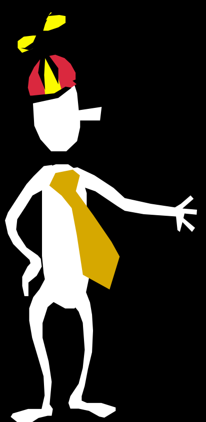
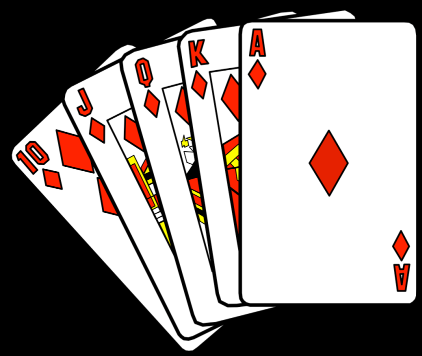
15-251

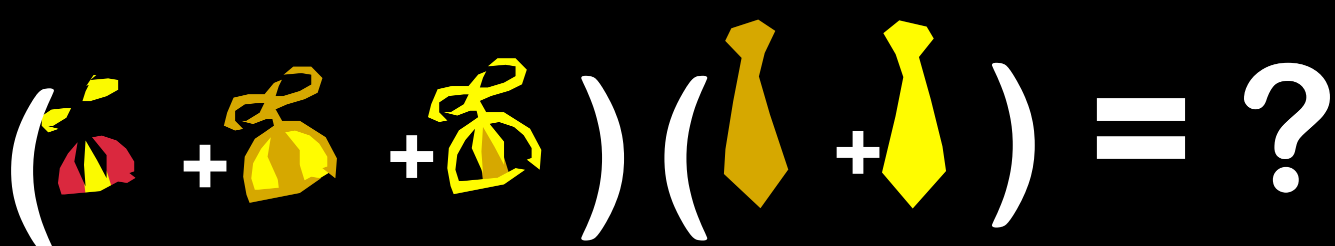
Great Theoretical Ideas in Computer Science

Counting II: Recurring Problems and Correspondences

Lecture 7, September 16, 2008

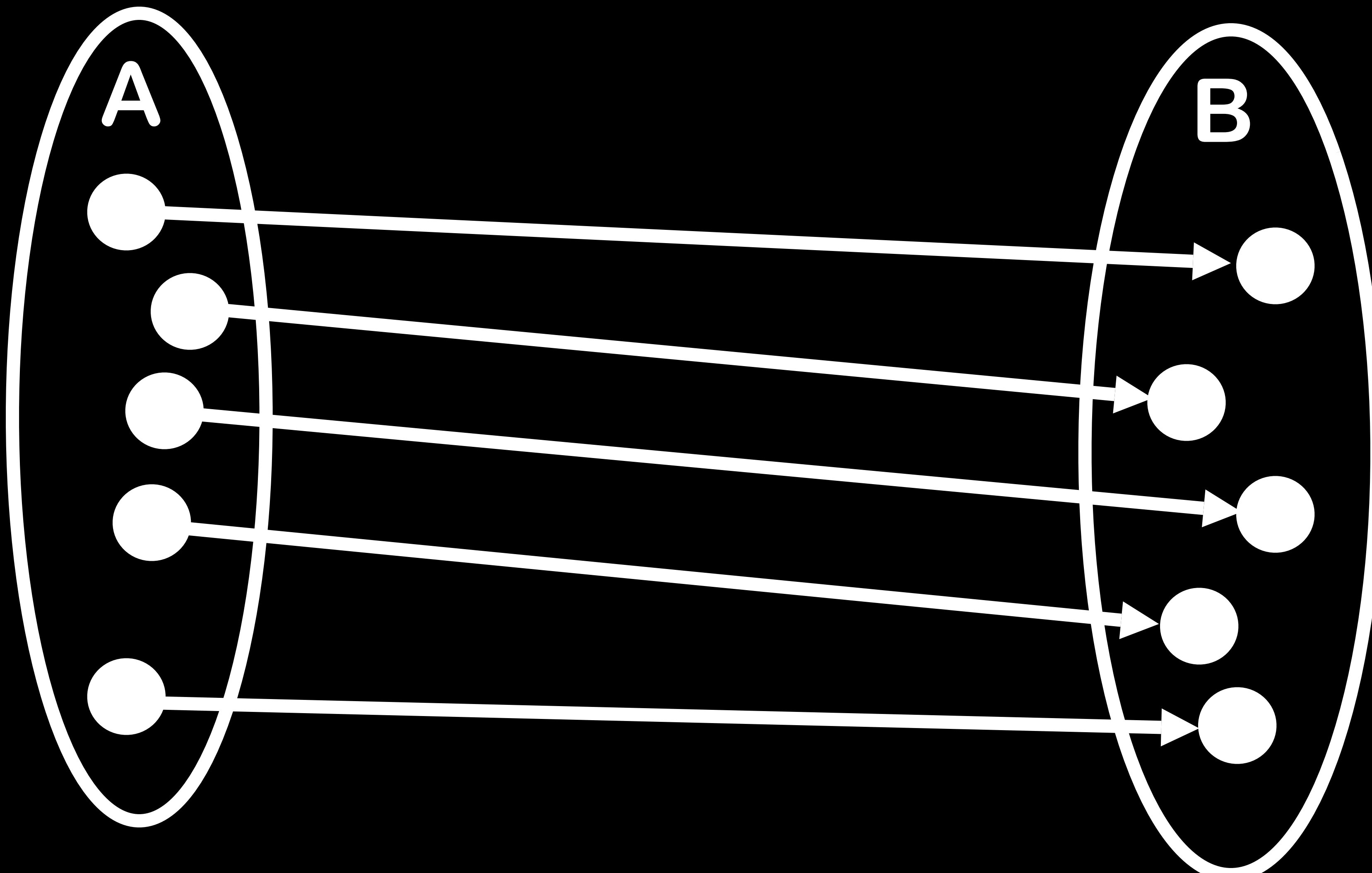






Review from last time…

1-1 onto Correspondence (just “correspondence” for short)



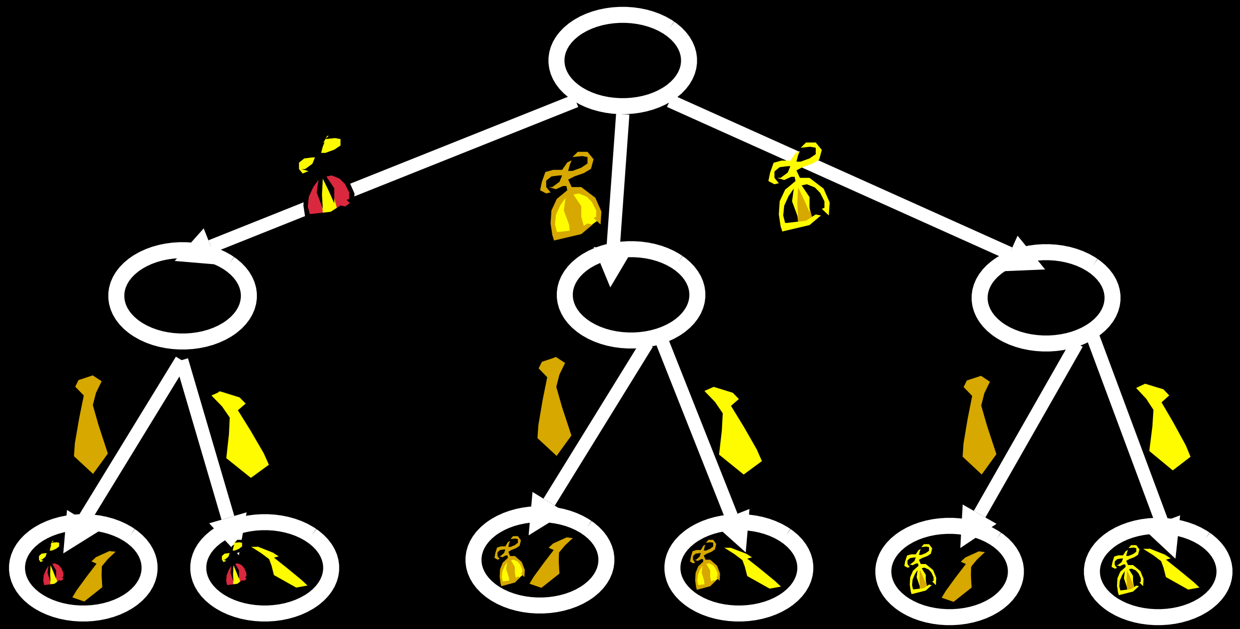
If a finite set A has a k-to-1 correspondence to finite set B, then |B| = |A|/k

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.

