Counting III
Lecture 8, September 18, 2008

Arrange n symbols: $r_1$ of type 1, $r_2$ of type 2, ..., $r_k$ of type k

\[
\frac{n!}{(n-r_1)!r_1! \cdots (n-r_1-r_2)!r_2! \cdots} = \frac{n!}{r_1!r_2! \cdots r_k!}
\]
How many different ways to divide up the loot?

Sequences with 20 G’s and 4 /’s

\[ \binom{24}{4} \]

How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

\[ \binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k} \]

5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?

\[ \frac{14!}{2!3!2!} = 3,632,428,800 \]
How many integer solutions to the following equations?

\[x_1 + x_2 + x_3 + \ldots + x_n = k\]

\[x_1, x_2, x_3, \ldots, x_n \geq 0\]

\[\binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k}\]

Identical/Distinct Dice

Suppose that we roll seven dice

\[\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & 6 & 6 & 6
\end{array}\]

How many different outcomes are there, if order matters?

6^7

What if order doesn’t matter? (E.g., Yahtzee)

\[\binom{12}{7}\]

(Corresponds to 6 pirates and 7 bars of gold)

Identical/Distinct Objects

If we are putting k objects into n distinct bins.

<table>
<thead>
<tr>
<th>Objects are distinguishable</th>
<th>(n^k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objects are indistinguishable</td>
<td>(\binom{k+n-1}{k})</td>
</tr>
</tbody>
</table>

The Binomial Formula

\[(1+X)^n = \binom{n}{0}X^0 + \binom{n}{1}X^1 + \ldots + \binom{n}{n}X^n\]

Binomial coefficients
What is the coefficient of $(X_1^{r_1}X_2^{r_2}...X_k^{r_k})$ in the expansion of $(X_1+X_2+X_3+...+X_k)^n$?

\[
\frac{n!}{r_1!r_2!...r_k!}
\]

And now for some more counting...

**Power Series Representation**

\[
(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k
\]

“Product form” or “Generating form”

\[
= \sum_{k=0}^{\infty} \binom{n}{k} X^k
\]  
For $k>n$, \[\binom{n}{k} = 0\]

“Power Series” or “Taylor Series” Expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

\[
(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k
\]

Let $x = 1$, \[2^n = \sum_{k=0}^{n} \binom{n}{k}\]

The number of subsets of an n-element set
By varying $x$, we can discover new identities:

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Let $x = -1$,

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Equivalently,

$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

The number of subsets with even size is the same as the number of subsets with odd size.

Proofs that work by manipulating algebraic forms are called “algebraic” arguments. Proofs that build a bijection are called “combinatorial” arguments.

Let $O_n$ be the set of binary strings of length $n$ with an odd number of ones.

Let $E_n$ be the set of binary strings of length $n$ with an even number of ones.

We just saw an algebraic proof that $|O_n| = |E_n|$.
A Combinatorial Proof

Let $O_n$ be the set of binary strings of length $n$ with an odd number of ones

Let $E_n$ be the set of binary strings of length $n$ with an even number of ones

A combinatorial proof must construct a bijection between $O_n$ and $E_n$

An Attempt at a Bijection

Let $f_n$ be the function that takes an $n$-bit string and flips all its bits

$f_n$ is clearly a one-to-one and onto function

for odd $n$. E.g. in $f_7$ we have:

<table>
<thead>
<tr>
<th>Original</th>
<th>Flipped</th>
</tr>
</thead>
<tbody>
<tr>
<td>0010011</td>
<td>1101100</td>
</tr>
<tr>
<td>1001101</td>
<td>0110010</td>
</tr>
</tbody>
</table>

...but do even $n$ work? In $f_6$ we have

<table>
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<th>Flipped</th>
</tr>
</thead>
<tbody>
<tr>
<td>0010011</td>
<td>110011</td>
</tr>
<tr>
<td>1001101</td>
<td>011010</td>
</tr>
</tbody>
</table>

Uh oh. Complementing maps evens to evens!

A Correspondence That Works for all $n$

Let $f_n$ be the function that takes an $n$-bit string and flips only the first bit. For example,

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>0010011</td>
<td>1010011</td>
</tr>
<tr>
<td>1001101</td>
<td>0001101</td>
</tr>
<tr>
<td>110011</td>
<td>010011</td>
</tr>
<tr>
<td>101010</td>
<td>001010</td>
</tr>
</tbody>
</table>

$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$

The binomial coefficients have so many representations that many fundamental mathematical identities emerge…
\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

Set of all \(k\)-subsets of \(\{1..n\}\)

Either we do not pick \(n\): then we have to pick \(k\) elements out of the remaining \(n-1\).

