15-251
Great Theoretical Ideas in Computer Science

Review from last time…

Counting III
Lecture 8, September 18, 2008

$$X^1 + \quad X^2 + \quad X^3$$

Arrange n symbols: r_1 of type 1, r_2 of type 2, …, r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\ldots-r_{k-1}}{r_k}$$

$$= \frac{n!}{(n-r_1)!r_1!}\frac{(n-r_1)!}{(n-r_1-r_2)!r_2!}\cdots$$

$$= \frac{n!}{r_1!r_2!\ldots r_k!}$$

CARNEGIEMELLON

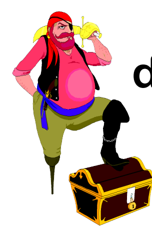
$$\frac{14!}{2!3!2!} = 3{,}632{,}428{,}800$$

How many different ways to divide up the loot?

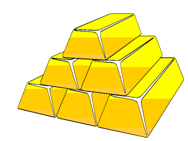
Sequences with 20 G's and 4 /'s

$$\binom{24}{4}$$

5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?



How many different ways can n distinct pirates divide k identical, indivisible bars of gold?



$$\binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + \ldots + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

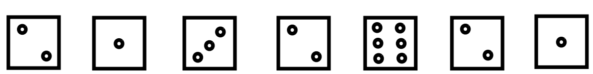
Identical/Distinct Objects

If we are putting k objects into n <u>distinct</u> bins.

Objects are distinguishable	n^k
Objects are indistinguishable	$\binom{k+n-1}{k}$

Identical/Distinct Dice

Suppose that we roll seven dice



How many different outcomes are there, if order matters? 6^7

What if order doesn't matter? (E.g., Yahtzee) $\binom{12}{7}$

(Corresponds to 6 pirates and 7 bars of gold)

The Binomial Formula

$$(1+X)^n = \binom{n}{0}X^0 + \binom{n}{1}X^1 + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

What is the coefficient of $(X_1^{r_1} X_2^{r_2} \ldots X_k^{r_k})$ in the expansion of $(X_1+X_2+X_3+\ldots+X_k)^n$?

$$\frac{n!}{r_1! r_2! \ldots r_k!}$$

Power Series Representation

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} X^k$$

For $k>n$, $\binom{n}{k} = 0$

"Product form" or "Generating form"

"Power Series" or "Taylor Series" Expansion

And now for some more counting…

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Let $x = 1$, $\quad 2^n = \sum_{k=0}^{n} \binom{n}{k}$

The number of subsets of an n-element set

By varying x, we can discover new identities:

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Let x = -1, $$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Equivalently, $$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Proofs that work by manipulating algebraic forms are called "algebraic" arguments.
Proofs that build a bijection are called "combinatorial" arguments

The number of subsets with even size is the same as the number of subsets with odd size

$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We just saw an algebraic proof that $|O_n| = |E_n|$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones

Let E_n be the set of binary strings of length n with an even number of ones

A combinatorial proof must construct a bijection between O_n and E_n

A Correspondence That Works for all n

Let f_n be the function that takes an n-bit string and flips only the first bit. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

An Attempt at a Bijection

Let f_n be the function that takes an n-bit string and flips all its bits

f_n is clearly a one-to-one and onto function

for odd n. E.g. in f_7 we have:

0010011 → 1101100
1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100
101010 → 010101

Uh oh. Complementing maps evens to evens!

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Set of all k-subsets of {1..n}

Either we do not pick n: then we have to pick k elements out of the remaining n-1.

Or we do pick n: then we have to pick k-1 elts. out of the remaining n-1.

"Pascal's Triangle"

$\binom{0}{0} = 1$

$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$

$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$

$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$

- Al-Karaji, Baghdad 953-1029
- Chu Shin-Chieh 1303
- Blaise Pascal 1654

The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle: k^{th} row are coefficients of $(1+X)^k$

Inductive definition of k^{th} entry of n^{th} row:
Pascal(n,0) = Pascal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand"

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1 = 1
1 + 1 = 2
1 + 2 + 1 = 4
1 + 3 + 3 + 1 = 8
1 + 4 + 6 + 4 + 1 = 16
1 + 5 + 10 + 10 + 5 + 1 = 32
1 + 6 + 15 + 20 + 15 + 6 + 1 = 64

Summing on 1st Avenue

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} \binom{i}{1} = \binom{n+1}{2}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Odds and Evens

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1 + 15 + 15 + 1 = 6 + 20 + 6

Summing on kth Avenue

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Fibonacci Numbers

1

1 1 = 2

1 2 1 = 3

1 3 3 1 = 5

1 4 6 4 1 = 8

1 5 10 10 5 1 = 13

1 6 15 20 15 6 1

Al-Karaji Squares

1	
1 1	= 1
1 $2+2\cdot 1$	= 4
1 $3+2\cdot 3$ 1	= 9
1 $4+2\cdot 6$ 4 1	= 16
1 $5+2\cdot 10$ 10 5 1	= 25
1 $6+2\cdot 15$ 20 15 6 1	= 36

Sums of Squares

1

1 1

1^2 2^2 1^2

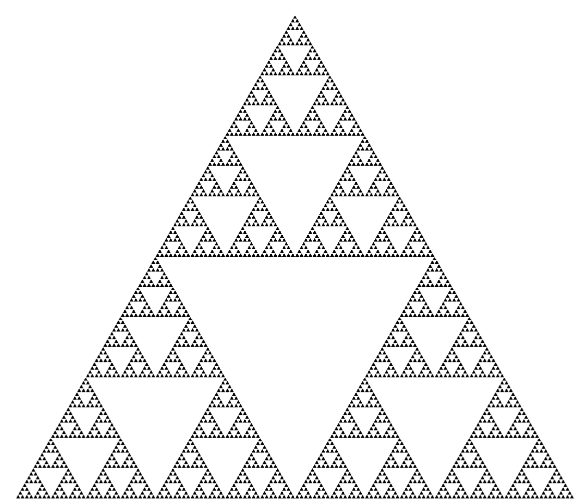
1^2 3^2 3^2 1^2

1 4 6 4 1

1 5 10 10 5 1

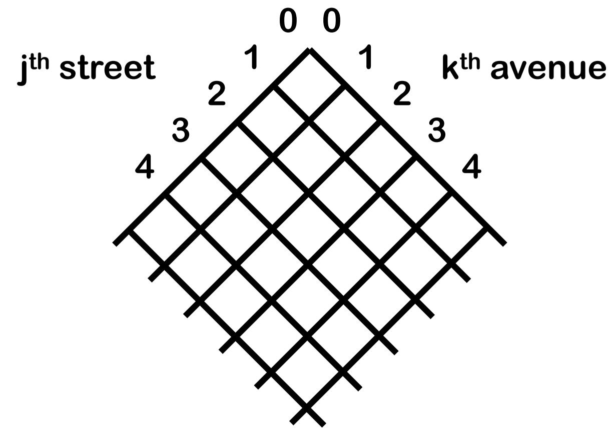
1 6 15 20 15 6 1

Pascal Mod 2



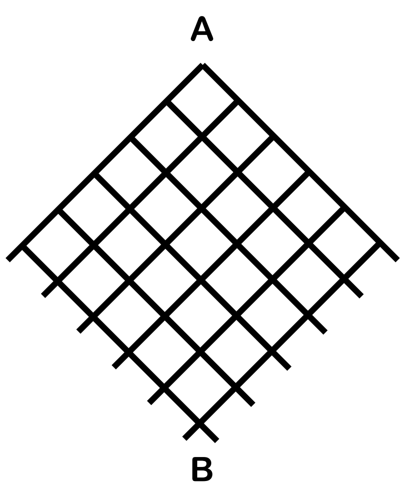
All these properties can be proved inductively and algebraically. We will give combinatorial proofs using the Manhattan block walking representation of binomial coefficients

Manhattan



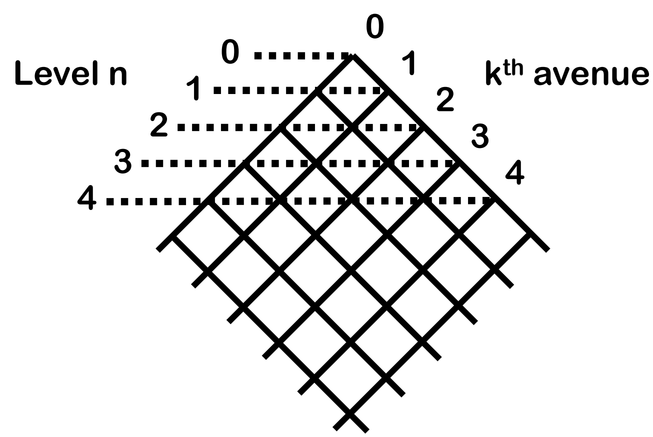
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k)

How many shortest routes from A to B?



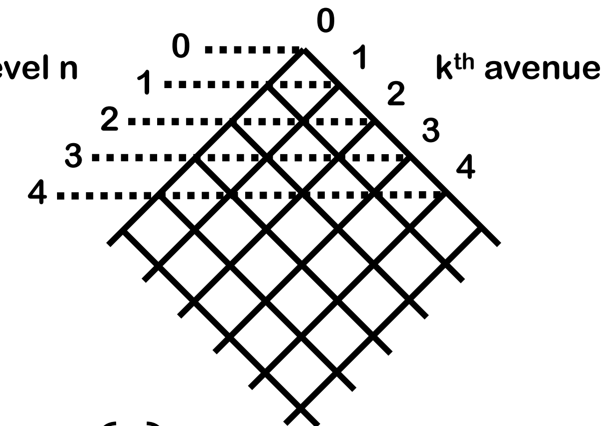
$\binom{10}{5}$

Manhattan



There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k)

Manhattan



There are $\binom{n}{k}$ shortest routes from (0,0) to level n and k^{th} avenue

Level n ... k^{th} avenue

0 1

1 ...... 1 1

2 ...... 1 2 1

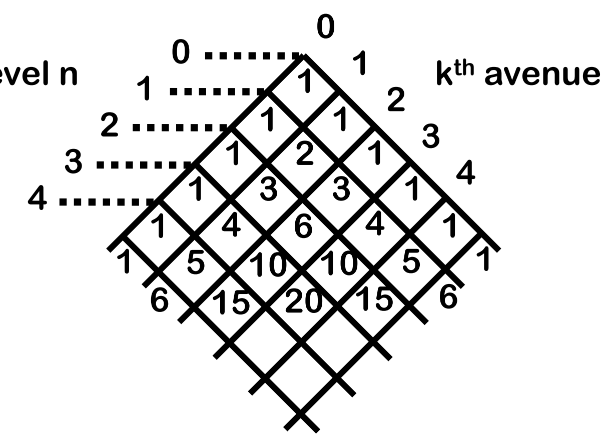
3 ...... 1 3 3 1

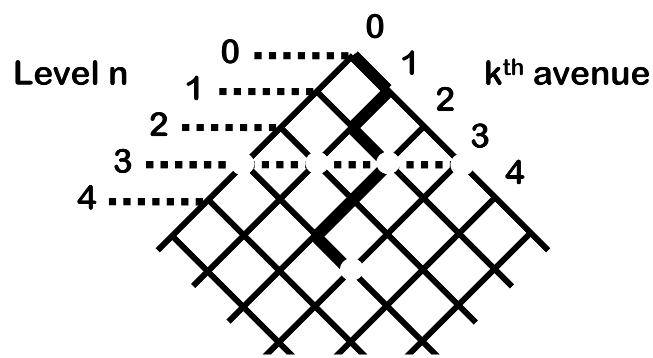
4 ...... 1 4 6 4 1

1 5 10 + 10 5 1

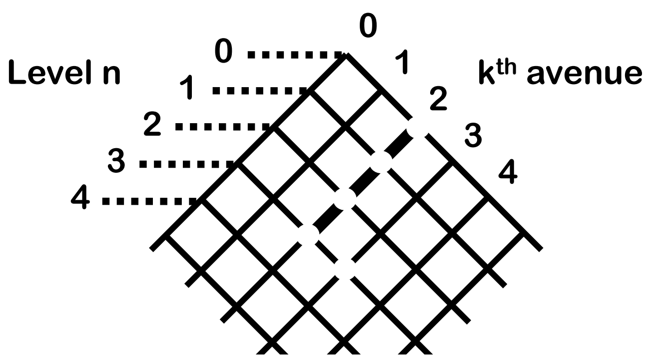
6 15 20 15 6

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

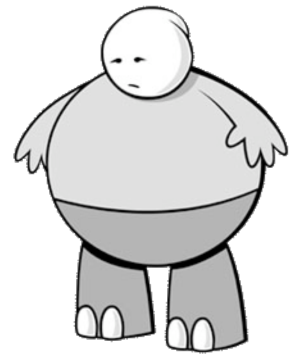




$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$



$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$



Here's What You Need to Know…

- Polynomials count
- Binomial formula
- Combinatorial proofs of binomial identities
- Basic generating functionology

Handout on generating functions