The number of subsets of an n-element set is 2^n.

The number of subsets of size r that can be formed from an n-element set is:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$



A choice tree provides a “choice tree representation” of a set S, if

1. Each leaf label is in S, and each element of S is some leaf label
2. No two leaf labels are the same

Product Rule (Rephrased)

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

THEN

There are $P_1 P_2 P_3 \ldots P_n$ objects of type S

How Many Different Orderings of Deck With 52 Cards?

What object are we making? Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

: :

1 possible choice for the 52^{nd} card.

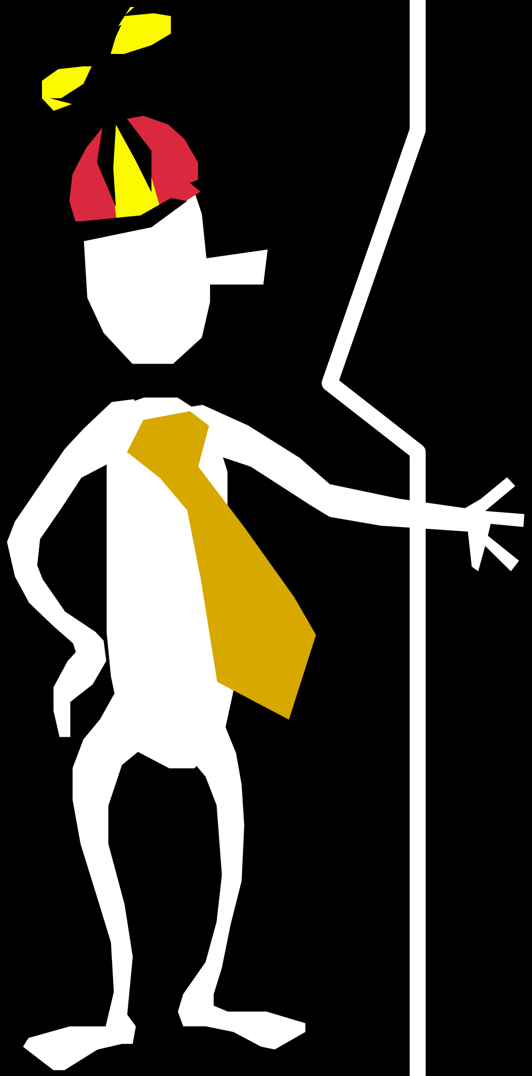
By product rule: $52 \times 51 \times 50 \times \ldots \times 2 \times 1 = 52!$

The Sleuth’s Criterion

There should be a unique way to create an object in S.

In other words:

For any object in S, it should be possible to reconstruct the (unique) sequence of choices which lead to it.

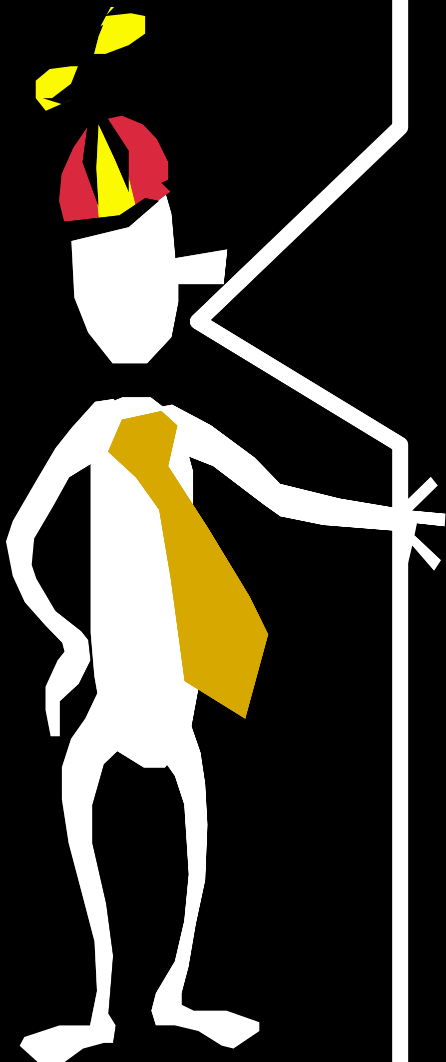


The three big mistakes people make in associating a choice tree with a set S are:

1. Creating objects not in S

2. Leaving out some objects from the set S

3. Creating the same object two different ways



DEFENSIVE THINKING
ask yourself:

Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?

Inclusion-Exclusion

If A and B are two finite sets, what is the size of $(A \cup B)$?

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$$|A| + |B| - |A \cap B|$$

Inclusion-Exclusion

If A, B, C are three finite sets, what is the size of $(A \cup B \cup C)$?

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If A, B, C are three finite sets, what is the size of $(A \cup B \cup C)$?

$$|A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

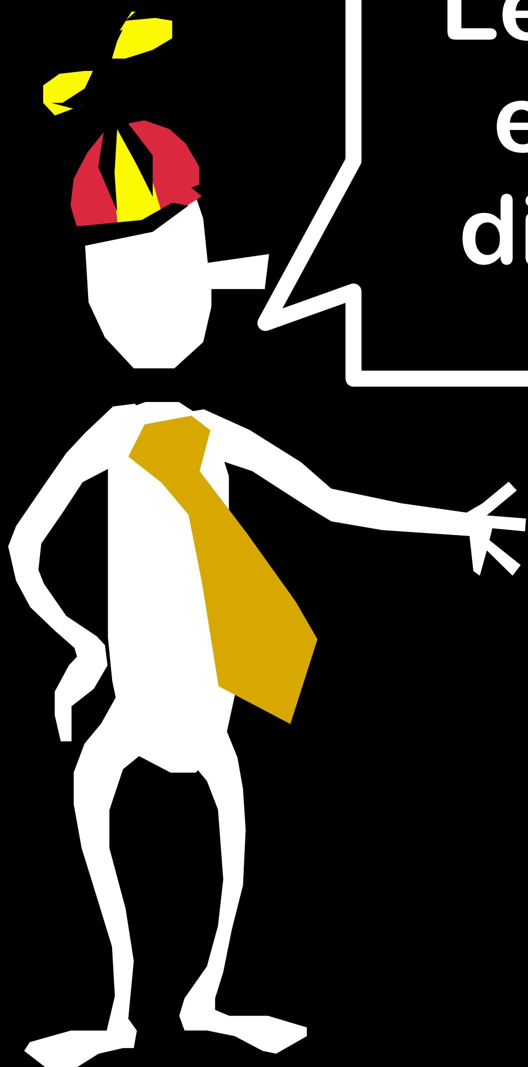
Inclusion-Exclusion

If $A_1, A_2, \ldots, A_n$ are n finite sets, what is the size of $(A_1 \cup A_2 \cup \ldots \cup A_n)$?

Inclusion-Exclusion

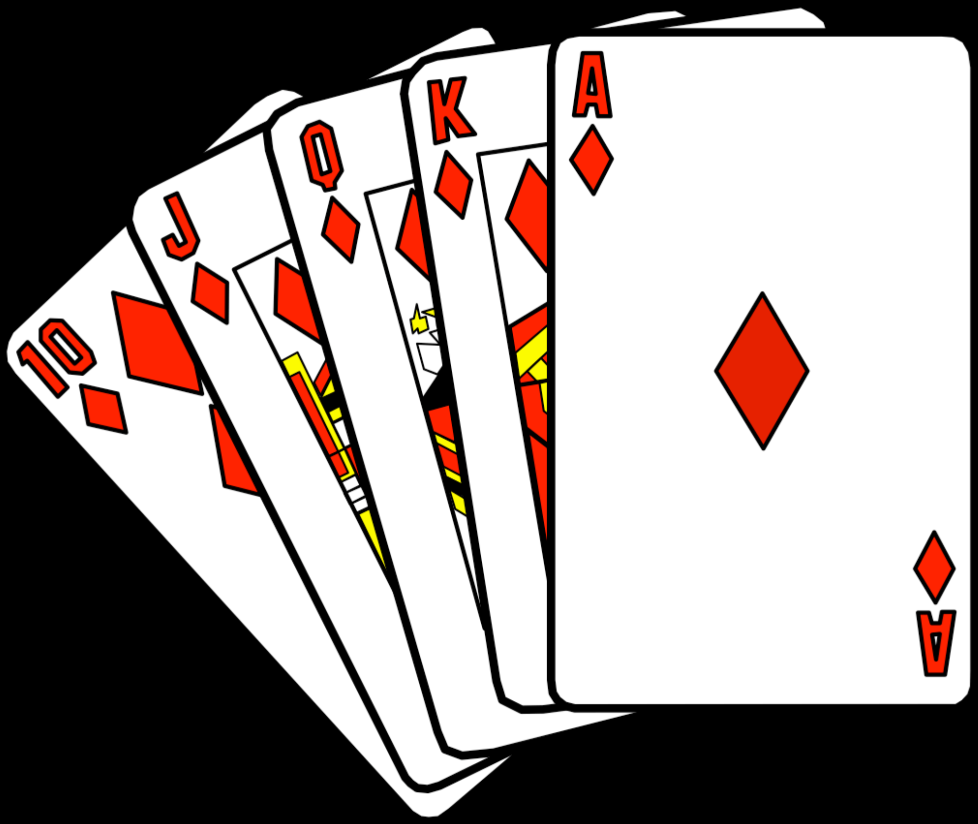
If $A_1, A_2, \ldots, A_n$ are **n** finite sets, what is the size of $(A_1 \cup A_2 \cup \ldots \cup A_n)$?

$$\Sigma_i |A_i|$$
$$- \Sigma_{i<j} |A_i \cap A_j|$$
$$+ \Sigma_{i<j<k} |A_i \cap A_j \cap A_k|$$
$$\ldots$$
$$+ (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$



Let's use our principles to extend our reasoning to different types of objects

Counting Poker Hands



52 Card Deck, 5 card hands

4 possible **suits**:
♥♦♣♠

13 possible **ranks**:
2,3,4,5,6,7,8,9,10,J,Q,K,A

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Pair: set of two cards of the same rank
Straight: 5 cards of consecutive rank
Flush: set of 5 cards with the same suit

Ranked Poker Hands

Straight Flush: a straight and a flush

4 of a kind: 4 cards of the same rank

Full House: 3 of one kind and 2 of another

Flush: a flush, but not a straight

Straight: a straight, but not a flush

3 of a kind: 3 of the same rank, but not a full house or 4 of a kind

2 Pair: 2 pairs, but not 4 of a kind or a full house

A Pair

Straight Flush

Straight Flush

9 choices for rank of lowest card at the start of the straight

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4 possible suits for the flush

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$9 \times 4 = 36$

Straight Flush

9 choices for rank of lowest card at the start of the straight

4 possible suits for the flush

$9 \times 4 = 36$

$$\frac{36}{\binom{52}{5}} = \frac{36}{2{,}598{,}960} = 1 \text{ in } 72{,}193.333\ldots$$

4 of a Kind

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13 choices of rank

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48 choices for remaining card

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$13 \times 48 = 624$

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48 choices for remaining card

$13 \times 48 = 624$

$$\frac{624}{\binom{52}{5}} = \frac{624}{2{,}598{,}960} = 1 \text{ in } 4{,}165$$

Flush

Flush

4 choices of suit

Flush

4 choices of suit

$\binom{13}{5}$ choices of cards

Flush

$$\left.\begin{array}{l} \text{4 choices of suit} \\ \dbinom{13}{5} \text{ choices of cards} \end{array}\right\} \begin{array}{r} 4 \times 1287 \\ = 5148 \end{array}$$

Flush

$$\left.\begin{array}{l} \text{4 choices of suit} \\ \binom{13}{5} \text{ choices of cards} \end{array}\right\} \begin{array}{r} 4 \times 1287 \\ = 5148 \end{array}$$

"but not a straight flush…"

Flush

4 choices of suit

$\binom{13}{5}$ choices of cards

4×1287
$= 5148$

"but not a straight flush…"

- 36 straight flushes

Flush

4 choices of suit

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- 36 straight flushes

5112 flushes

Flush

4 choices of suit

$\binom{13}{5}$ choices of cards

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"but not a straight flush…"

- 36 straight flushes

5112 flushes

$$\frac{5{,}112}{\binom{52}{5}} = 1 \text{ in } 508.4\ldots$$

Straight

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9 choices of lowest card

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9 choices of lowest card

4^5 choices of suits for 5 cards

9×1024
$= 9216$

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Straight

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$$\left.\begin{array}{l} \\ \end{array}\right\} \begin{array}{r} 9 \times 1024 \\ = 9216 \end{array}$$

"but not a straight flush…"

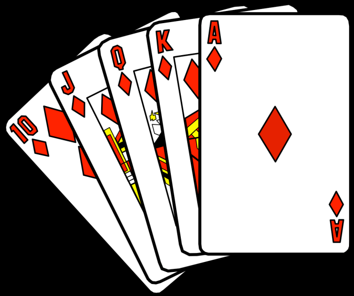
- 36 straight flushes

9180 straights

$$\frac{9,180}{\binom{52}{5}} = 1 \text{ in } 283.06\ldots$$

Ranking

Straight Flush	36
4-of-a-kind	624
Full House	3,744
Flush	5,112
Straight	9,180
3-of-a-kind	54,912
2-pair	123,552
A pair	1,098,240
Nothing	1,302,540



Storing Poker Hands: How many bits per hand?

I want to store a 5 card poker hand using the smallest number of bits (space efficient)

Order the 2,598,560 Poker Hands Lexicographically (or in any fixed way)

To store a hand all I need is to store its index of size $\lceil \log_2(2{,}598{,}560) \rceil = 22$ bits

Hand 0000000000000000000000
Hand 0000000000000000000001
Hand 0000000000000000000010

.

.

.

Is 22 Bits OPTIMAL?

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$2^{21} = 2{,}097{,}152 < 2{,}598{,}560$

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Thus there are more poker hands than there are 21-bit strings

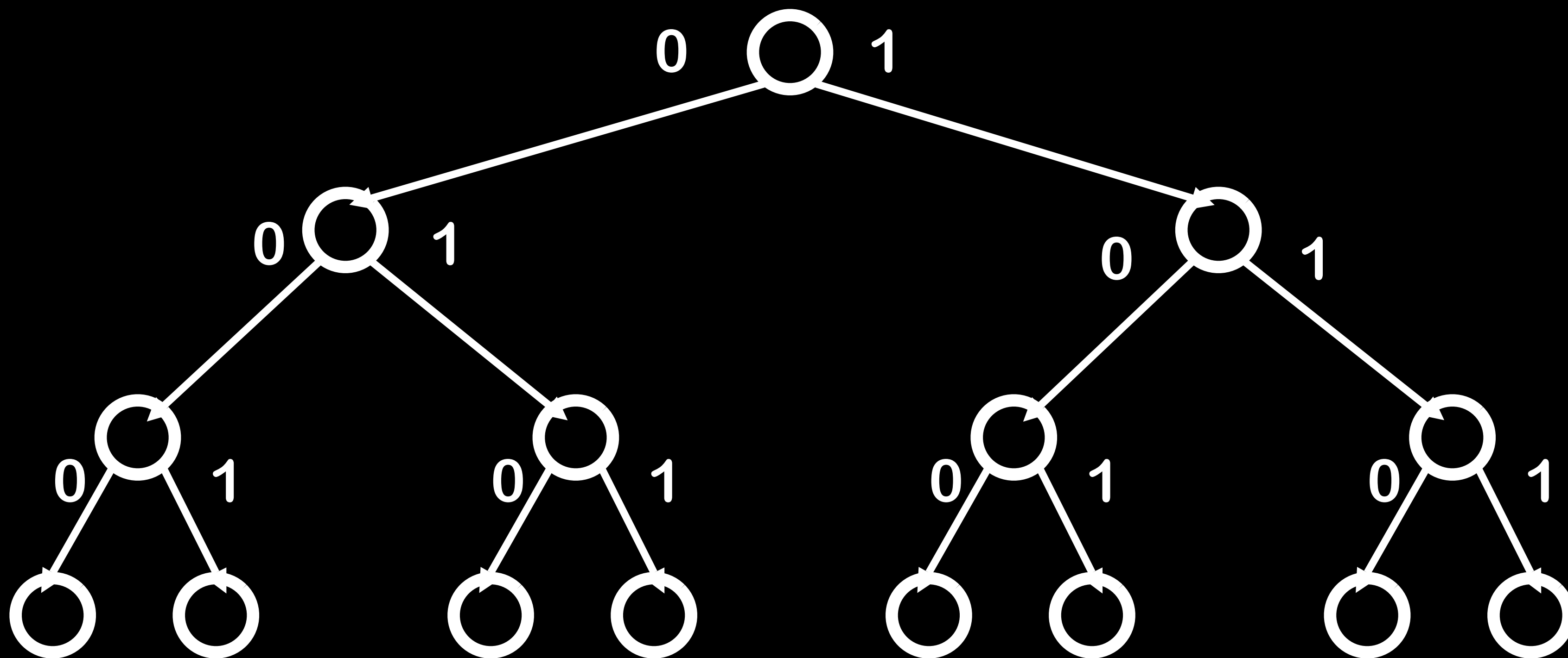
Is 22 Bits OPTIMAL?

$2^{21} = 2{,}097{,}152 < 2{,}598{,}560$

Thus there are more poker hands than there are 21-bit strings

Hence, you can't have a 21-bit string for each hand

Binary (Boolean) Choice Tree



A binary (Boolean) choice tree is a choice tree where each internal node has degree 2

Usually the choices are labeled 0 and 1

22 Bits is OPTIMAL

$2^{21} = 2{,}097{,}152 < 2{,}598{,}560$

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A binary choice tree of depth 21 can have at most 2^{21} leaves.

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22 Bits is OPTIMAL

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A binary choice tree of depth 21 can have at most 2^{21} leaves.

Hence, there are not enough leaves for all 5-card hands.

*But we can encode with one fewer bit on average, if we take the probabilities into account

An n-element set can be stored so that each element uses $\lceil \log_2(n) \rceil$ bits

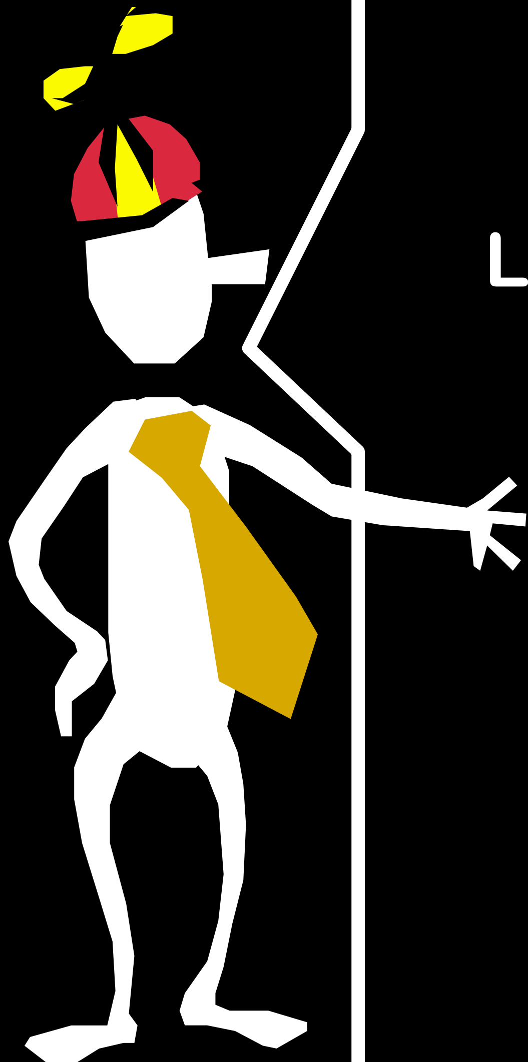
Furthermore, any representation of the set will have **some** string of at least that length

Information Counting Principle:

If each element of a set can be represented using **k** bits, the size of the set is bounded by 2^k

Information Counting Principle:

Let S be a set represented by a depth-k binary choice tree, the size of the set is bounded by 2^k

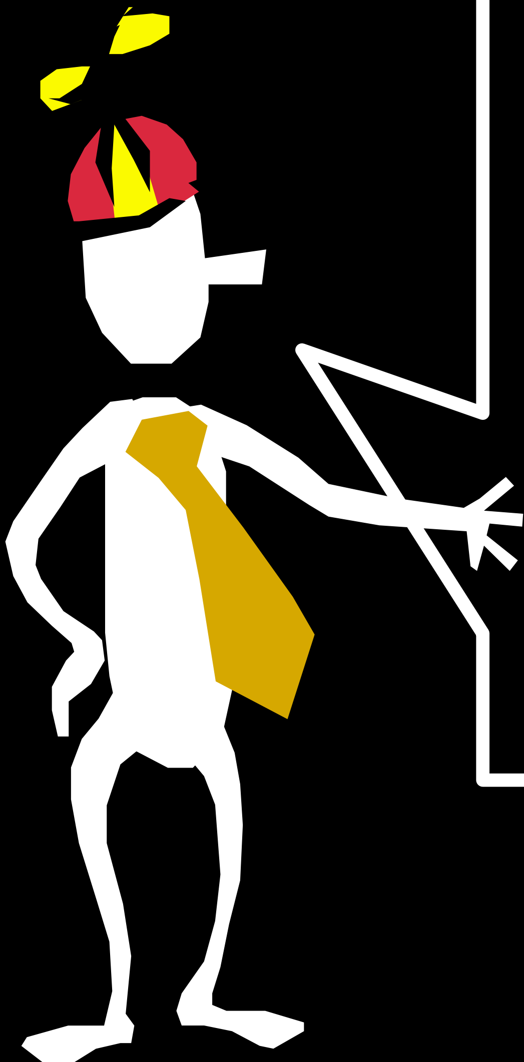


ONGOING MEDITATION:

Let S be any set and T be a binary choice tree representation of S

Think of each element of S being encoded by binary sequences of choices that lead to its leaf

We can also start with a binary encoding of a set and make a corresponding binary choice tree



Now, for something completely different…

How many ways to rearrange the letters in the word "SYSTEMS"?

SYSTEMS

7 places to put the Y,
6 places to put the T,
5 places to put the E,
4 places to put the M,
and the S's are forced

$$7 \times 6 \times 5 \times 4 = 840$$

SYSTEMS

Let's pretend that the S's are distinct:

$S_1YS_2TEMS_3$

There are 7! permutations of $S_1YS_2TEMS_3$

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$$\frac{7!}{3!} = 840$$

Arrange n symbols: r_1 of type 1, r_2 of type 2, …, r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\ldots-r_{k-1}}{r_k}$$

$$= \frac{n!}{(n-r_1)!r_1!}\ \frac{(n-r_1)!}{(n-r_1-r_2)!r_2!}\ \cdots$$

$$= \frac{n!}{r_1!r_2!\ \ldots\ r_k!}$$

CARNEGIEMELLON

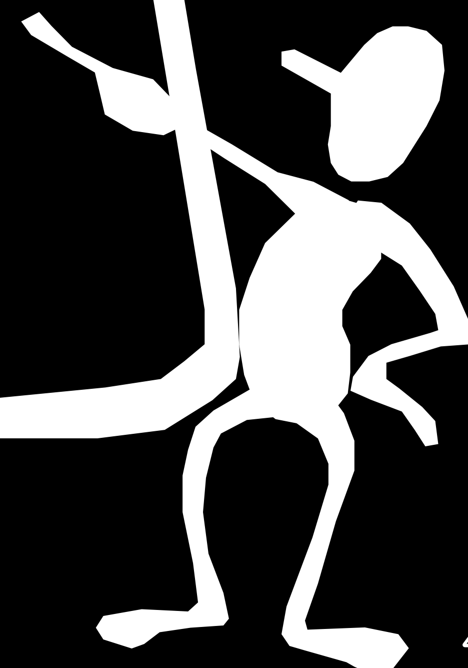
CARNEGIEMELLON

$$\frac{14!}{2!3!2!} = 3,632,428,800$$

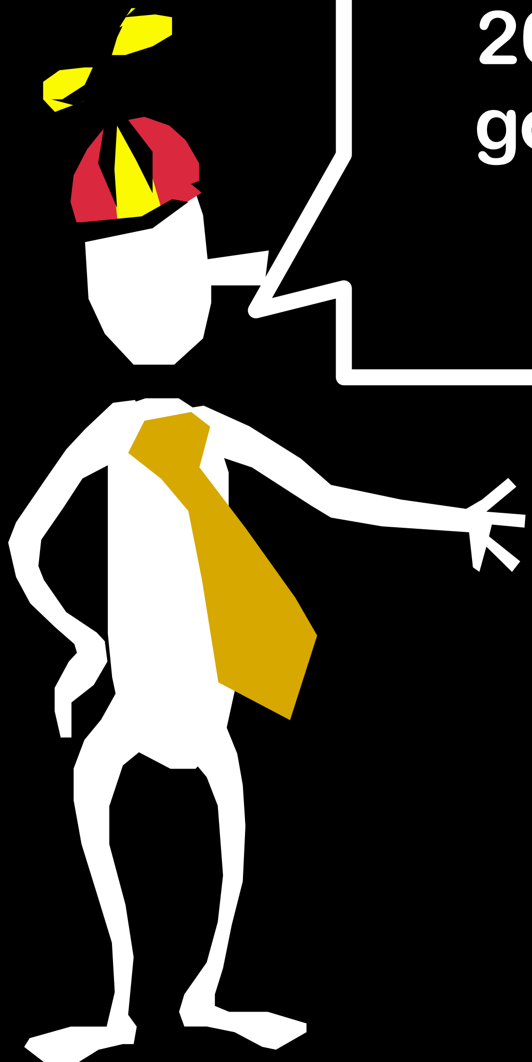
Remember:

The number of ways to arrange n symbols with r_1 of type 1, r_2 of type 2, …, r_k of type k is:

$$\frac{n!}{r_1!r_2! \ldots r_k!}$$



5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?



Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/

represents the following division among the pirates: 2, 1, 0, 17, 0

In general, the ith pirate gets the number of G's after the i-1st / and before the ith /

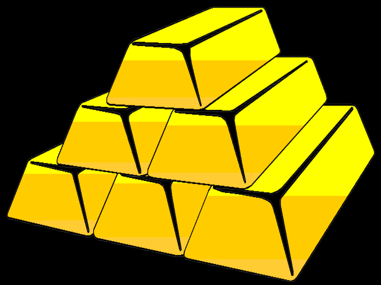
This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s

How many different ways to divide up the loot?

Sequences with 20 G’s and 4 /’s

$$\binom{24}{4}$$

How many different ways can n distinct pirates divide k identical, indivisible bars of gold?



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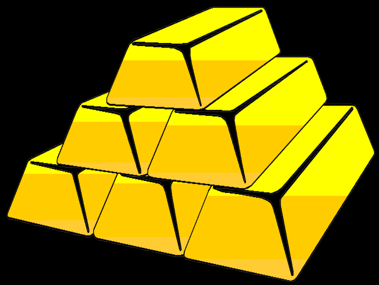

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

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Think of x_k are being the number of gold bars that are allotted to pirate k

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Think of x_k are being the number of gold bars that are allotted to pirate k

$$\binom{24}{4}$$

How many integer solutions to the following equations?

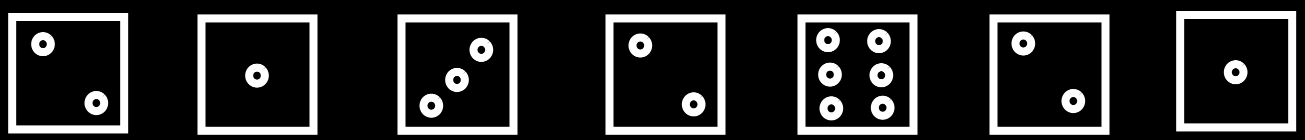
$$x_1 + x_2 + x_3 + \ldots + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

$$\binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k}$$

Identical/Distinct Dice

Suppose that we roll seven dice



How many different outcomes are there, if order matters?

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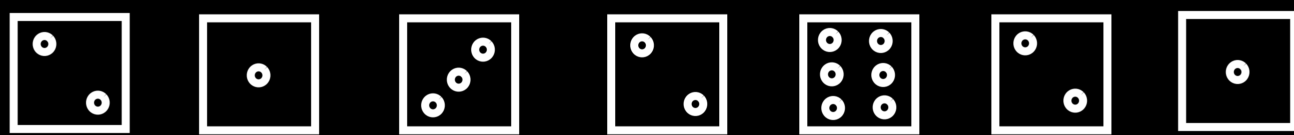

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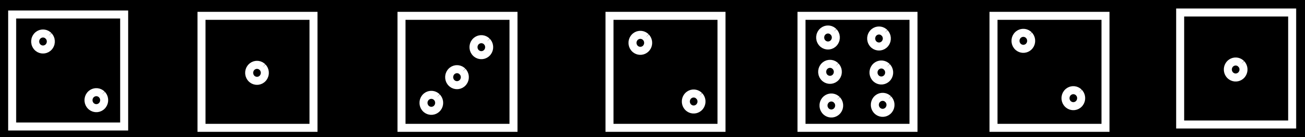

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What if order doesn’t matter? (E.g., Yahtzee)

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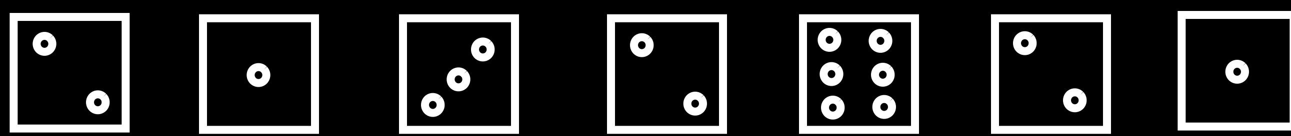

How many different outcomes are there, if order matters?

$$6^7$$

What if order doesn't matter? (E.g., Yahtzee)

$$\binom{12}{5} = \binom{12}{7}$$

Back to the Pirates

How many ways are there of choosing 20 pirates from a set of 5 distinct pirates, with repetitions allowed?

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How many ways are there of choosing 20 pirates from a set of 5 distinct pirates, with repetitions allowed?

$$\binom{5 + 20 - 1}{20} = \binom{24}{20} = \binom{24}{4}$$

Multisets

A multiset is a set of elements, each of which has a multiplicity

The size of the multiset is the sum of the multiplicities of all the elements

Example:

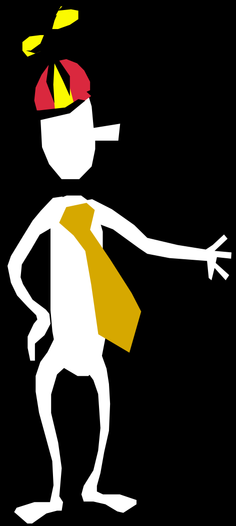
{X, Y, Z} with m(X)=0 m(Y)=3, m(Z)=2

Unary visualization: {Y, Y, Y, Z, Z}

Counting Multisets

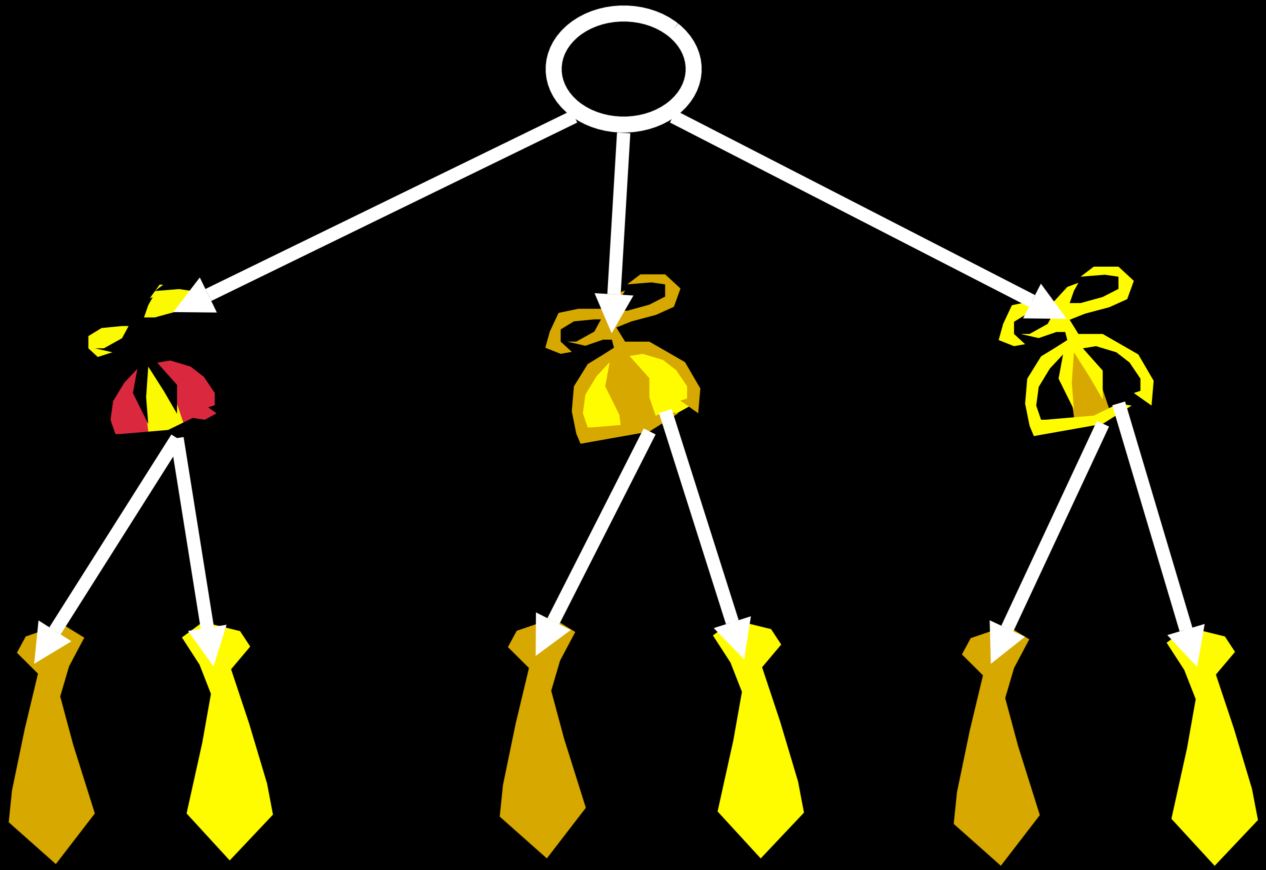
There number of ways
to choose a multiset of
size k from n types of elements is:

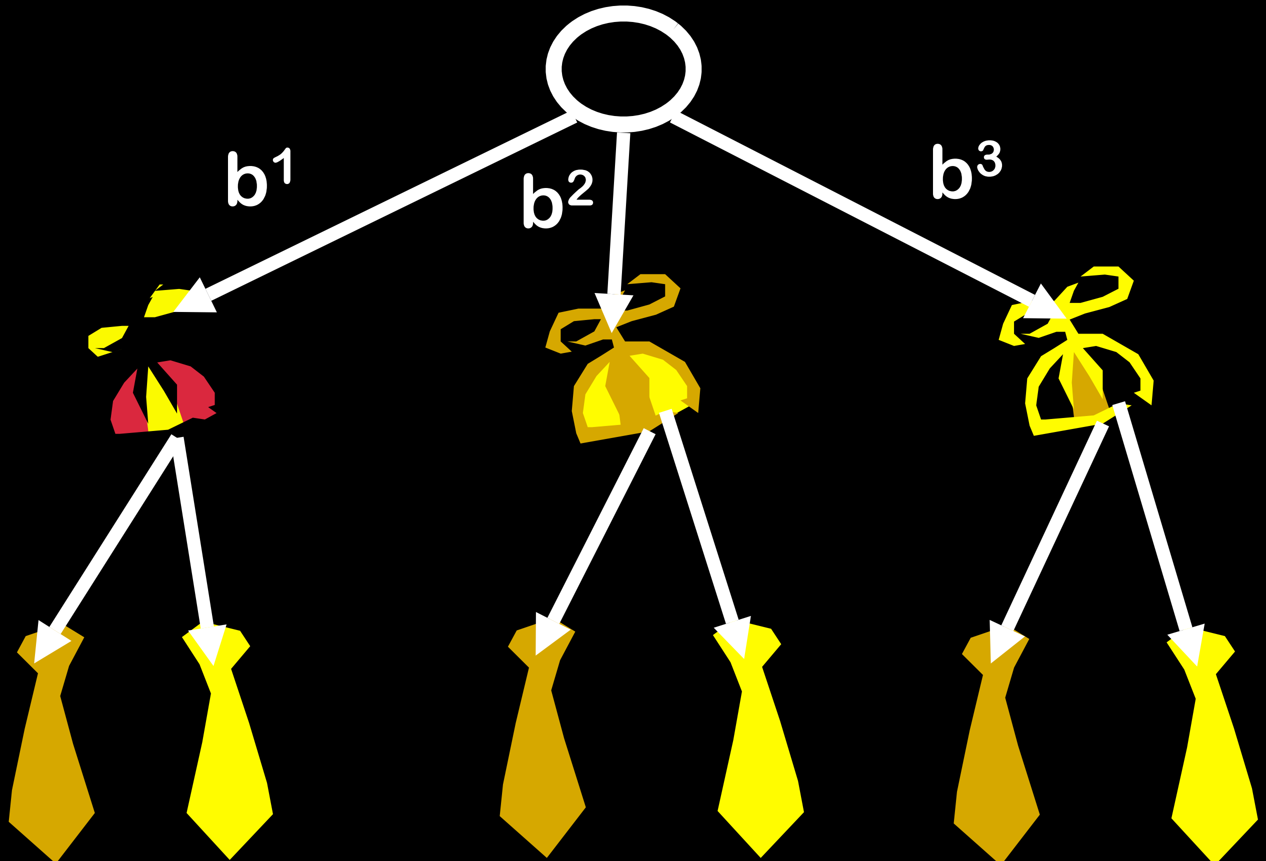
$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$



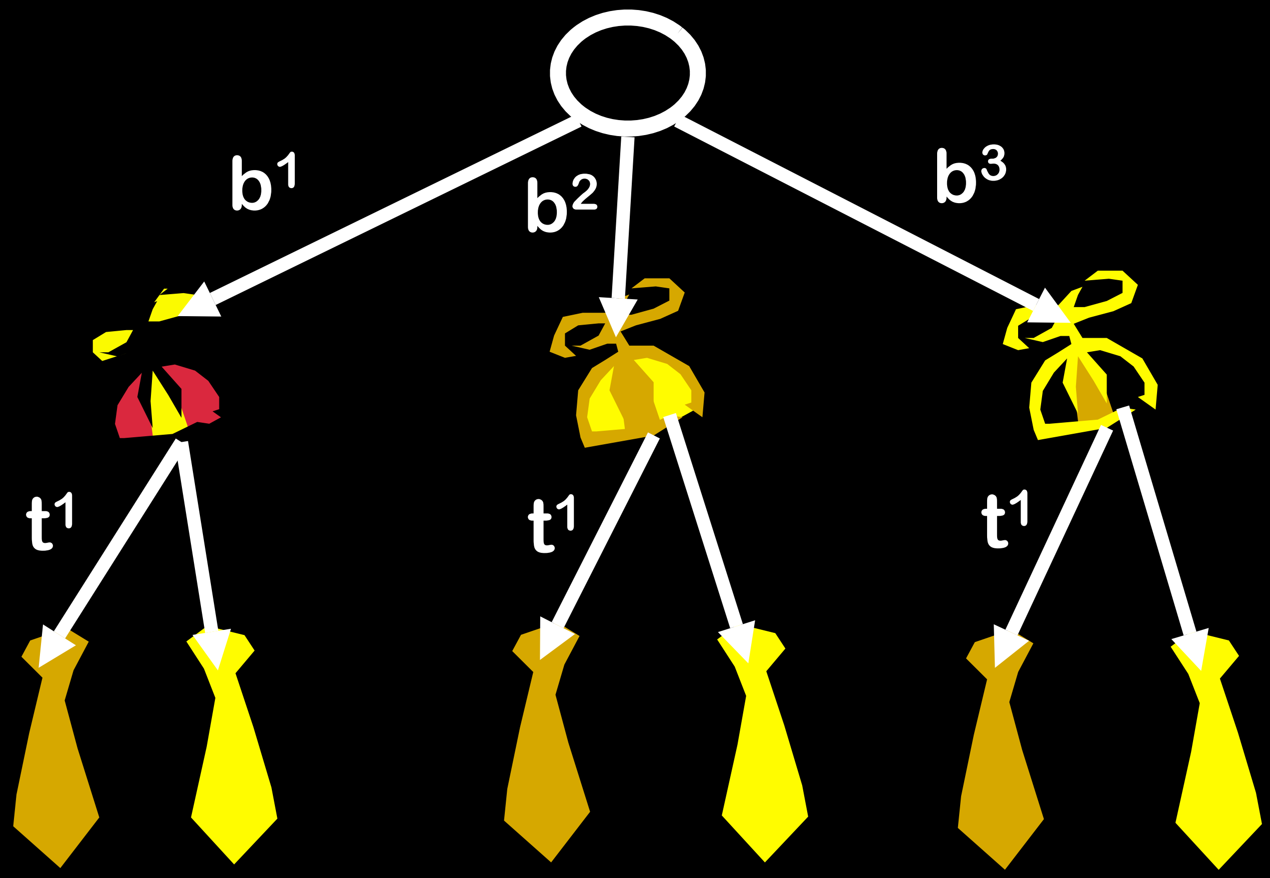
Polynomials Express Choices and Outcomes