Or we do pick \(n\): then we have to pick \(k-1\) elts. out of the remaining \(n-1\).

**The Binomial Formula**

\[
(1+X)^0 = 1
\]

\[
(1+X)^1 = 1 + 1X
\]

\[
(1+X)^2 = 1 + 2X + 1X^2
\]

\[
(1+X)^3 = 1 + 3X + 3X^2 + 1X^3
\]

\[
(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4
\]

Pascal’s Triangle: \(k\)th row are coefficients of \((1+X)^k\)

Inductive definition of \(k\)th entry of \(n\)th row:

\[
Pascal(n,0) = Pascal(n,n) = 1;
\]

\[
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)
\]

**“Pascal’s Triangle”**

\[
\begin{array}{cccccc}
0 & 1 & & & & \\
1 & 1 & & & & \\
2 & 1 & 1 & & & \\
3 & 3 & 3 & 1 & & \\
4 & 6 & 4 & 1 & & \\
5 & 10 & 10 & 5 & 1 & \\
6 & 15 & 20 & 15 & 6 & 1
\end{array}
\]

**“It is extraordinary how fertile in properties the triangle is. Everyone can try his hand”**

- Al-Karaji, Baghdad 953-1029
- Chu Shin-Chieh 1303
- Blaise Pascal 1654
Summing the Rows

\[ 2^n = \sum_{k=0}^{n} \binom{n}{k} = 1 \]
\[ 1 + 1 \]
\[ 1 + 2 + 1 \]
\[ 1 + 3 + 3 + 1 \]
\[ 1 + 4 + 6 + 4 + 1 \]
\[ 1 + 5 + 10 + 10 + 5 + 1 \]
\[ 1 + 6 + 15 + 20 + 15 + 6 + 1 \]
\[ = 1 \]
\[ = 2 \]
\[ = 4 \]
\[ = 8 \]
\[ = 16 \]
\[ = 32 \]
\[ = 64 \]

Odds and Evens

\[ 1 \]
\[ 1 \]
\[ 1 \]
\[ 1 \]
\[ 1 + 1 \]
\[ 1 + 2 + 1 \]
\[ 1 + 3 + 3 + 1 \]
\[ 1 + 4 + 6 + 4 + 1 \]
\[ 1 + 5 + 10 + 10 + 5 + 1 \]
\[ 1 + 6 + 15 + 20 + 15 + 6 + 1 \]
\[ = 1 \]
\[ = 2 \]
\[ = 4 \]
\[ = 8 \]
\[ = 16 \]
\[ = 32 \]
\[ = 64 \]

Summing on 1\(^{st}\) Avenue

\[ \sum_{i=1}^{n} i = \sum_{i=1}^{n} \binom{i}{2} = \binom{n+1}{2} \]

Summing on k\(^{th}\) Avenue

\[ \sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \]
Fibonacci Numbers

1 1 1 1 1 1
1 2 3 5
1 3 6 10 15 20
1 4 6 10 15 20 27

Al-Karaji Squares

1
1 1 = 1
1 2+2 = 4
1 3+2 = 9
1 4+2 = 16
1 5+2 = 25
1 6+2 = 36

Sums of Squares

1 1 1 1
1 2 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Pascal Mod 2

1
1 1
1 2 2
1 2 3 3
1 2 3 4 4
1 2 3 4 5 5
1 2 3 4 5 6 6
1 2 3 4 5 6 7 7
1 2 3 4 5 6 7 8 8

Pascal Mod 2

1
1 1
1 2 2
1 2 3 3
1 2 3 4 4
1 2 3 4 5 5
1 2 3 4 5 6 6
1 2 3 4 5 6 7 7
1 2 3 4 5 6 7 8 8

Pascal Mod 2

1
1 1
1 2 2
1 2 3 3
1 2 3 4 4
1 2 3 4 5 5
1 2 3 4 5 6 6
1 2 3 4 5 6 7 7
1 2 3 4 5 6 7 8 8
All these properties can be proved inductively and algebraically. We will give combinatorial proofs using the Manhattan block walking representation of binomial coefficients.

How many shortest routes from A to B?

There are \( \binom{j+k}{k} \) shortest routes from (0,0) to (j,k).

There are \( \binom{n}{k} \) shortest routes from (0,0) to (n-k,k).
There are $\binom{n}{k}$ shortest routes from (0,0) to level n and k\textsuperscript{th} avenue.
Handout on generating functions

\[ \sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \]