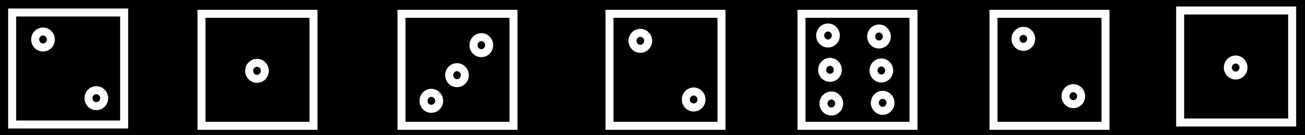


How many different outcomes are there, if order matters?

6^7

What if order doesn't matter? (E.g., Yahtzee)

$\binom{12}{7}$

(Corresponds to 6 pirates and 7 bars of gold)

Identical/Distinct Objects

If we are putting k objects into n distinct bins.

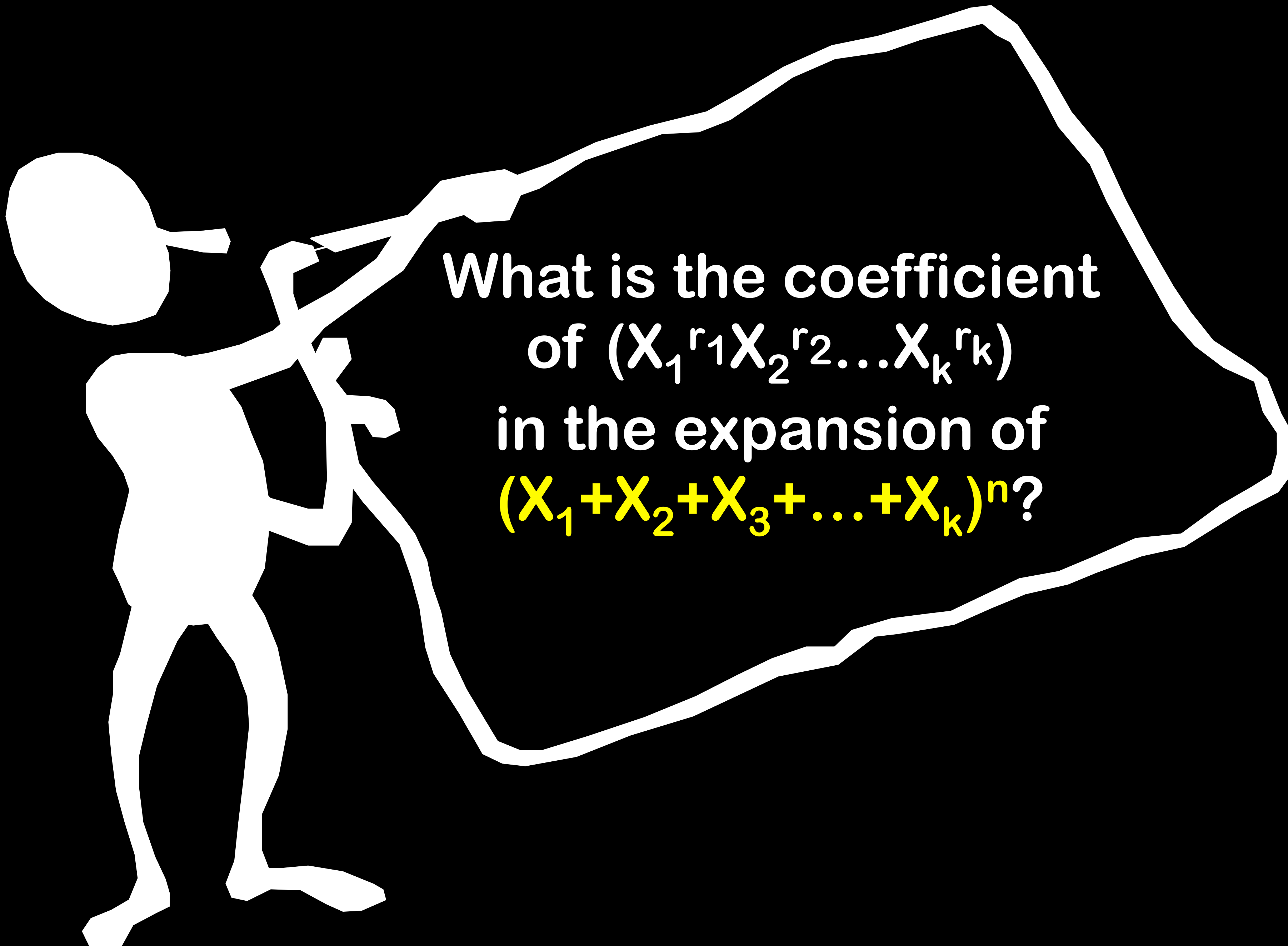
Objects are distinguishable	n^k
Objects are indistinguishable	$\binom{k+n-1}{k}$

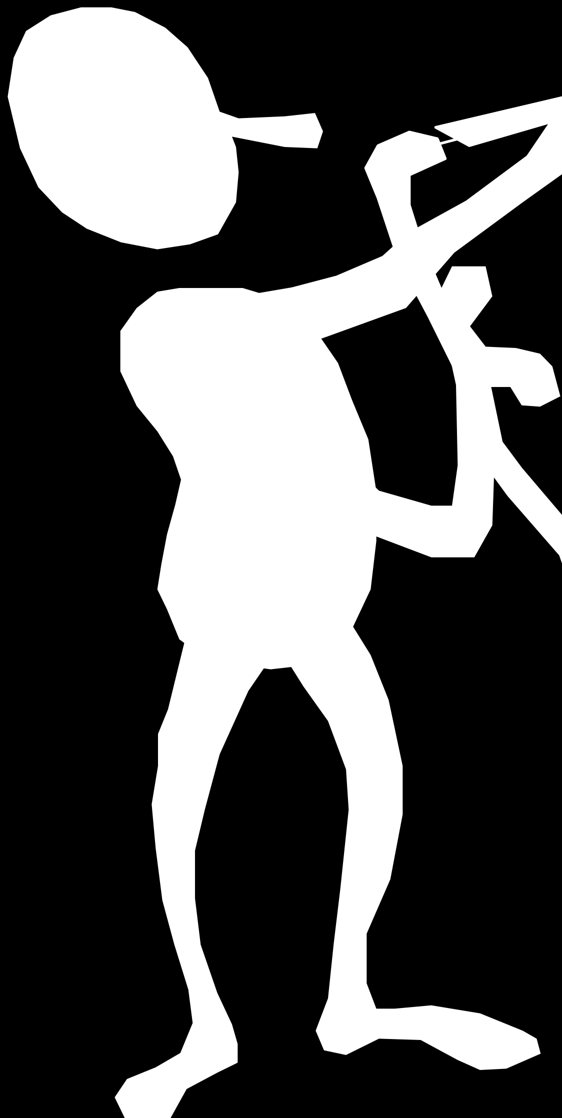
The Binomial Formula

$$(1+X)^n = \binom{n}{0} X^0 + \binom{n}{1} X^1 + \dots + \binom{n}{n} X^n$$

Binomial Coefficients

binomial
expression


What is the coefficient
of $(X_1^{r_1}X_2^{r_2}\ldots X_k^{r_k})$
in the expansion of
$(X_1+X_2+X_3+\ldots+X_k)^n$?



What is the coefficient of $(X_1^{r_1}X_2^{r_2}\ldots X_k^{r_k})$ in the expansion of $(X_1+X_2+X_3+\ldots+X_k)^n$?

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

And now for some more counting…

Power Series Representation

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

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For k>n,

$$\binom{n}{k} = 0$$

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"Product form" or
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For k>n,

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For k>n,

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"Product form" or
"Generating form"

"Power Series" or "Taylor Series" Expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

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The number of subsets of an n-element set

By varying x, we can discover new identities:

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

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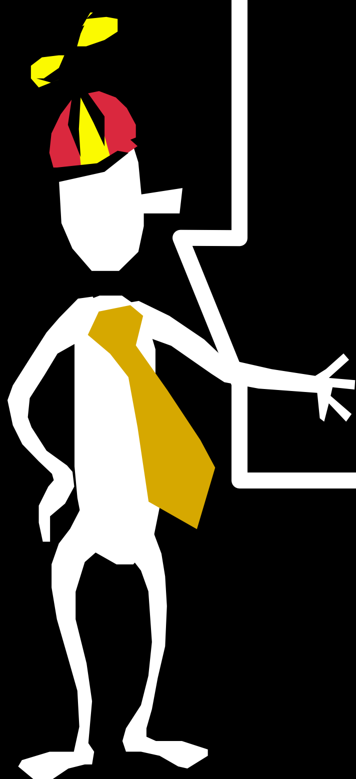
Let x = -1, $$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Equivalently, $$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

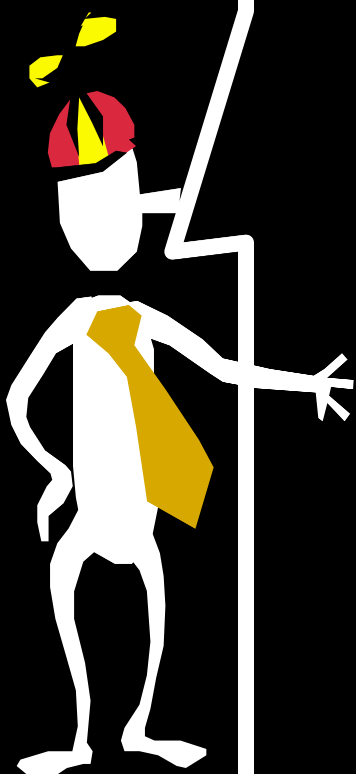
The number of subsets with even size is the same as the number of subsets with odd size

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$



Proofs that work by manipulating algebraic forms are called “algebraic” arguments.
Proofs that build a bijection are called “combinatorial” arguments

$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We just saw an algebraic proof that

$$|O_n| = |E_n|$$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones

Let E_n be the set of binary strings of length n with an even number of ones

A combinatorial proof must construct a bijection between O_n and E_n

An Attempt at a Bijection

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Let f_n be the function that takes an n-bit string and flips all its bits

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110011 → 001100
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110011 → 001100
101010 → 010101

Uh oh. Complementing maps evens to evens!

A Correspondence That Works for all n

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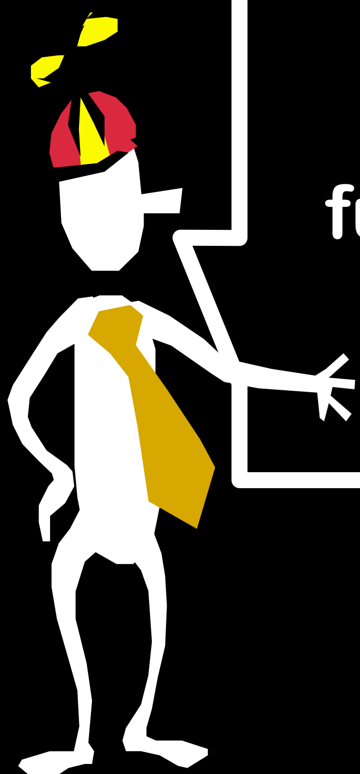
0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

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Set of all k-subsets of {1..n}

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Set of all
k-subsets
of $\{1..n\}$

Either we
do not pick n:
then we have to
pick k elements
out of the
remaining n-1.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Set of all k-subsets of {1..n}

Either we do not pick n: then we have to pick k elements out of the remaining n-1.

Or we do pick n: then we have to pick k-1 elts. out of the remaining n-1.

The Binomial Formula

$$(1+X)^0 = 1$$

$$(1+X)^1 = 1 + 1X$$

$$(1+X)^2 = 1 + 2X + 1X^2$$

$$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$$

$$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$$

The Binomial Formula

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$$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$$

Pascal's Triangle: k^{th} row are coefficients of $(1+X)^k$

The Binomial Formula

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle: k^{th} row are coefficients of $(1+X)^k$

Inductive definition of k^{th} entry of n^{th} row:
Pascal(n,0) = Pascal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

“Pascal’s Triangle”

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3} = 1$$

“Pascal’s Triangle”

$$\binom{0}{0} = 1$$

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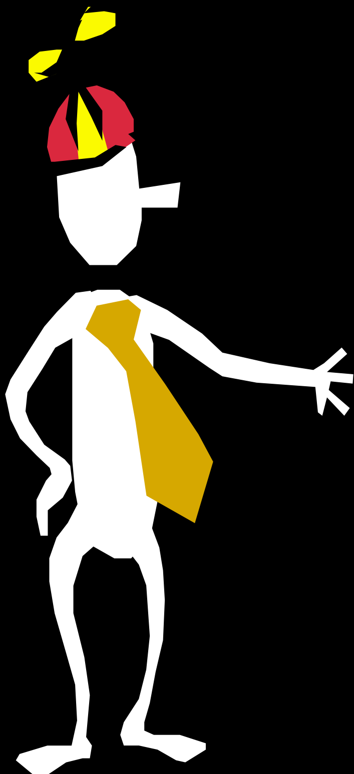
$$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$$

- Al-Karaji, Baghdad 953-1029
- Chu Shin-Chieh 1303
- Blaise Pascal 1654

Pascal's Triangle



"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand"

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1

1 + 1

1 + 2 + 1

1 + 3 + 3 + 1

1 + 4 + 6 + 4 + 1

1 + 5 + 10 + 10 + 5 + 1

1 + 6 + 15 + 20 + 15 + 6 + 1

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1 = 1

1 + 1

1 + 2 + 1

1 + 3 + 3 + 1

1 + 4 + 6 + 4 + 1

1 + 5 + 10 + 10 + 5 + 1

1 + 6 + 15 + 20 + 15 + 6 + 1

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$1 = 1$

$1 + 1 = 2$

$1 + 2 + 1$

$1 + 3 + 3 + 1$

$1 + 4 + 6 + 4 + 1$

$1 + 5 + 10 + 10 + 5 + 1$

$1 + 6 + 15 + 20 + 15 + 6 + 1$

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$$1 = 1$$

$$1 + 1 = 2$$

$$1 + 2 + 1 = 4$$

$$1 + 3 + 3 + 1$$

$$1 + 4 + 6 + 4 + 1$$

$$1 + 5 + 10 + 10 + 5 + 1$$

$$1 + 6 + 15 + 20 + 15 + 6 + 1$$

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1	$= 1$
$1 + 1$	$= 2$
$1 + 2 + 1$	$= 4$
$1 + 3 + 3 + 1$	$= 8$
$1 + 4 + 6 + 4 + 1$	
$1 + 5 + 10 + 10 + 5 + 1$	
$1 + 6 + 15 + 20 + 15 + 6 + 1$	

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1	$= 1$
$1 + 1$	$= 2$
$1 + 2 + 1$	$= 4$
$1 + 3 + 3 + 1$	$= 8$
$1 + 4 + 6 + 4 + 1$	$= 16$
$1 + 5 + 10 + 10 + 5 + 1$	
$1 + 6 + 15 + 20 + 15 + 6 + 1$	

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$$1 = 1$$

$$1 + 1 = 2$$

$$1 + 2 + 1 = 4$$

$$1 + 3 + 3 + 1 = 8$$

$$1 + 4 + 6 + 4 + 1 = 16$$

$$1 + 5 + 10 + 10 + 5 + 1 = 32$$

$$1 + 6 + 15 + 20 + 15 + 6 + 1$$

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$$1 = 1$$

$$1 + 1 = 2$$

$$1 + 2 + 1 = 4$$

$$1 + 3 + 3 + 1 = 8$$

$$1 + 4 + 6 + 4 + 1 = 16$$

$$1 + 5 + 10 + 10 + 5 + 1 = 32$$

$$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$$

Odds and Evens

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Odds and Evens

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Odds and Evens

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

1 + 15 + 15 + 1

Odds and Evens

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 + 15 + 15 + 1

Odds and Evens

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 + 15 + 15 + 1 = 6 + 20 + 6

Summing on 1st Avenue

$$\sum_{i=1}^{n} i \qquad \sum_{i=1}^{n} \binom{i}{1}$$

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Summing on 1st Avenue

$$\sum_{i=1}^{n} i \qquad \sum_{i=1}^{n} \binom{i}{1}$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Summing on 1st Avenue

$$\sum_{i=1}^{n} i \qquad \sum_{i=1}^{n} \binom{i}{1}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Summing on 1st Avenue

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} \binom{i}{1}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Summing on 1st Avenue

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} \binom{i}{1} = \binom{n+1}{2}$$

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Summing on k^{th} Avenue

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

$$\sum_{i=k}^{n} \binom{i}{k}$$

Summing on kth Avenue

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$$\sum_{i=k}^{n} \binom{i}{k}$$

Summing on kth Avenue

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$$\sum_{i=k}^{n} \binom{i}{k}$$

Summing on k^{th} Avenue

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1

1 1 = 2

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1

1 1 = 2

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1

1 1 = 2

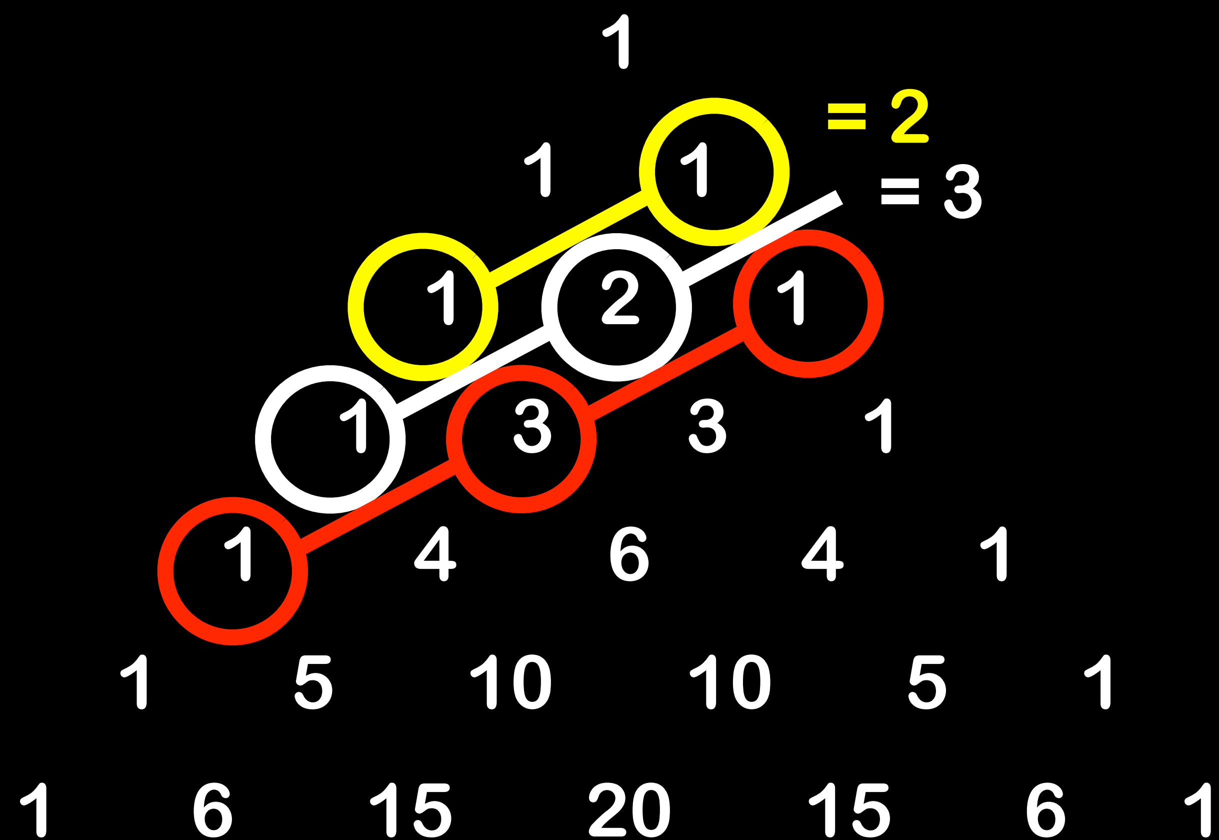
1 2 1 = 3

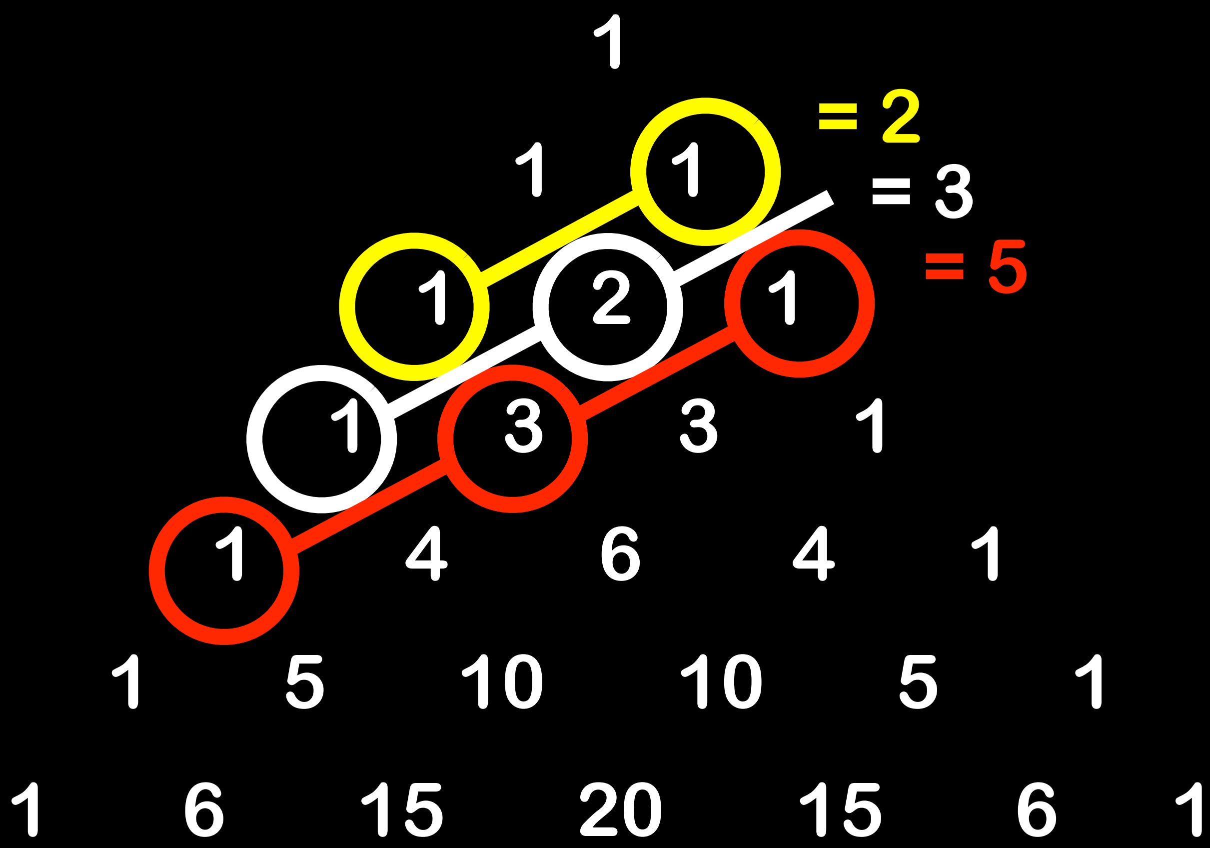
1 3 3 1

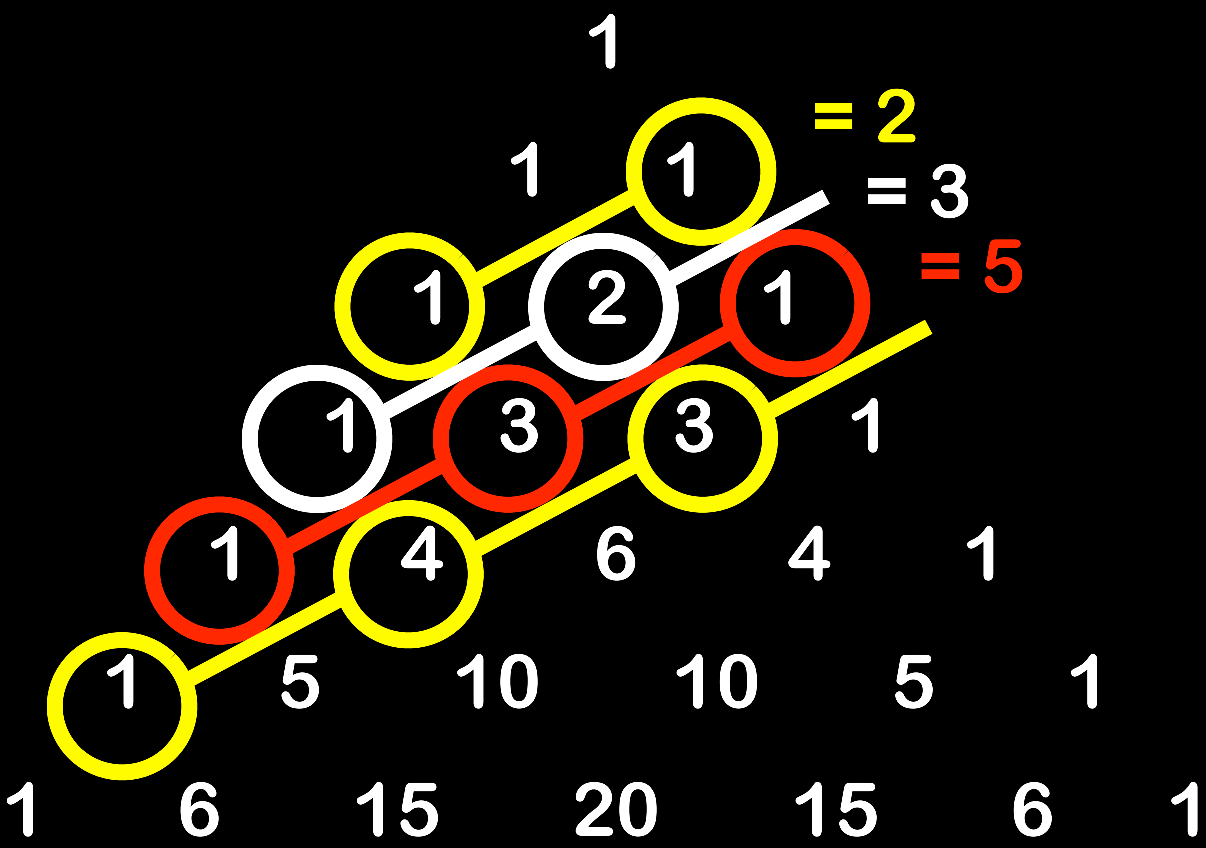
1 4 6 4 1

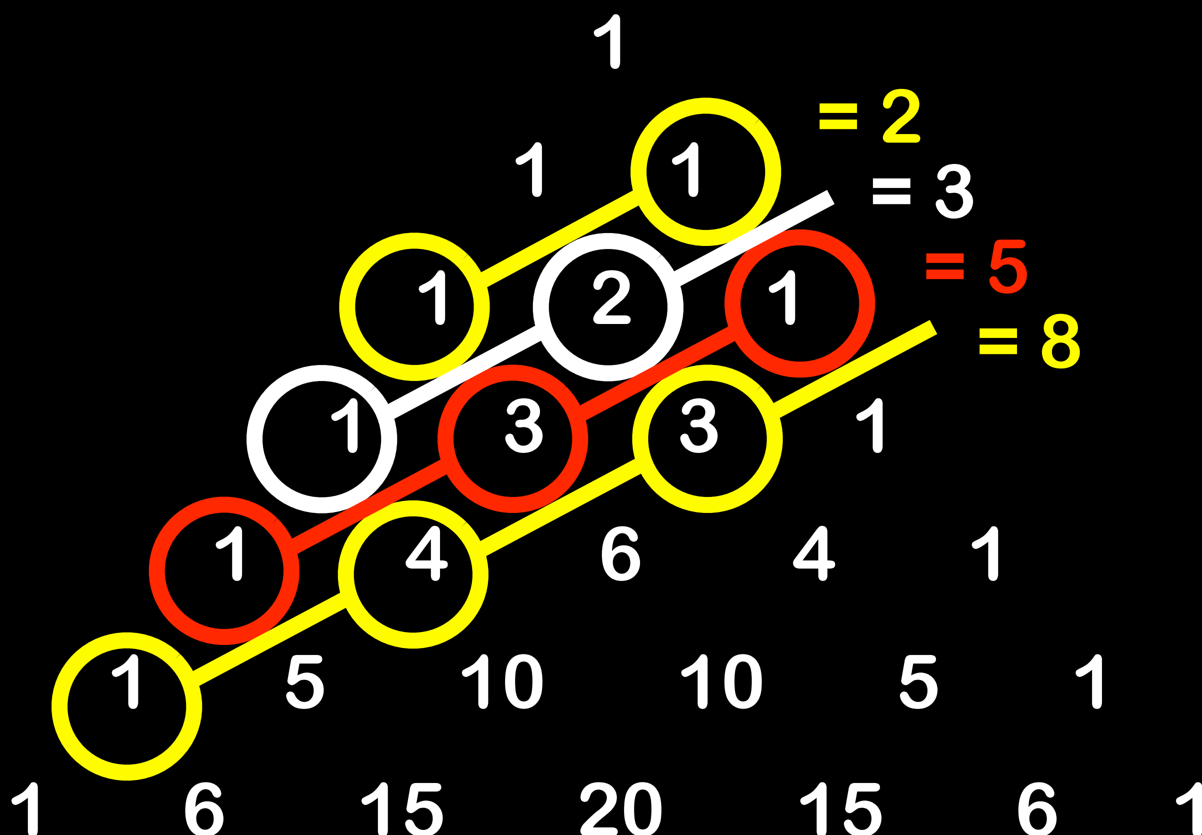
1 5 10 10 5 1

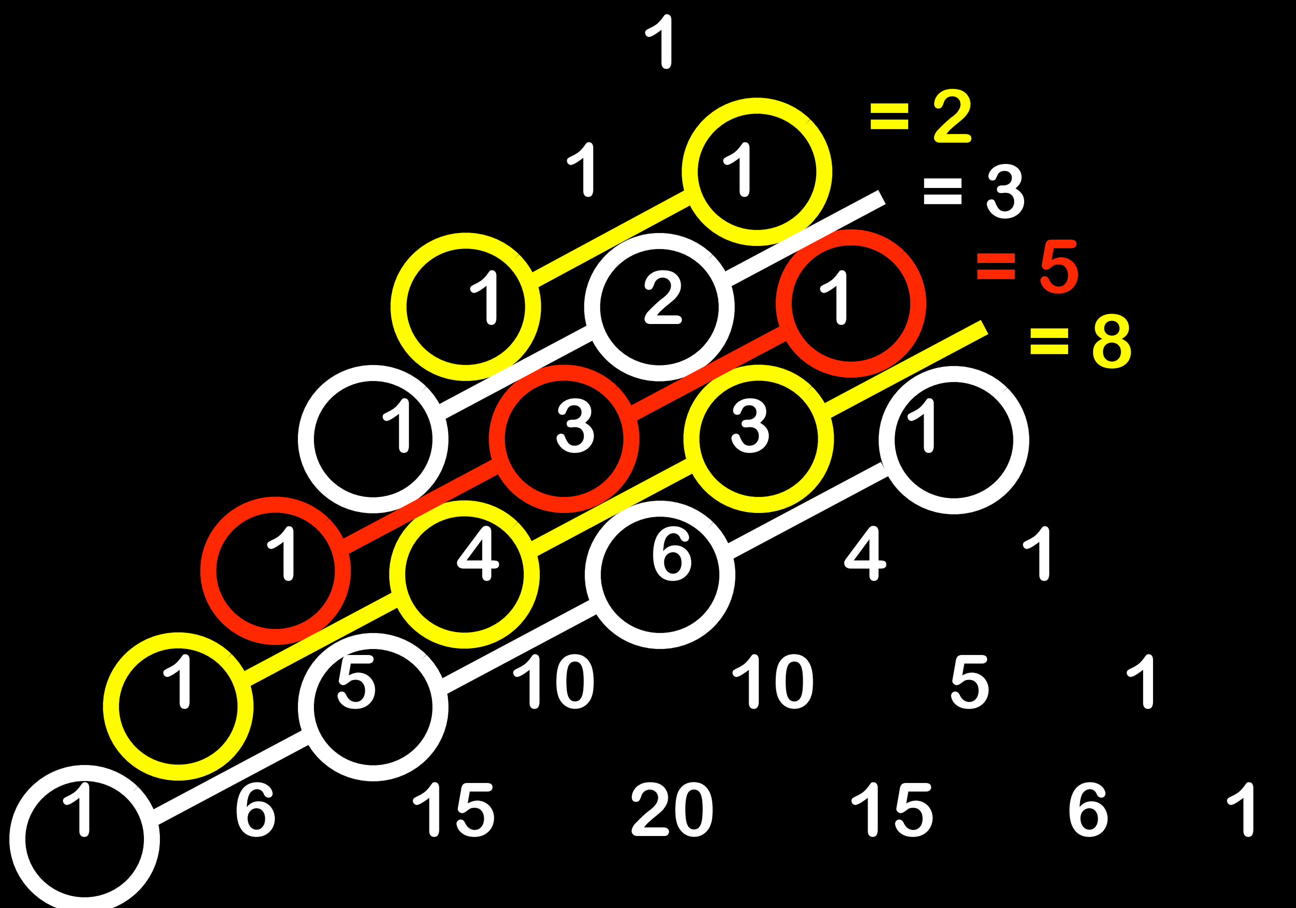
1 6 15 20 15 6 1

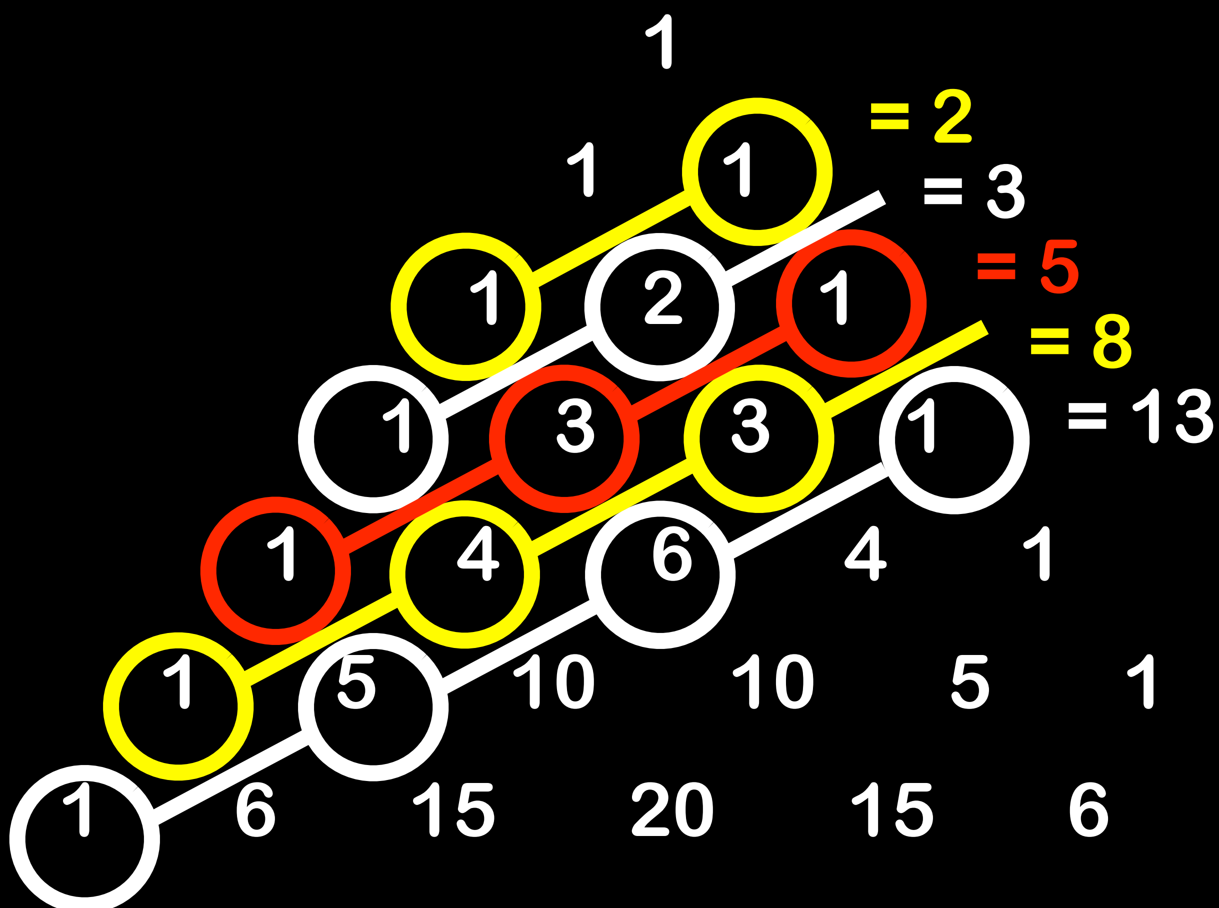

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
= 2
= 3


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
= 2
= 3
= 5

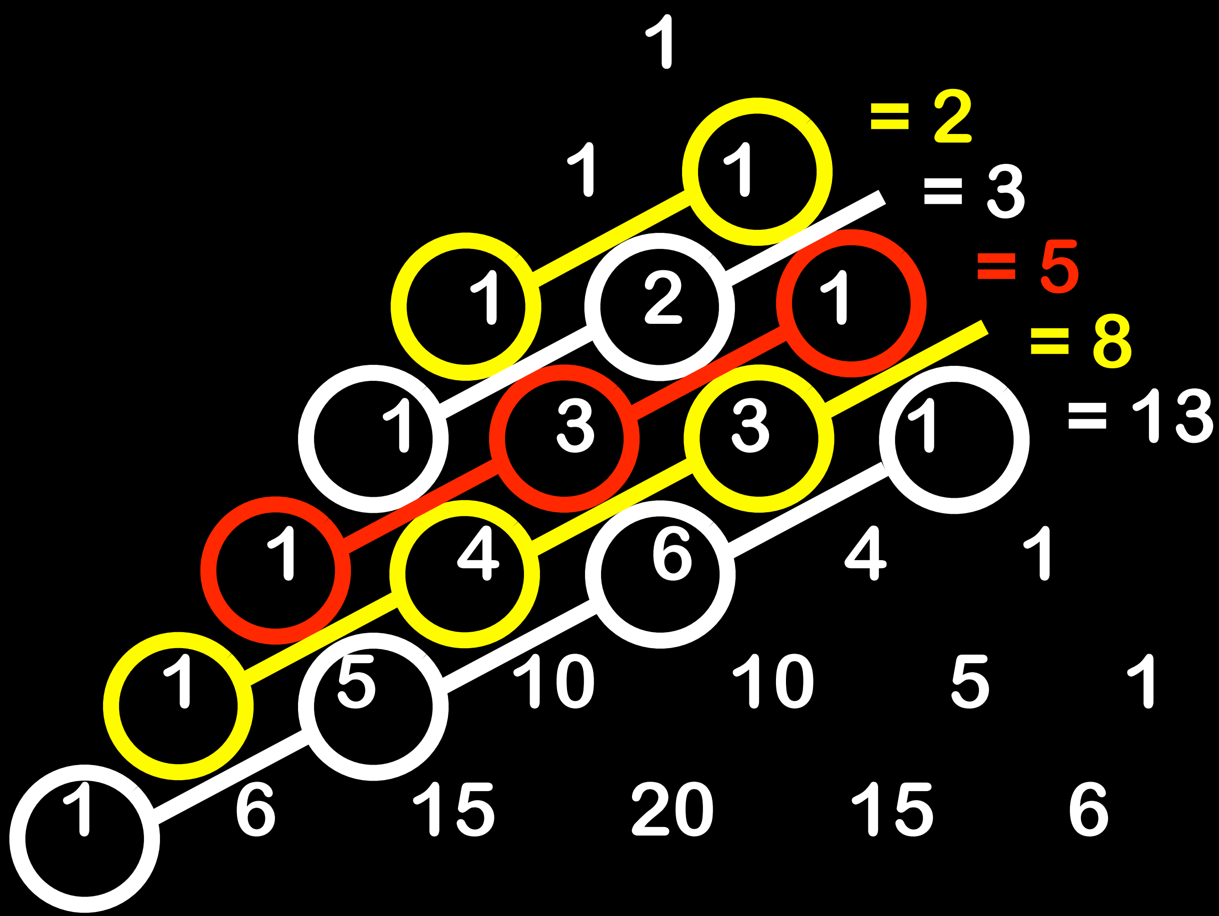

1
1 1
= 2
= 3
1 2 1
= 5
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
= 2
= 3
= 5
= 8


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
= 2
= 3
= 5
= 8


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
= 2
= 3
= 5
= 8
= 13

Fibonacci Numbers



Sums of Squares

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Sums of Squares

1

1 1

1^2 2^2 1^2

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Sums of Squares

1

1 1

$$\boxed{1^2 \quad 2^2 \quad 1^2}$$

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Sums of Squares

$$1$$

$$1 \quad 1$$

$$\boxed{1^2 \quad 2^2 \quad 1^2}$$

$$1^2 \quad 3^2 \quad 3^2 \quad 1^2$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

Sums of Squares

1

1 1

$1^2 \quad 2^2 \quad 1^2$

$1^2 \quad 3^2 \quad 3^2 \quad 1^2$

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Al-Karaji Squares

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Al-Karaji Squares

						1						
					1		1					
				1		2 +2·1		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Al-Karaji Squares

						1							
					1		1						
				1		2 +2·1							= 4
			1		3		3		1				
		1		4		6		4		1			
	1		5		10		10		5		1		
1		6		15		20		15		6		1	

Al-Karaji Squares

1

1 1

1 2 $+2\cdot$ 1 $= 4$

1 3 $+2\cdot$ 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Al-Karaji Squares

						1							
					1		1						
				1		2 +2·1		1					= 4
			1		3 +2·		3		1				= 9
		1		4		6		4		1			
	1		5		10		10		5		1		
1		6		15		20		15		6		1	

Al-Karaji Squares

						1						
					1		1					
				1		2 +2·		1				= 4
			1		3 +2·		3		1			= 9
		1		4 +2·		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Al-Karaji Squares

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 + 2 \cdot 1 \qquad = 4 \\
1 \quad 3 + 2 \cdot 3 \quad 1 \qquad = 9 \\
1 \quad 4 + 2 \cdot 6 \quad 4 \quad 1 \qquad = 16 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}
$$

Al-Karaji Squares

$$1$$

$$1 \quad 1$$

$$1 \quad 2 + 2 \cdot 1 \qquad = 4$$

$$1 \quad 3 + 2 \cdot 3 \quad 1 \qquad = 9$$

$$1 \quad 4 + 2 \cdot 6 \quad 4 \quad 1 \qquad = 16$$

$$1 \quad 5 + 2 \cdot 10 \quad 10 \quad 5 \quad 1$$

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

Al-Karaji Squares

1

1 1

1 $2 + 2 \cdot 1$ $= 4$

1 $3 + 2 \cdot 3$ 1 $= 9$

1 $4 + 2 \cdot 6$ 4 1 $= 16$

1 $5 + 2 \cdot 10$ 10 5 1 $= 25$

1 6 15 20 15 6 1

Al-Karaji Squares

$$1$$

$$1 \quad 1$$

$$1 \quad 2 + 2 \cdot 1 \qquad = 4$$

$$1 \quad 3 + 2 \cdot 3 \quad 1 \qquad = 9$$

$$1 \quad 4 + 2 \cdot 6 \quad 4 \quad 1 \qquad = 16$$

$$1 \quad 5 + 2 \cdot 10 \quad 10 \quad 5 \quad 1 \qquad = 25$$

$$1 \quad 6 + 2 \cdot 15 \quad 20 \quad 15 \quad 6 \quad 1$$

Al-Karaji Squares

$$1$$

$$1 \quad 1$$

$$1 \quad 2 + 2 \cdot 1 \qquad = 4$$

$$1 \quad 3 + 2 \cdot 3 \quad 1 \qquad = 9$$

$$1 \quad 4 + 2 \cdot 6 \quad 4 \quad 1 \qquad = 16$$

$$1 \quad 5 + 2 \cdot 10 \quad 10 \quad 5 \quad 1 \qquad = 25$$

$$1 \quad 6 + 2 \cdot 15 \quad 20 \quad 15 \quad 6 \quad 1 \qquad = 36$$

Al-Karaji Squares

$$1$$

$$1 \quad 1 \qquad = 1$$

$$1 \quad 2 \, {+2\cdot} 1 \qquad = 4$$

$$1 \quad 3 \, {+2\cdot} 3 \quad 1 \qquad = 9$$

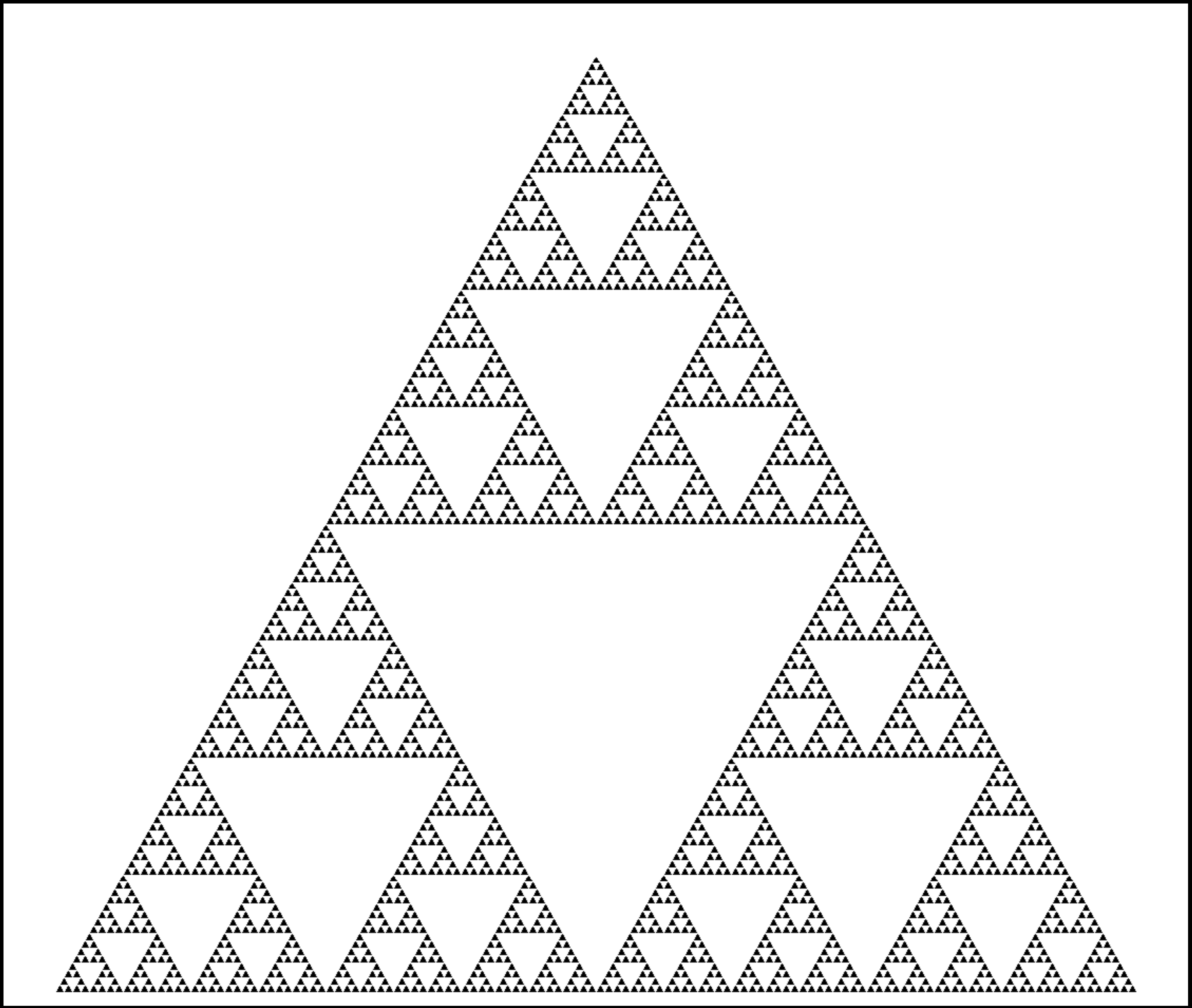
$$1 \quad 4 \, {+2\cdot} 6 \quad 4 \quad 1 \qquad = 16$$

$$1 \quad 5 \, {+2\cdot} 10 \quad 10 \quad 5 \quad 1 \qquad = 25$$

$$1 \quad 6 \, {+2\cdot} 15 \quad 20 \quad 15 \quad 6 \quad 1 \qquad = 36$$

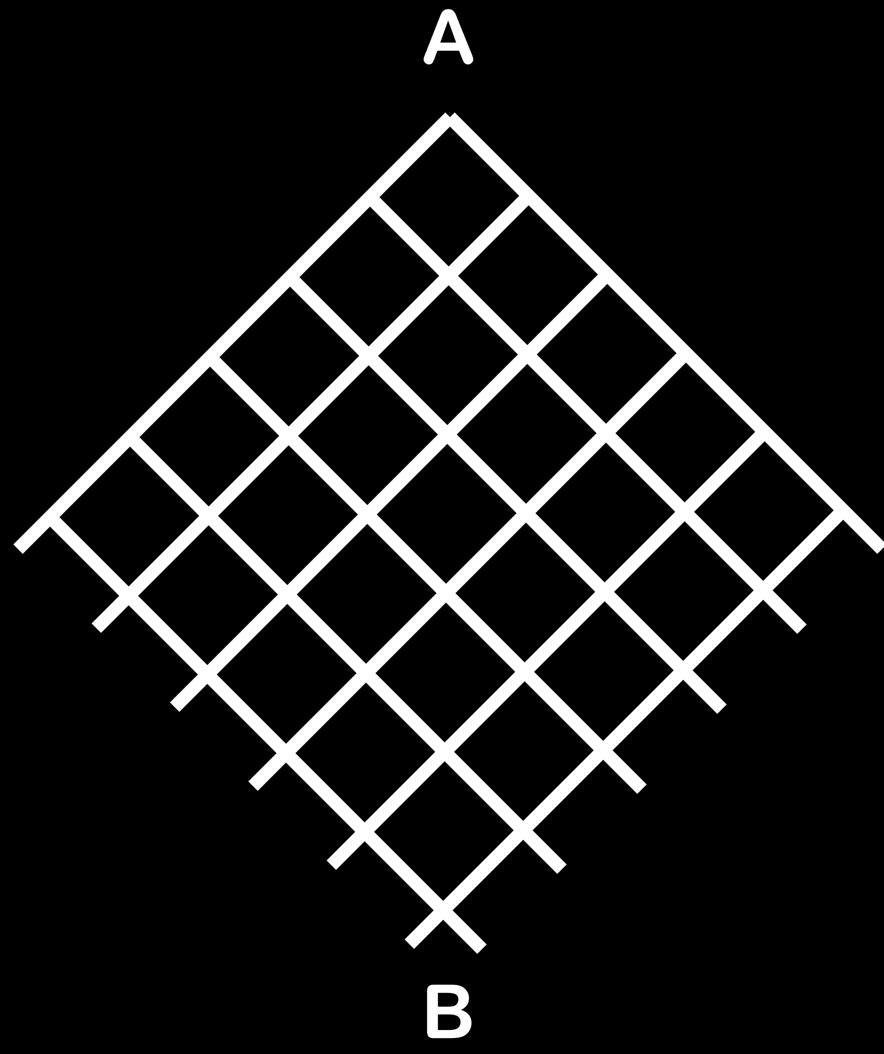
Pascal Mod 2

Pascal Mod 2

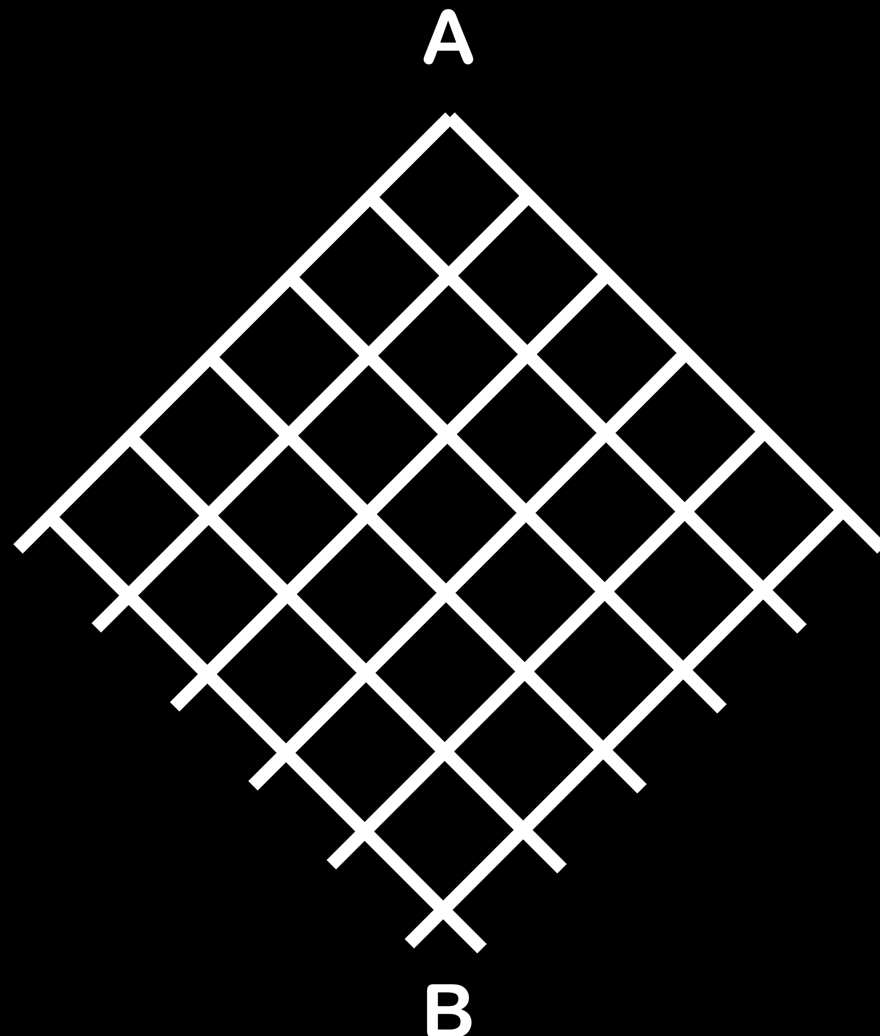


All these properties can be proved inductively and algebraically. We will give combinatorial proofs using the **Manhattan block walking representation of binomial coefficients**

How many shortest routes from A to B?

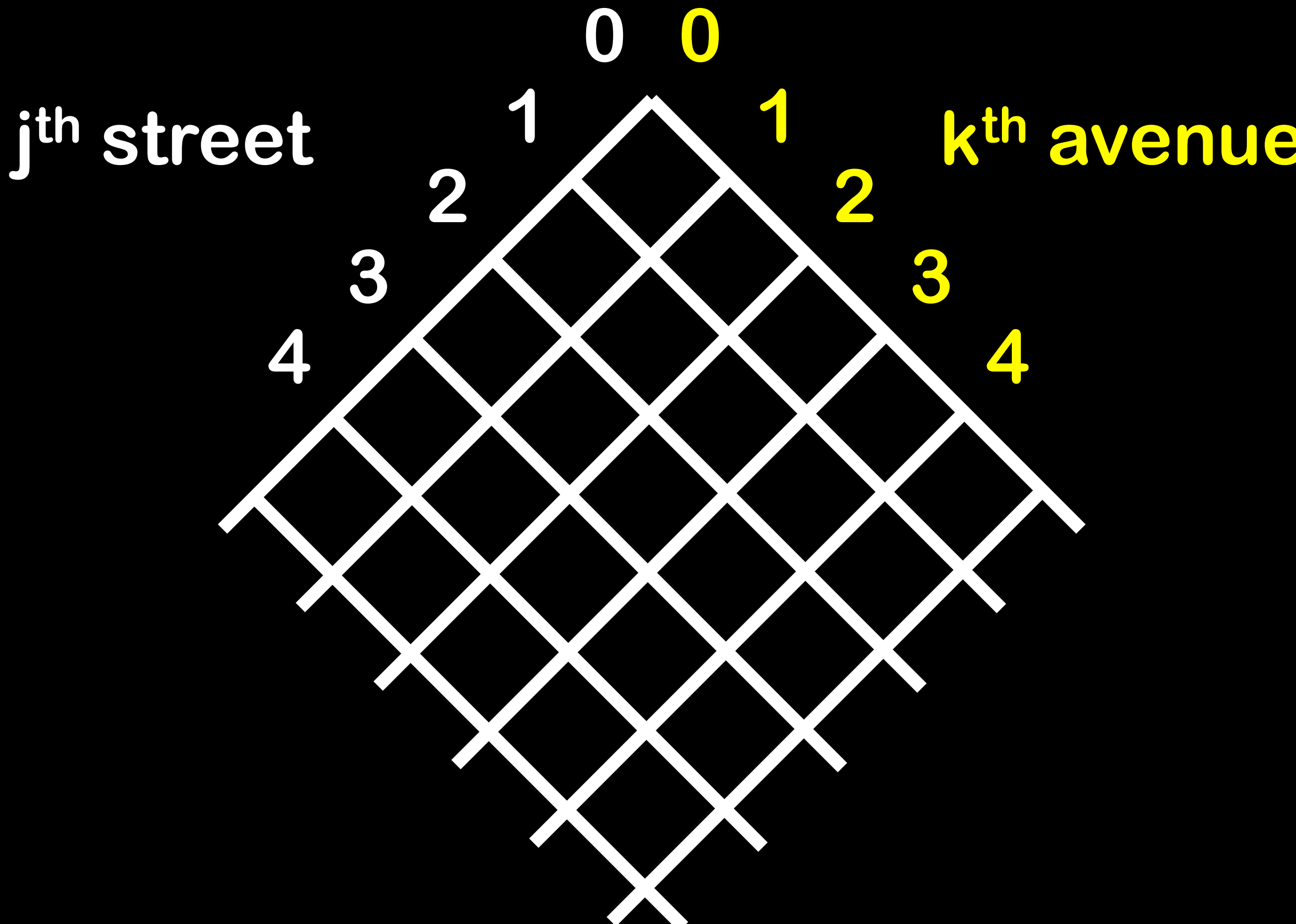


How many shortest routes from A to B?



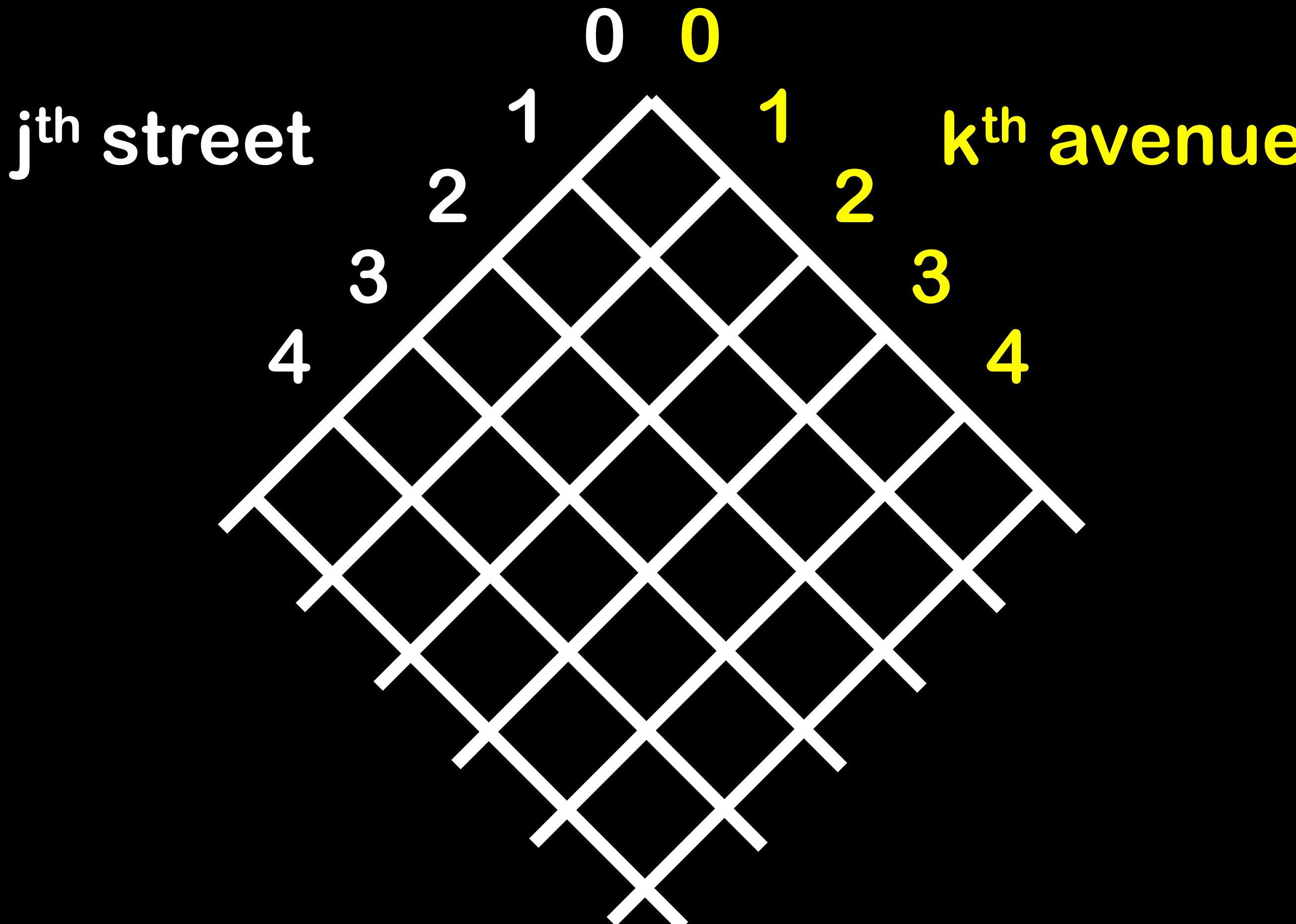
$$\binom{10}{5}$$

Manhattan



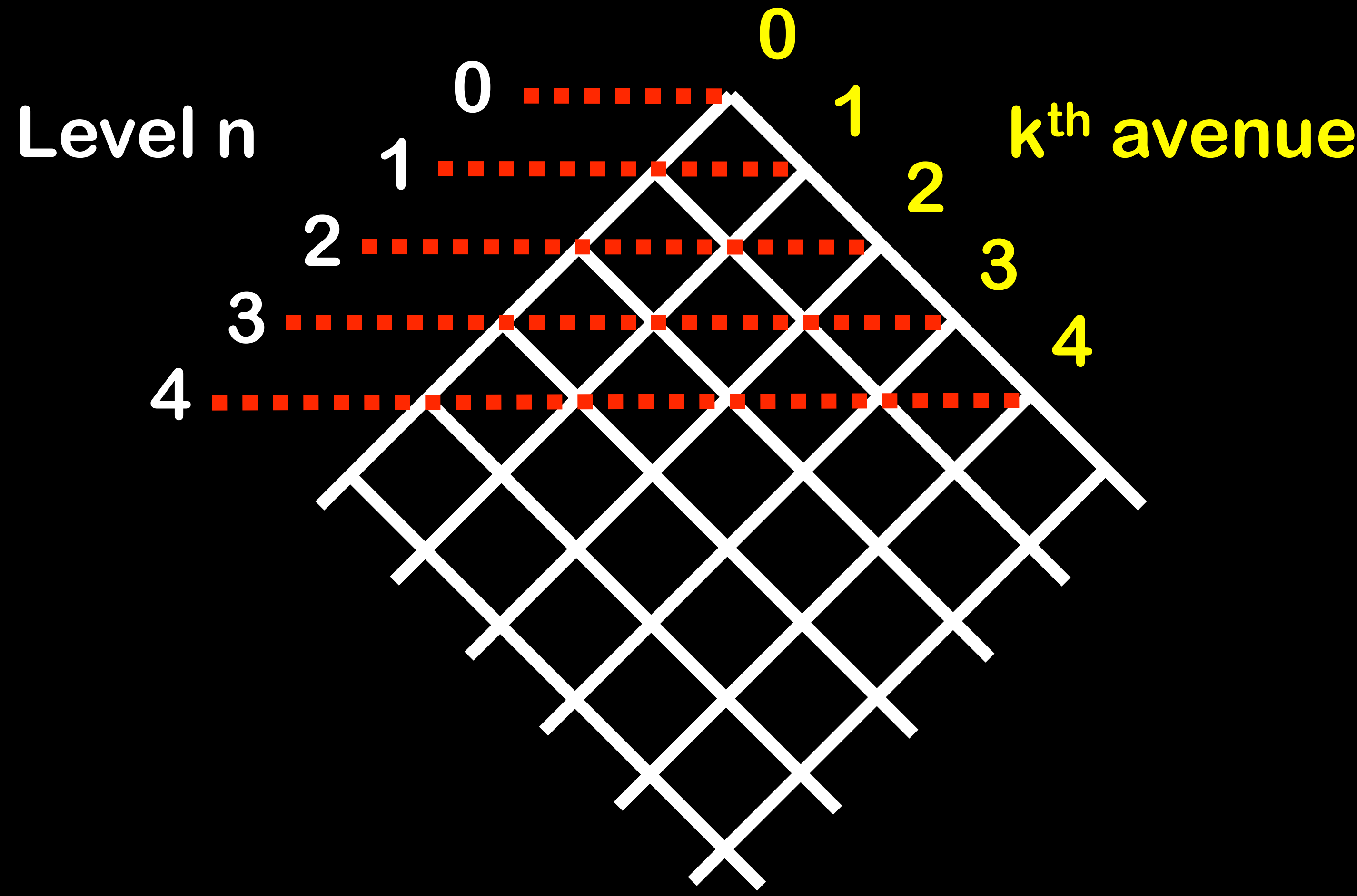
There are shortest routes from (0,0) to (j,k)

Manhattan



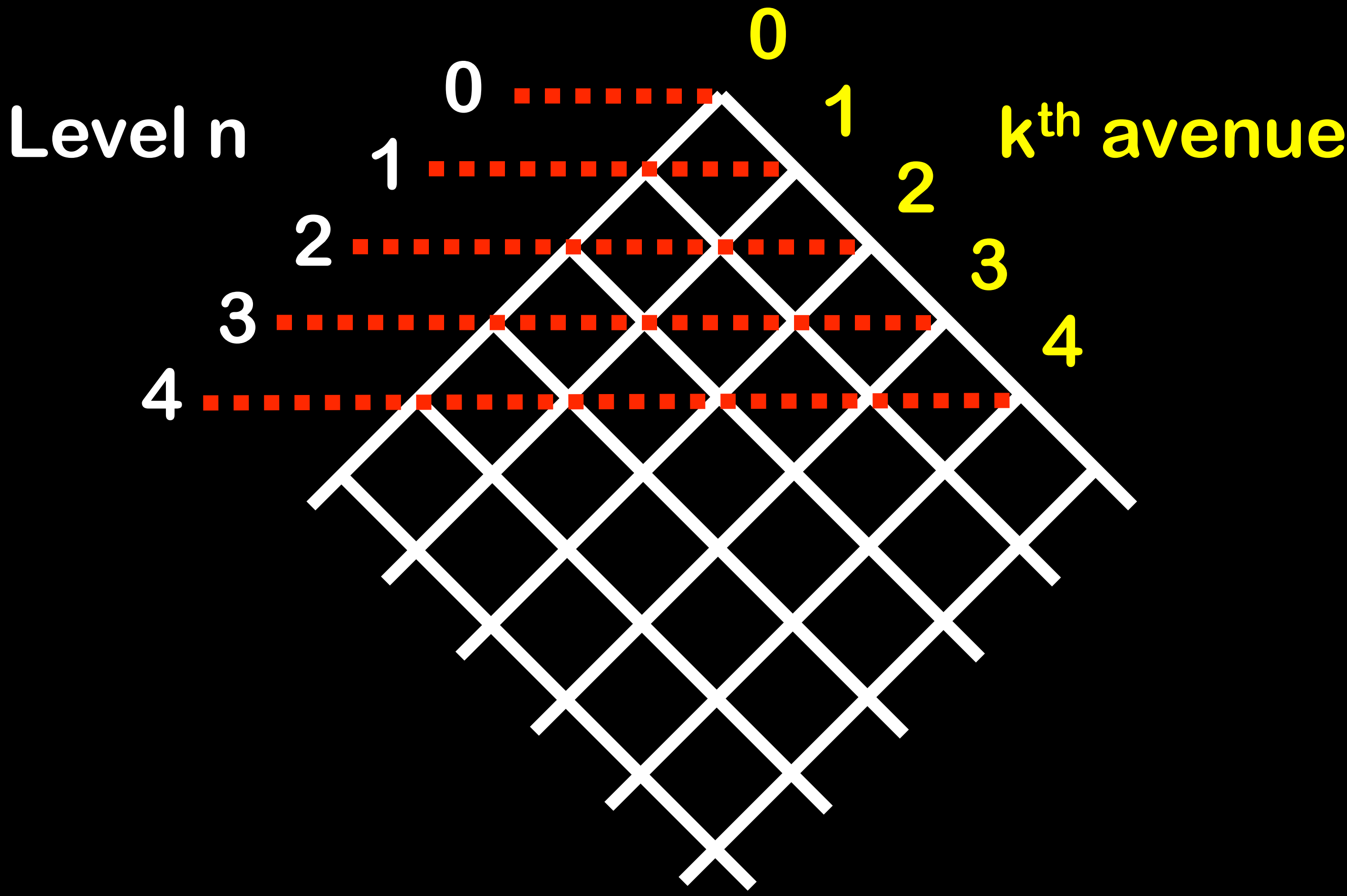
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k)

Manhattan



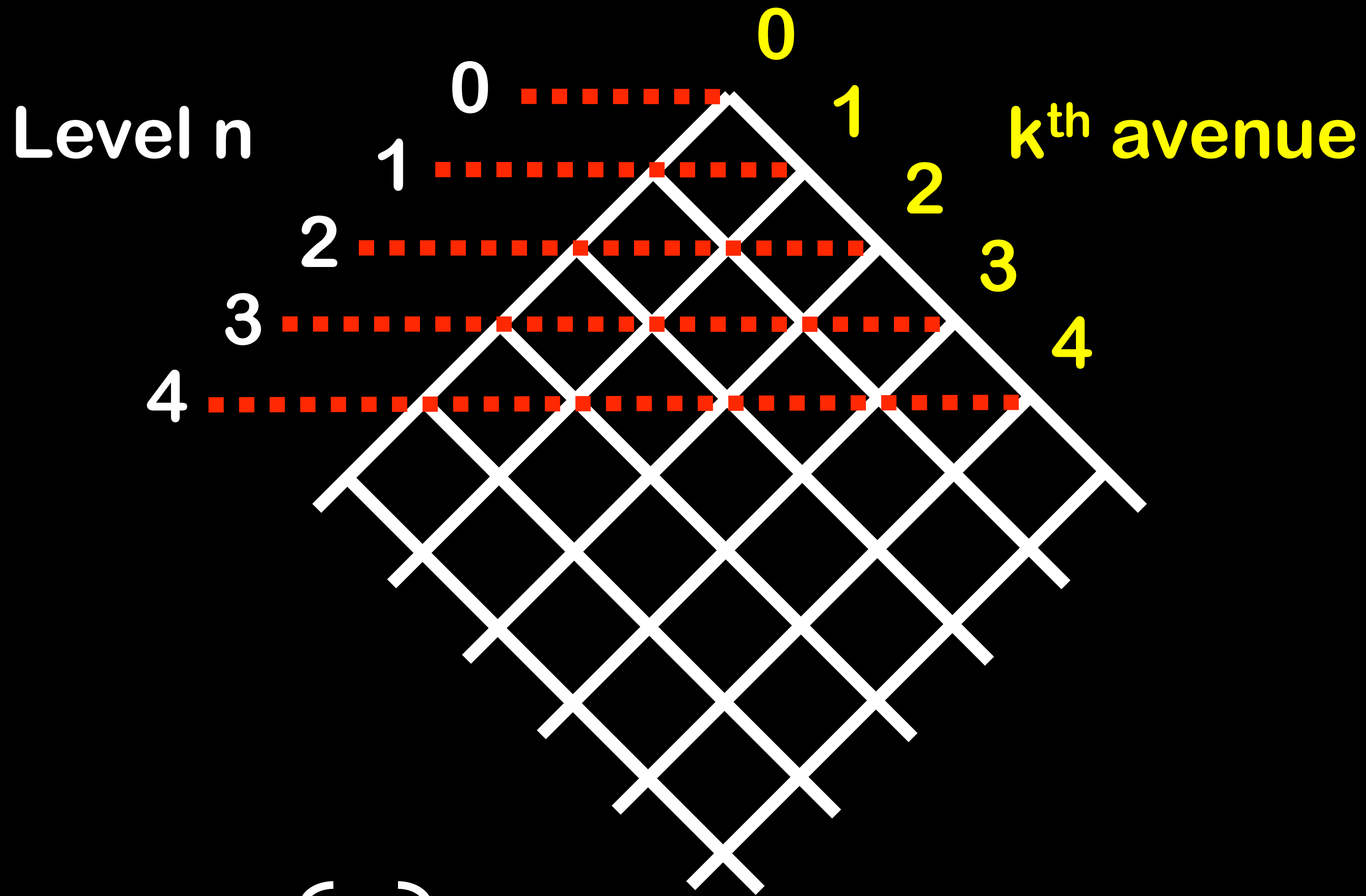
There are shortest routes from (0,0) to (n-k,k)

Manhattan

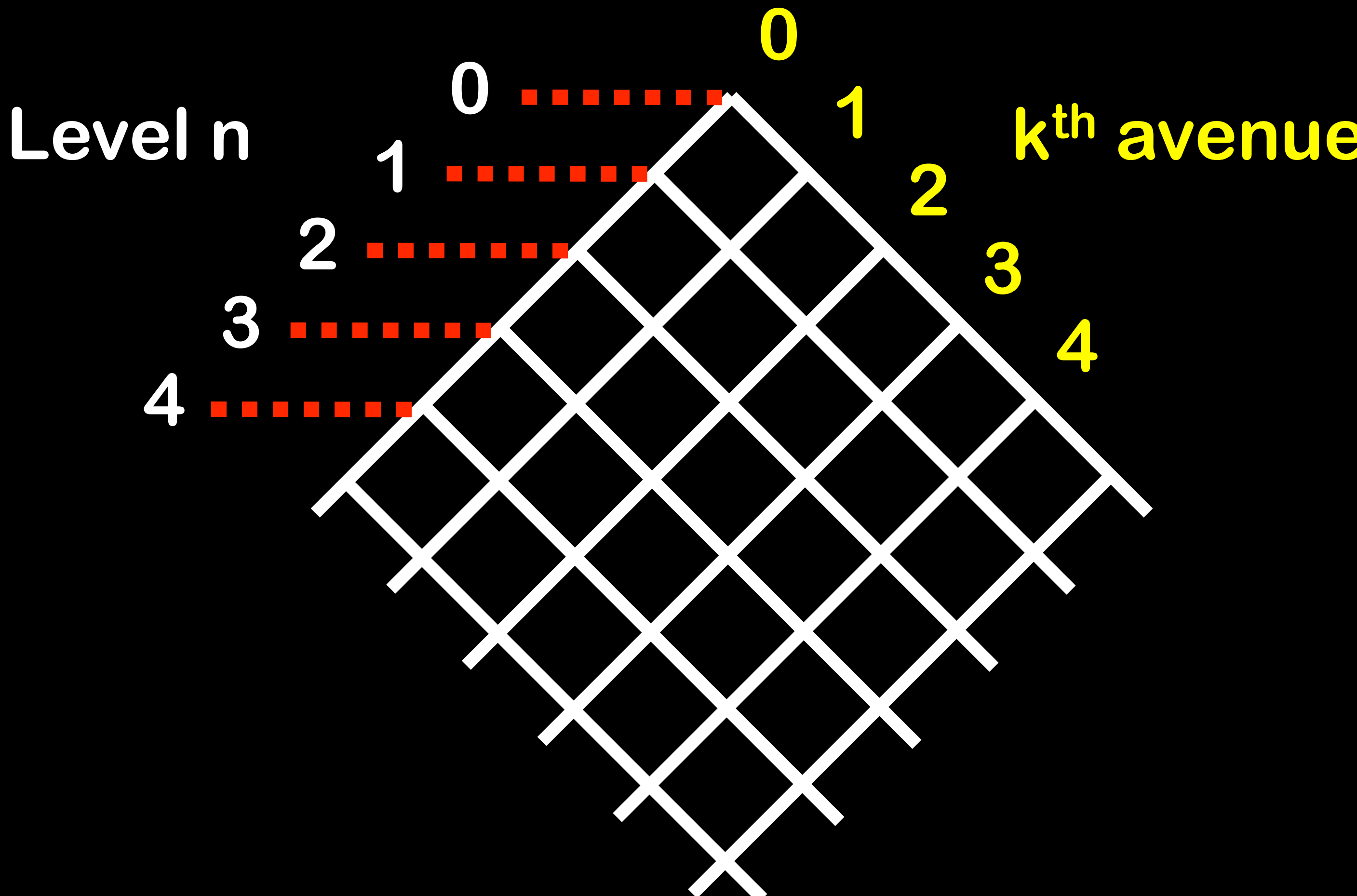


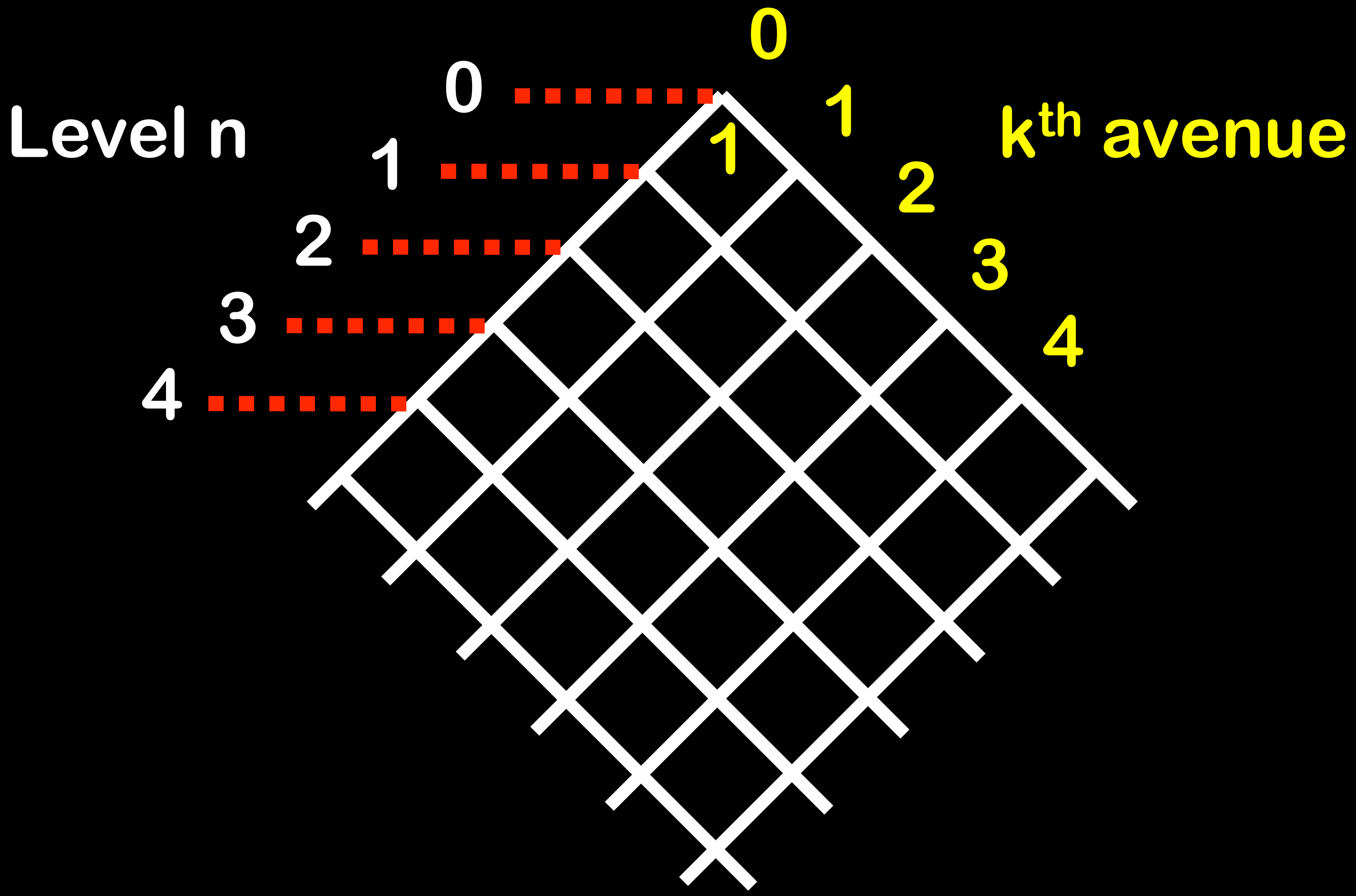
There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k)

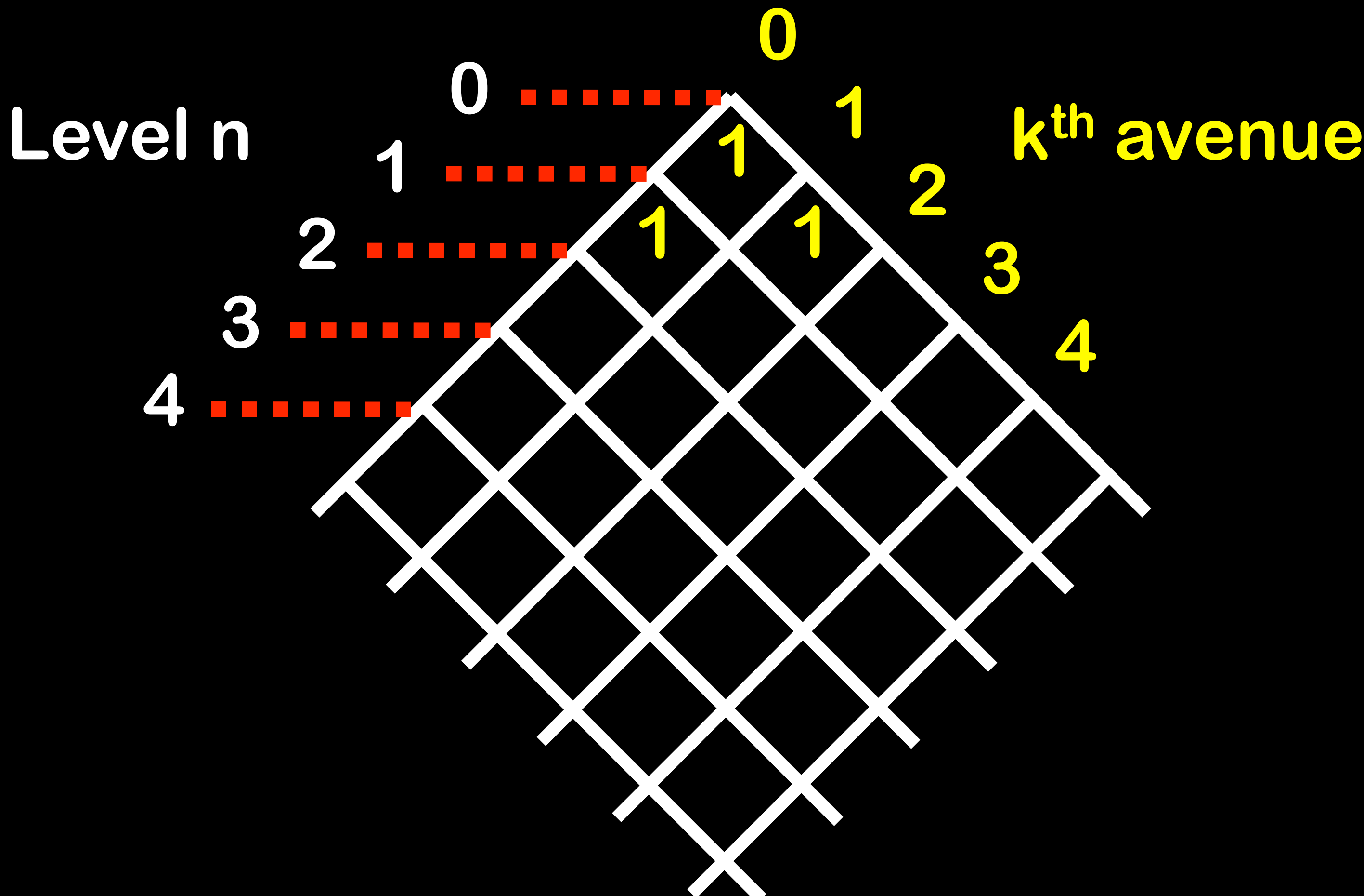
Manhattan



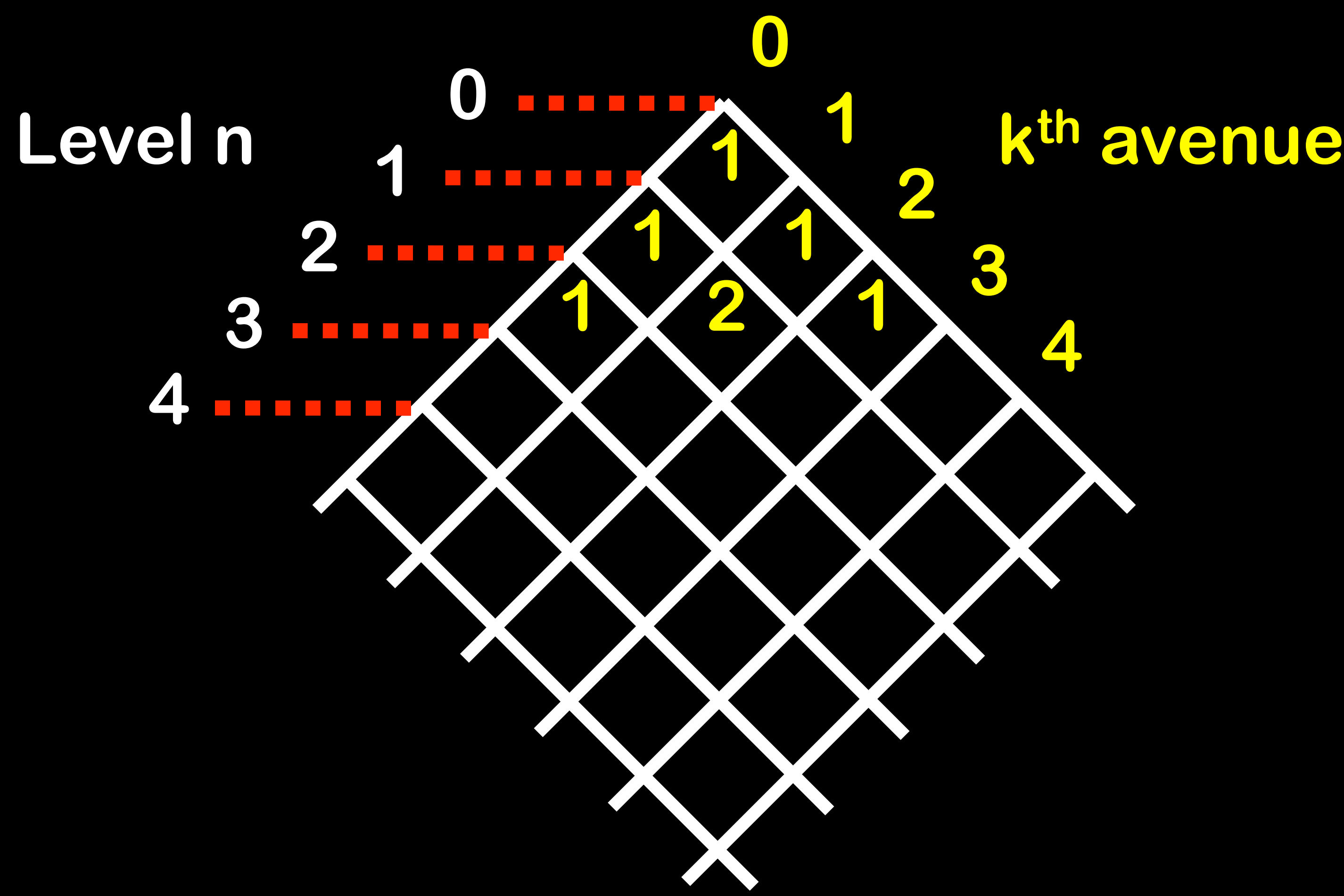
There are $\binom{n}{k}$ shortest routes from (0,0) to level n and k^{th} avenue


Level n
0
1
2
3
4
0
1
2
3
4
k^th avenue

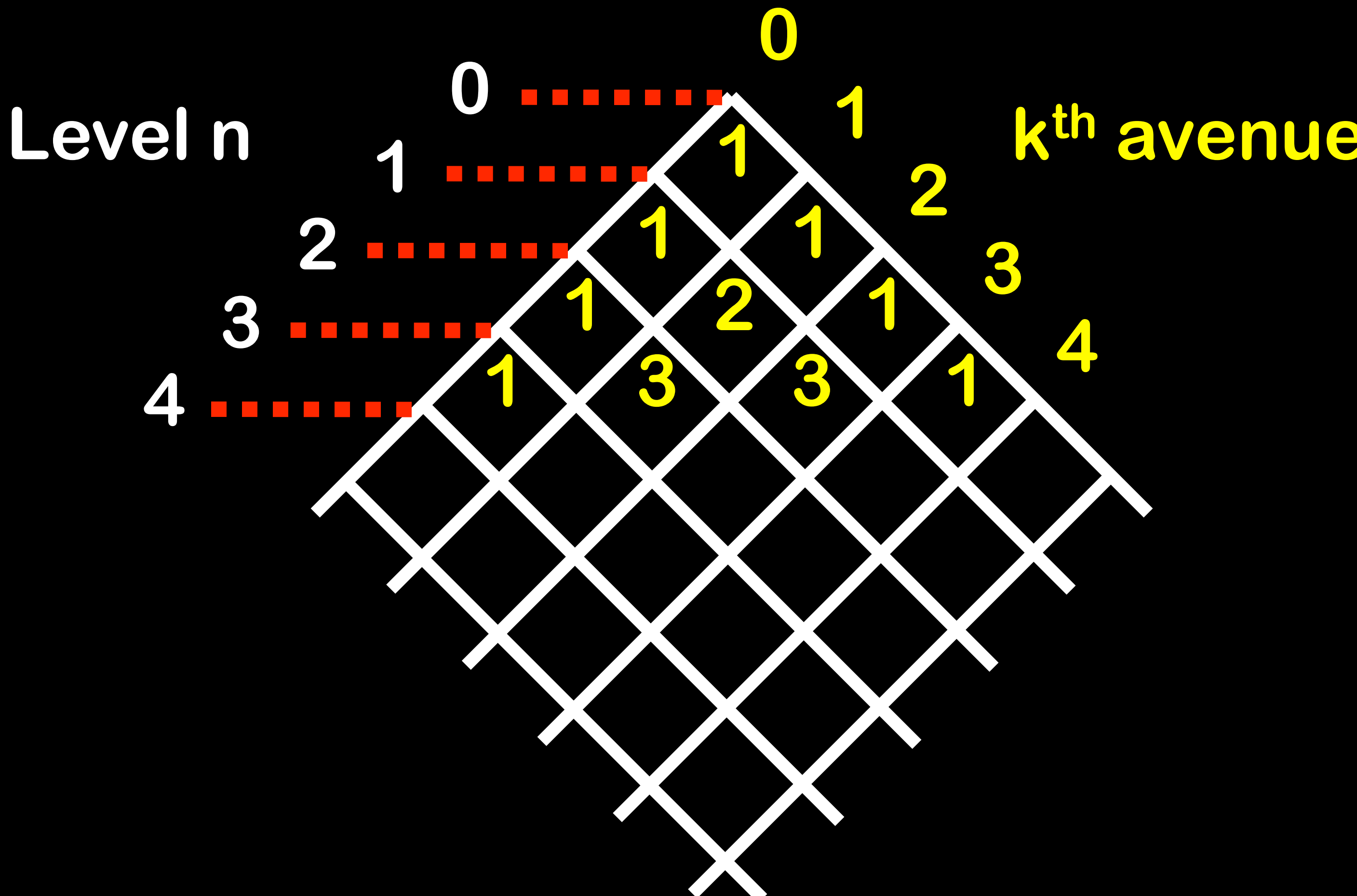

Level n
0
1
2
3
4
0
1
1
2
3
4
k^th avenue

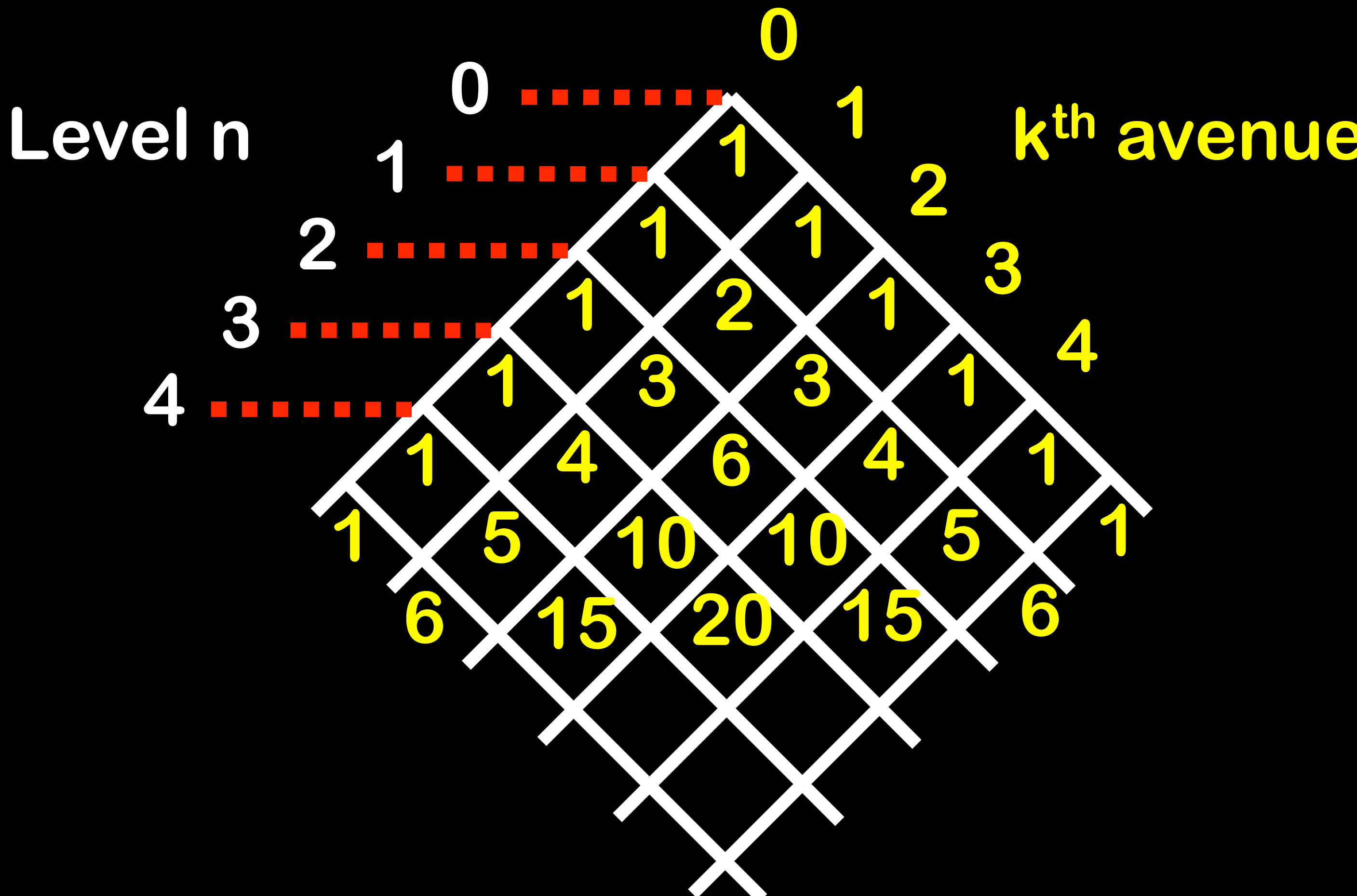


Level n
0
1
2
3
4
kth avenue
0
1
2
3
4
1
1
1



Level n
0
1
2
3
4
kth avenue
0
1
2
3
4
1
1 1
1 2 1


Level n
0
1
2
3
4
0
1
2
3
4
k^th avenue
1
1 1
1 2 1
1 3 3 1



Level n
0
1
2
3
4
kth avenue
0
1
2
3
4
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6

Level n

k^{th} avenue

0
1   1
2   1   1
3   1   2   1
4   1   3   3   1

1   4   6   4   1
1   5   10 + 10   5   1
6   15   20   15   6

Level n

k^{th} avenue

0 1

1 1 1

2 1 2 1

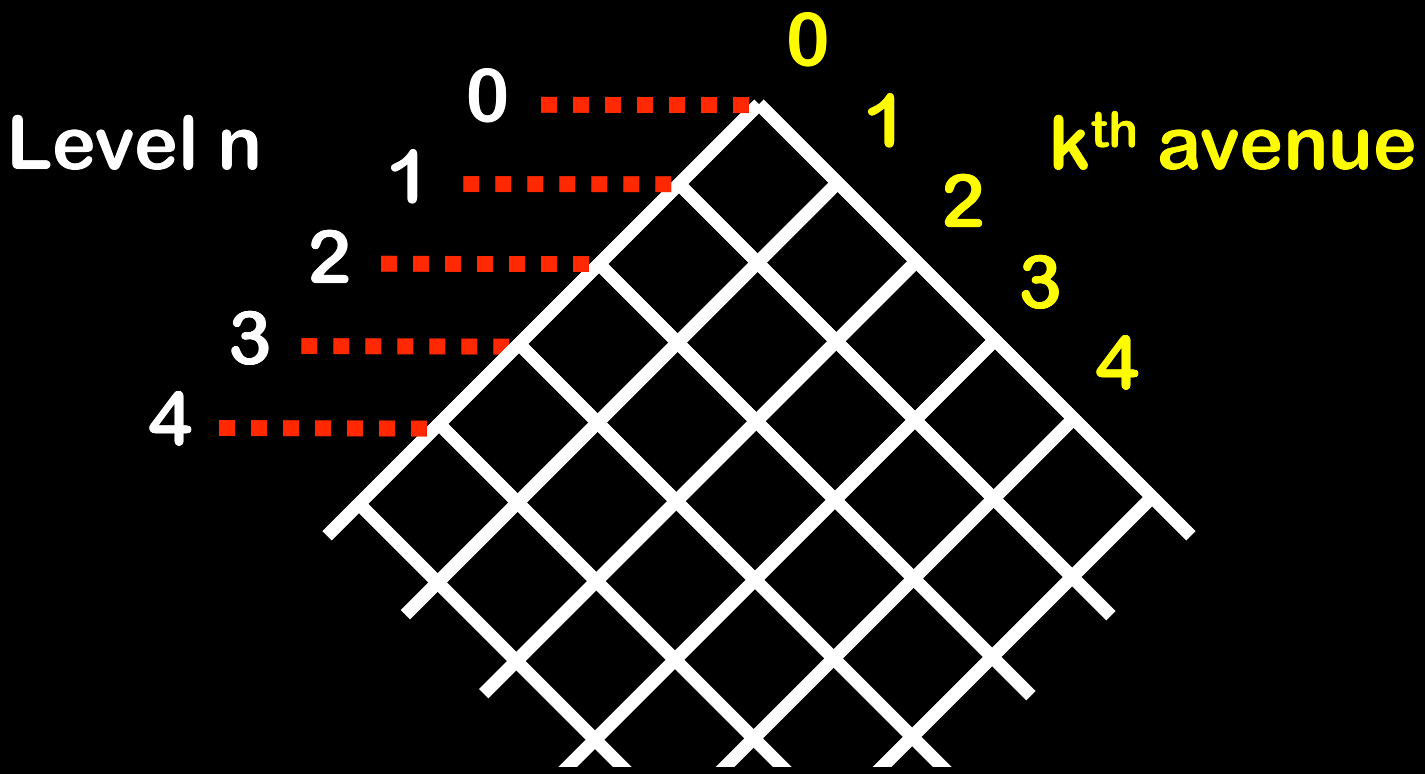
3 1 3 3 1

4 1 4 6 4 1

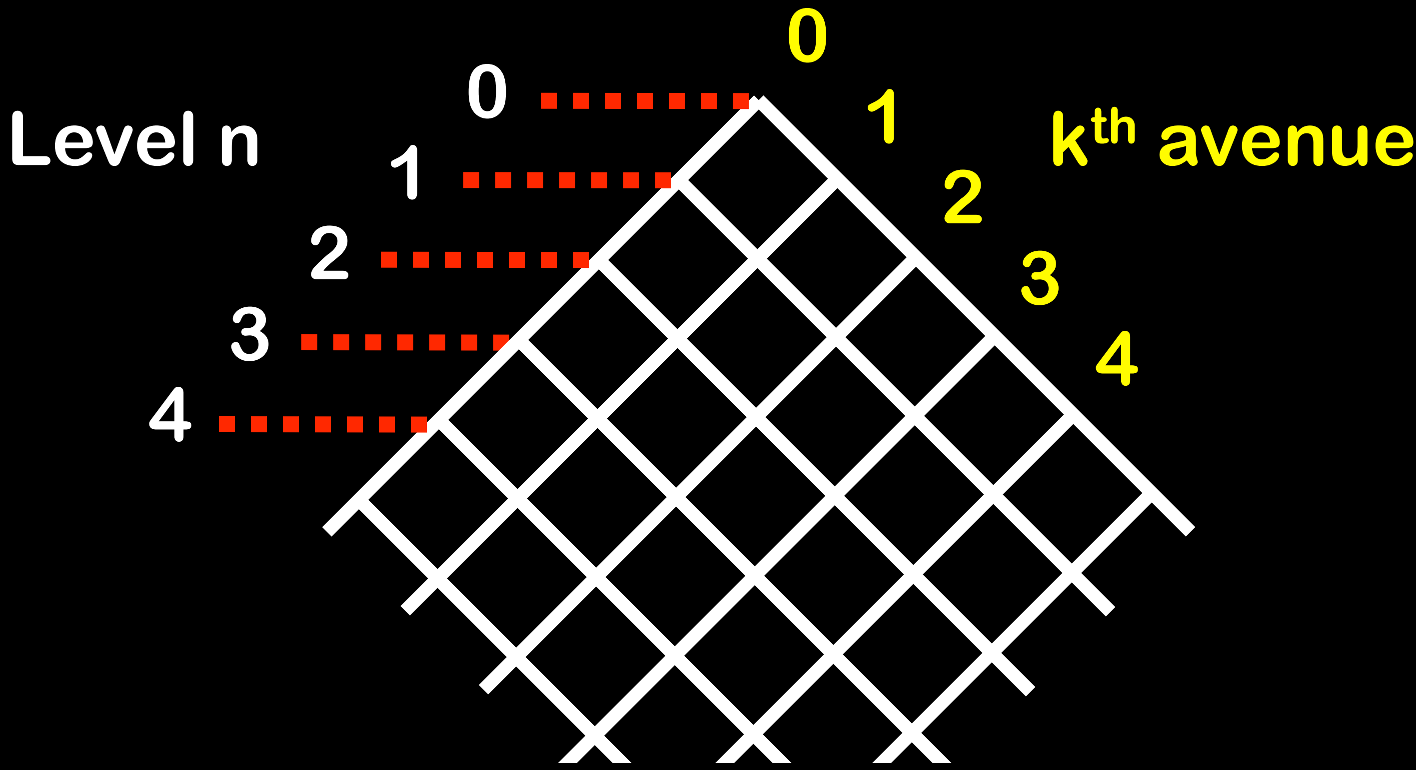
1 5 10 + 10 5 1

6 15 20 15 6

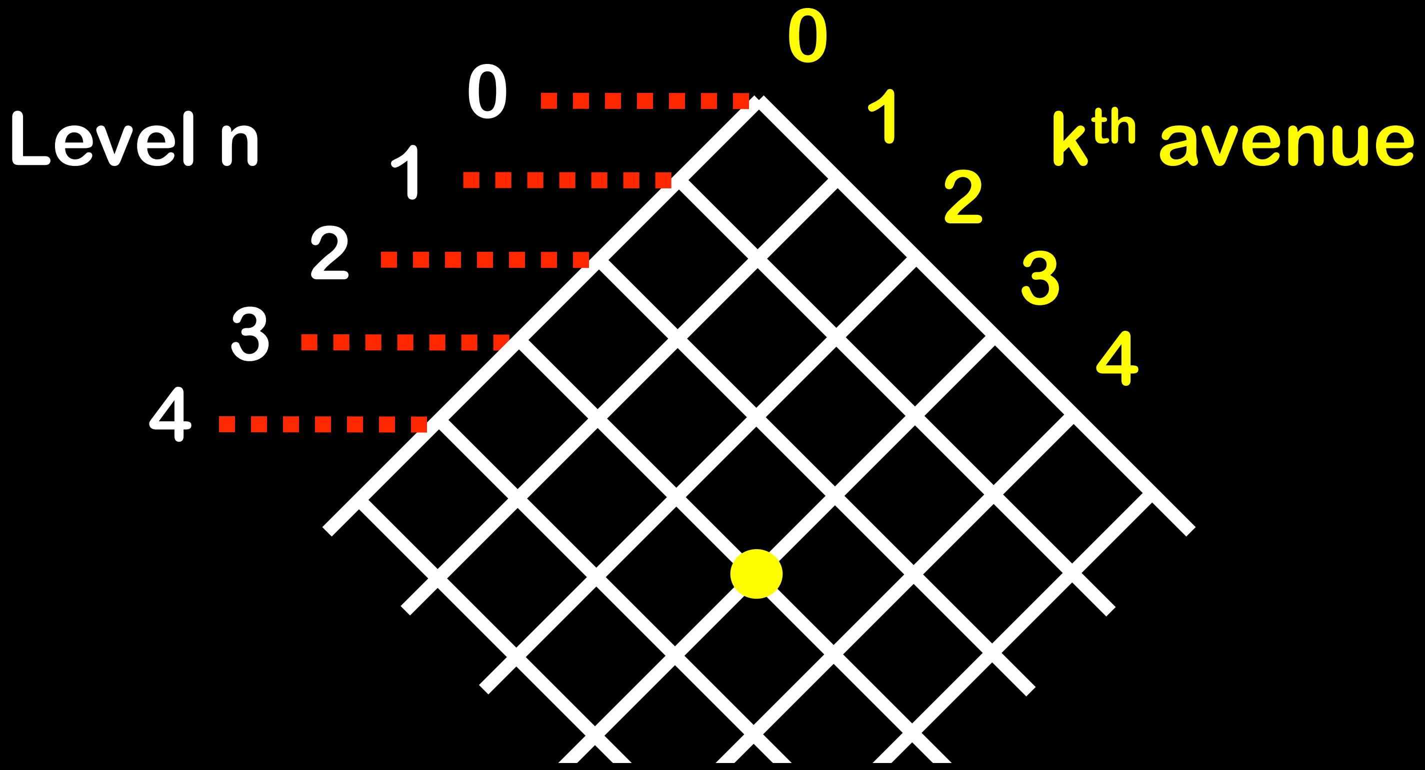
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



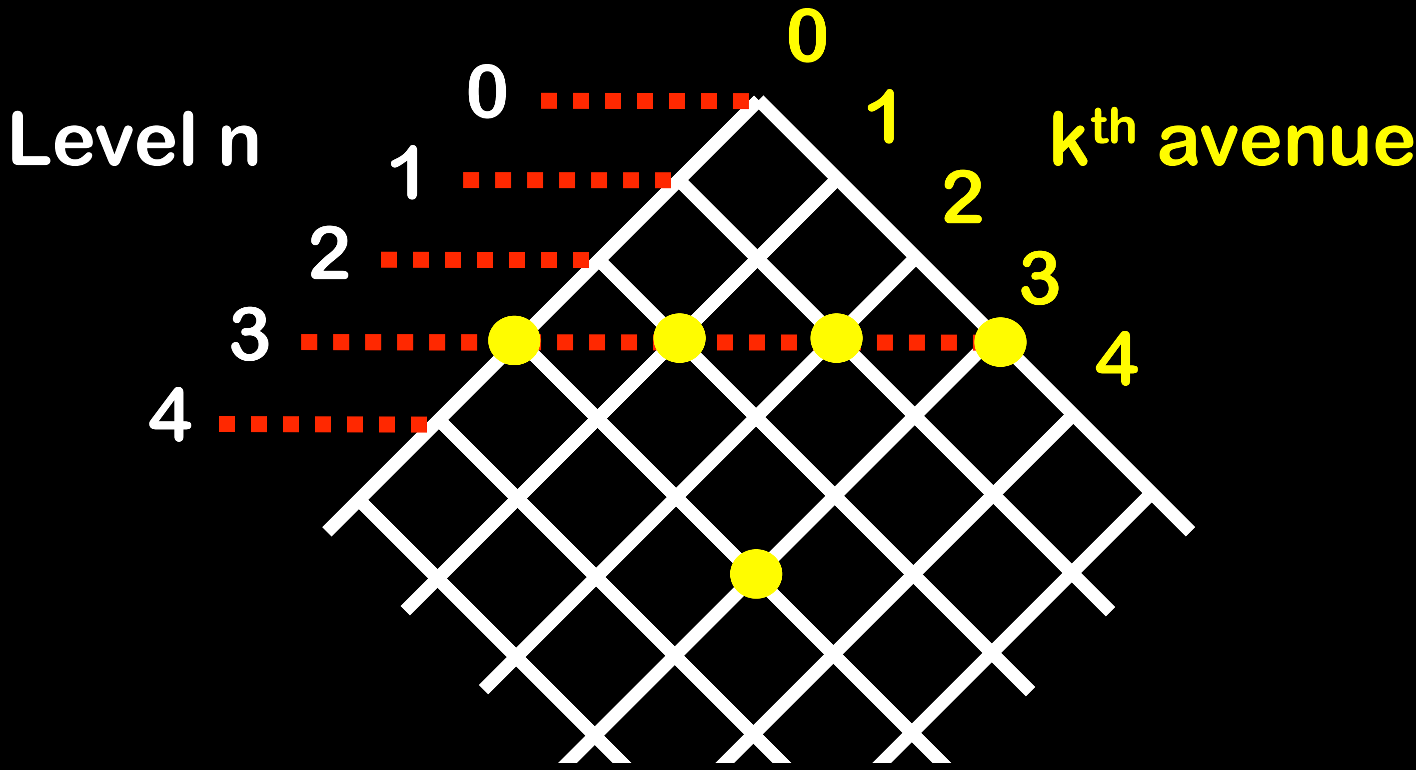
$$\sum_{k=0}^{n} \binom{n}{k}^2$$



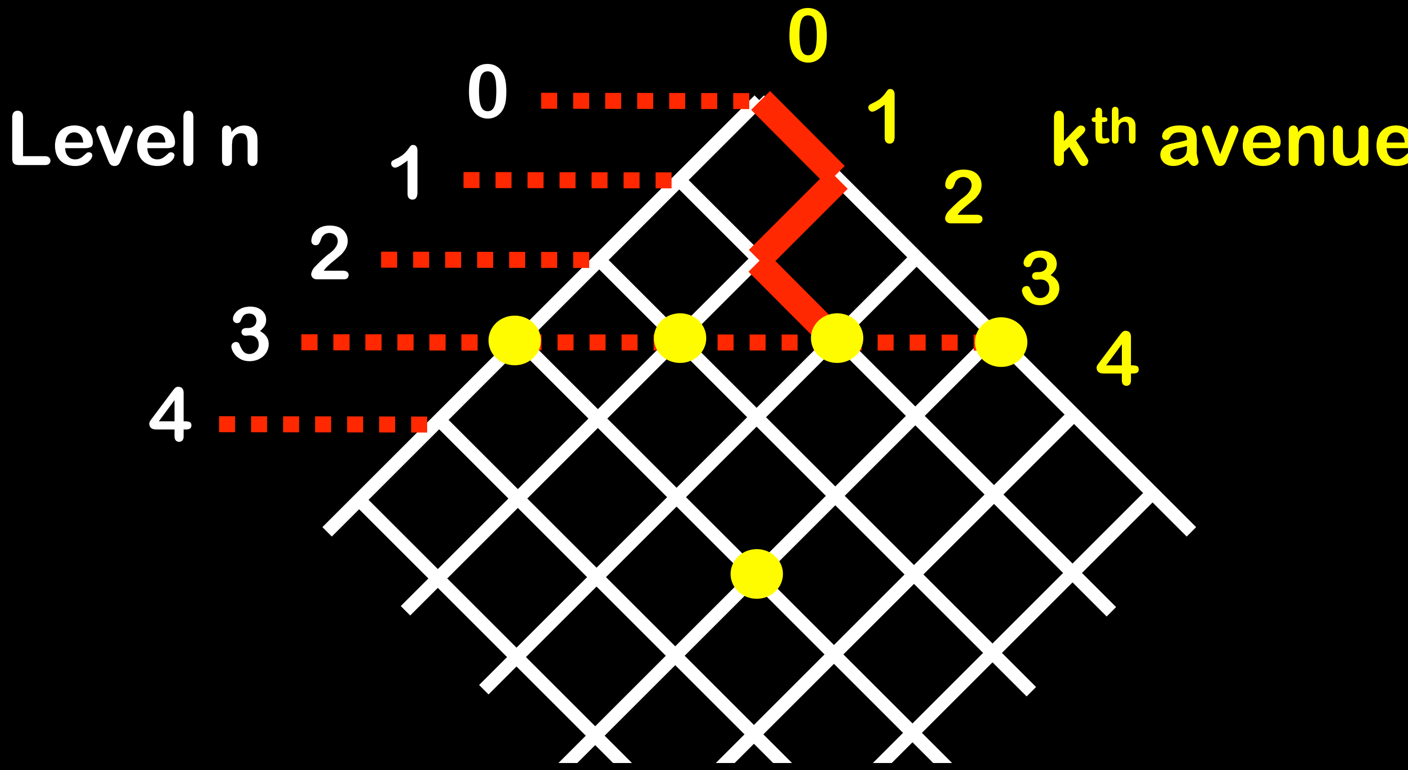
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$



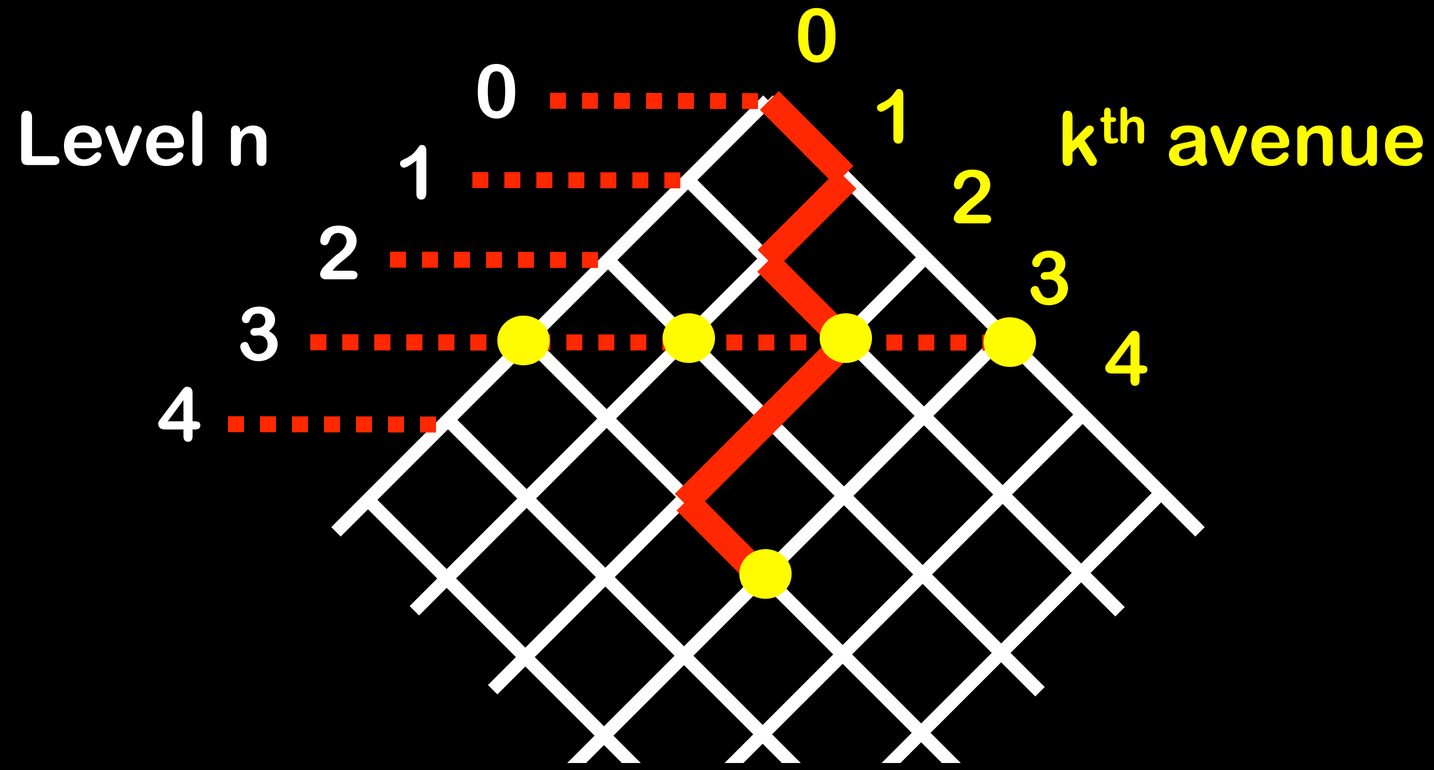
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$



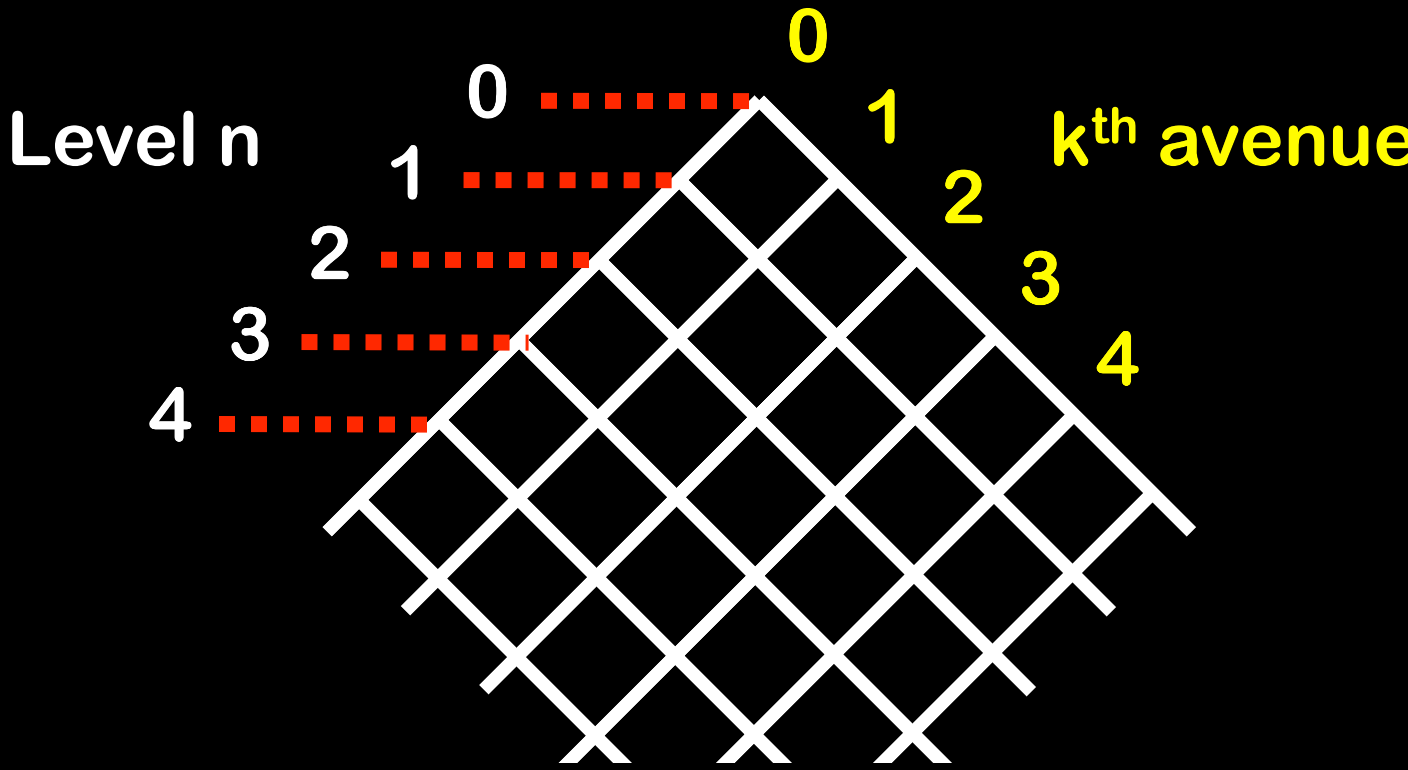
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$



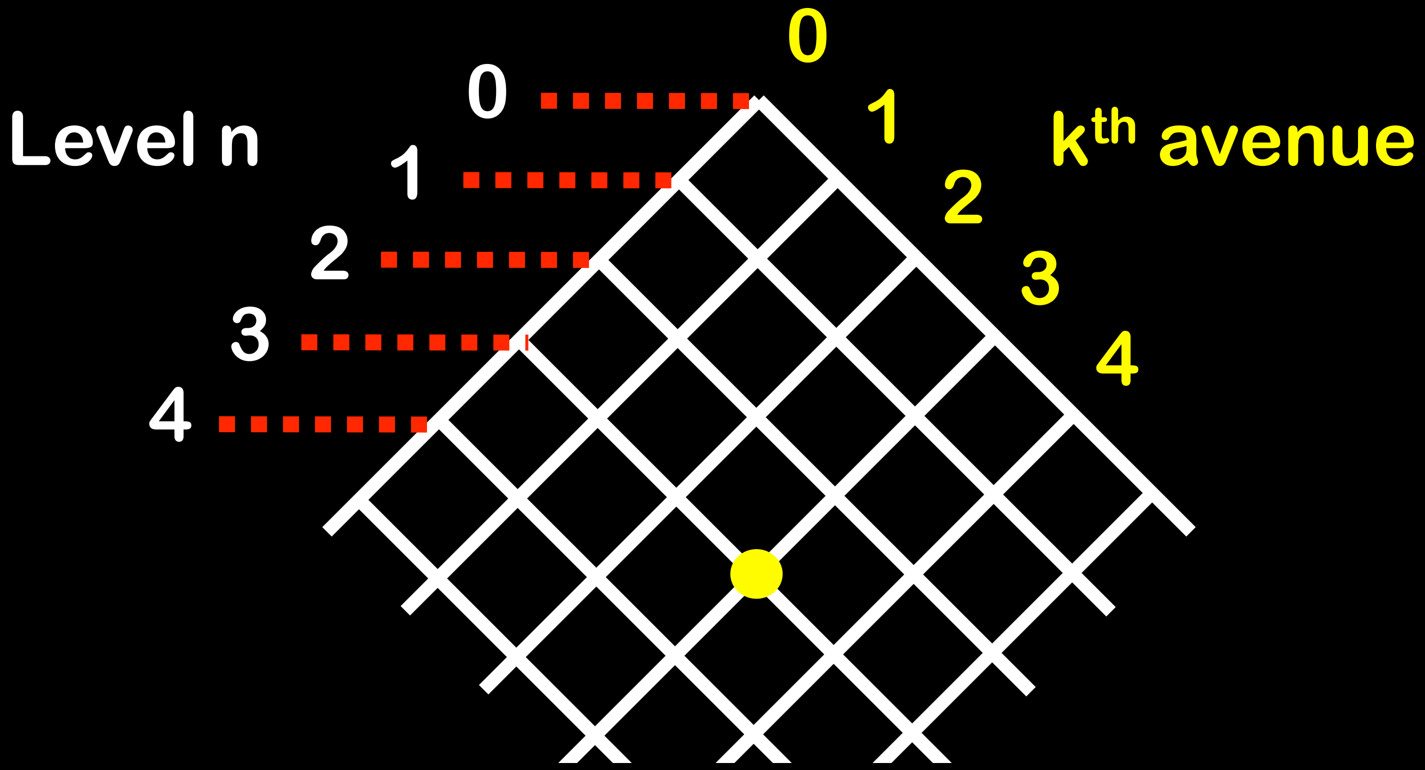
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$



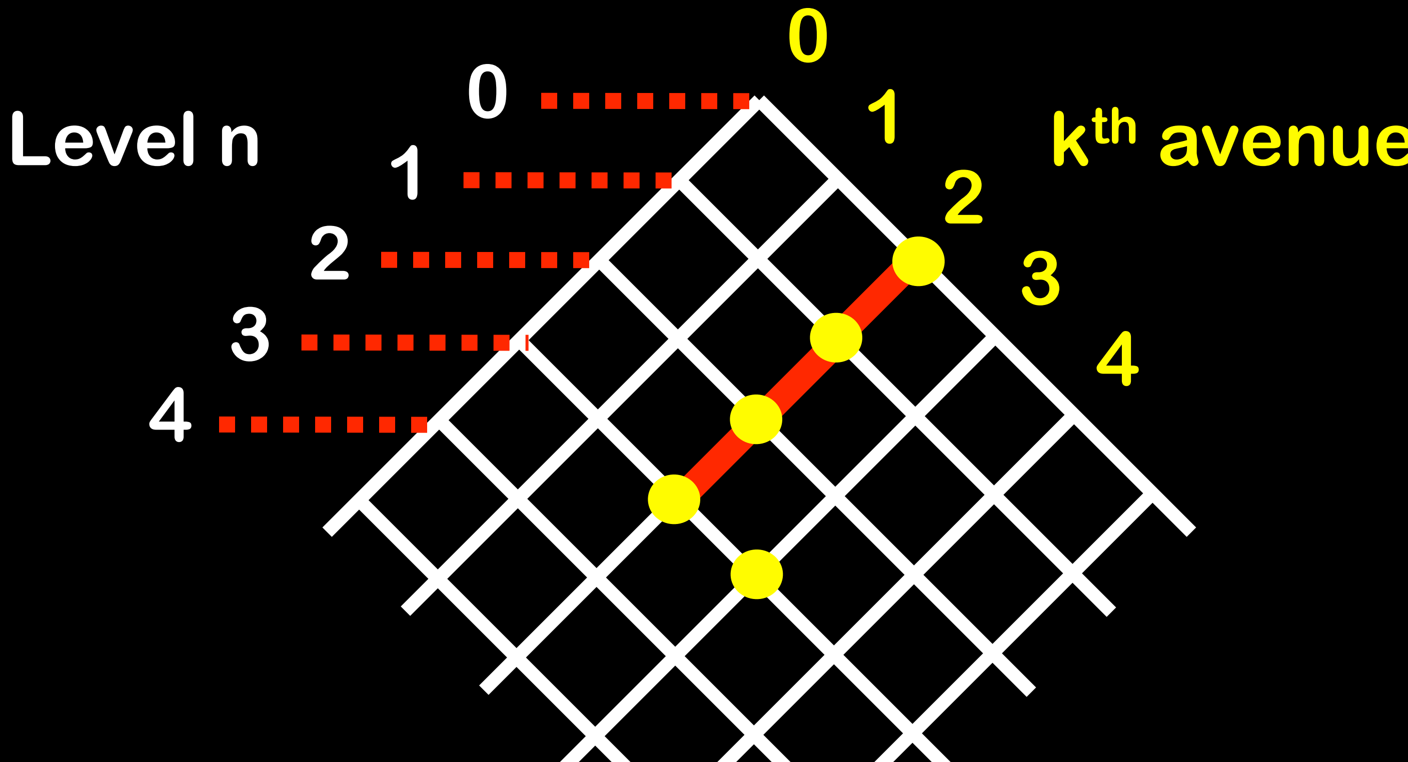
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$



$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

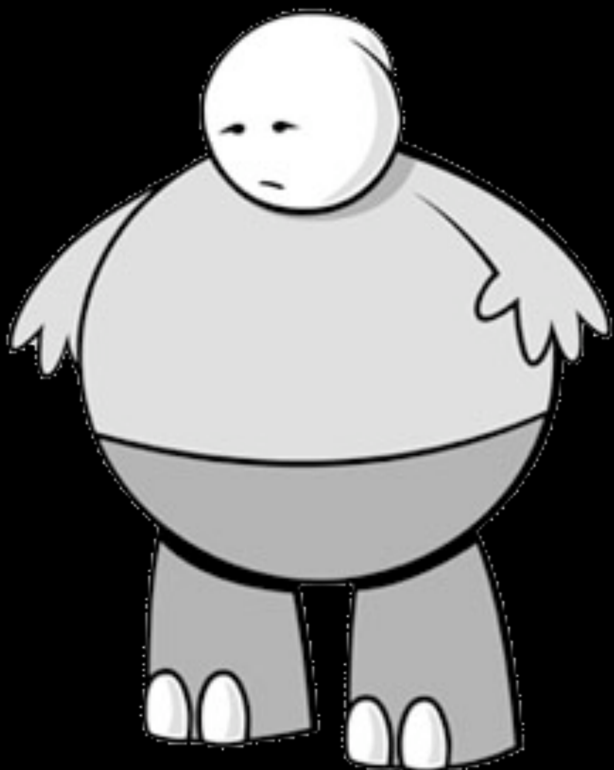


$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$



$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Handout on generating functions



Here's What You Need to Know…

- Polynomials count
- Binomial formula
- Combinatorial proofs of binomial identities
- Basic generating functionology