Products of Sum = Sums of Products

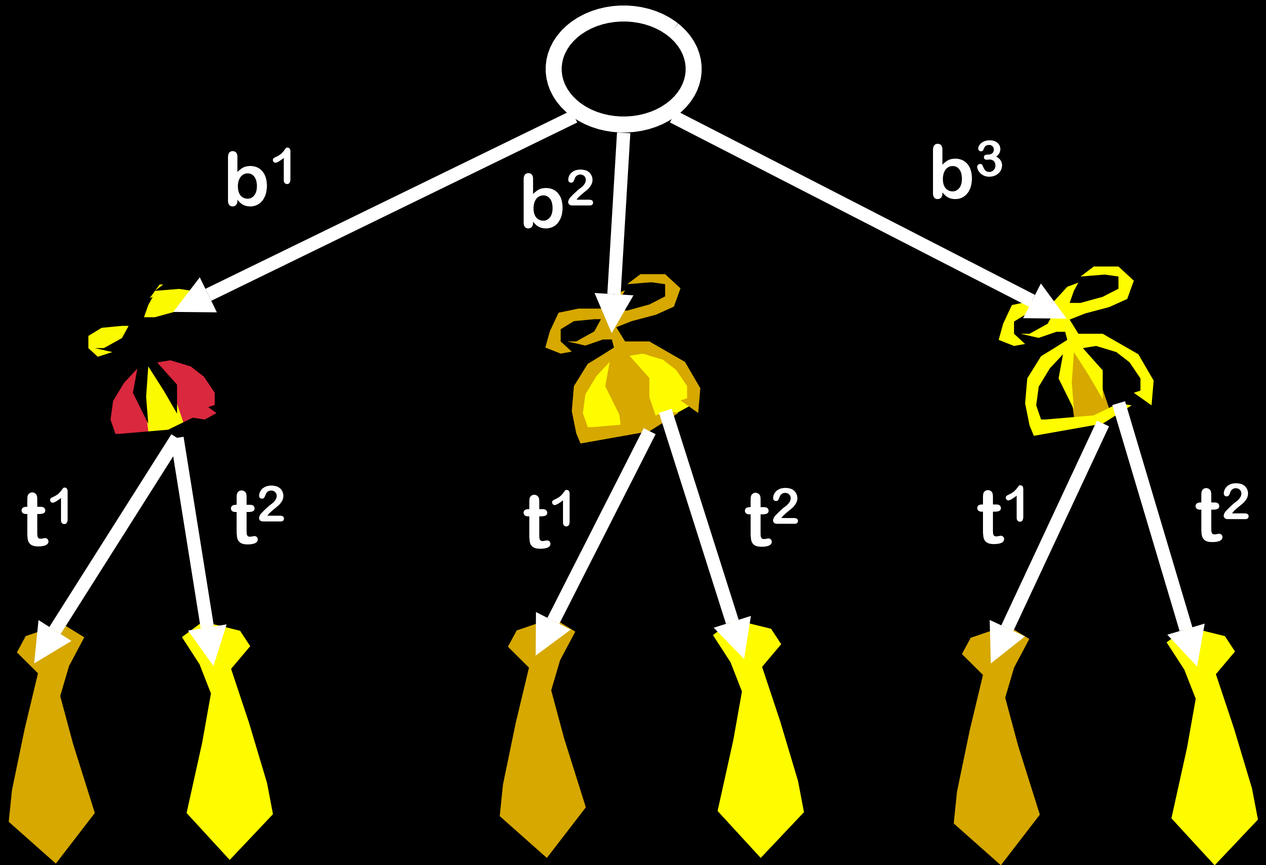




b^1
b^2
b^3



b^1
b^2
b^3
t^1
t^1
t^1



b^1
b^2
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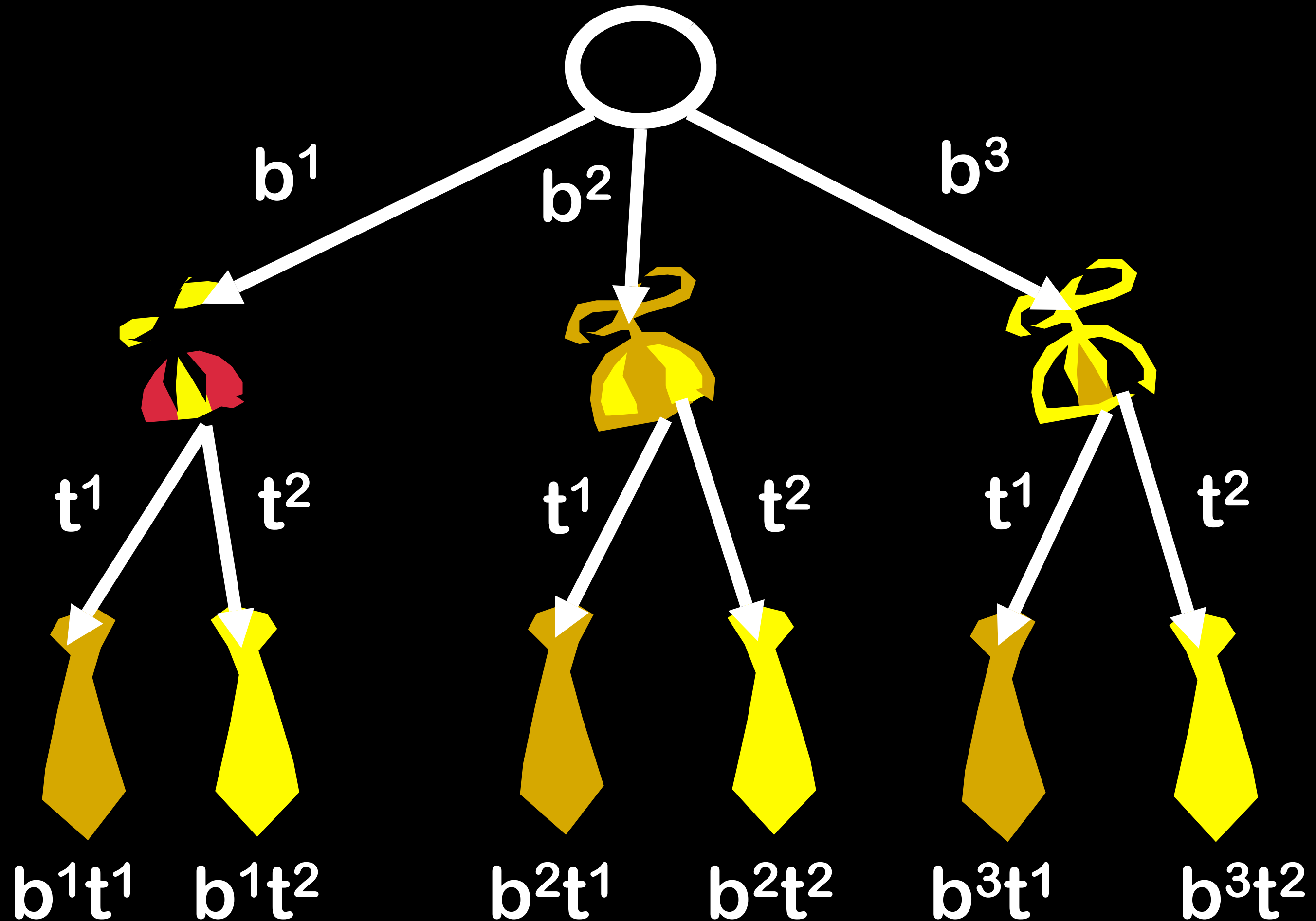
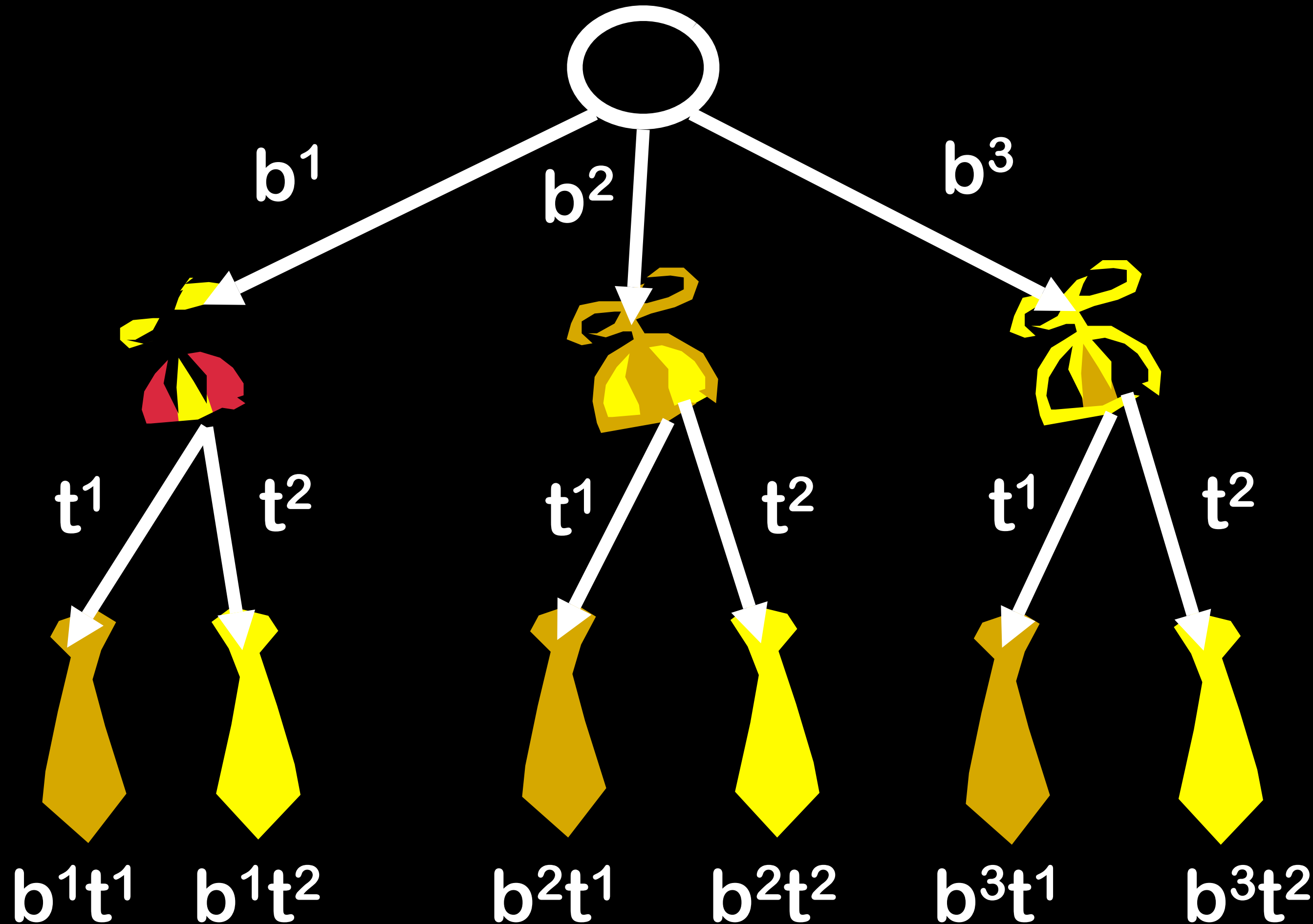
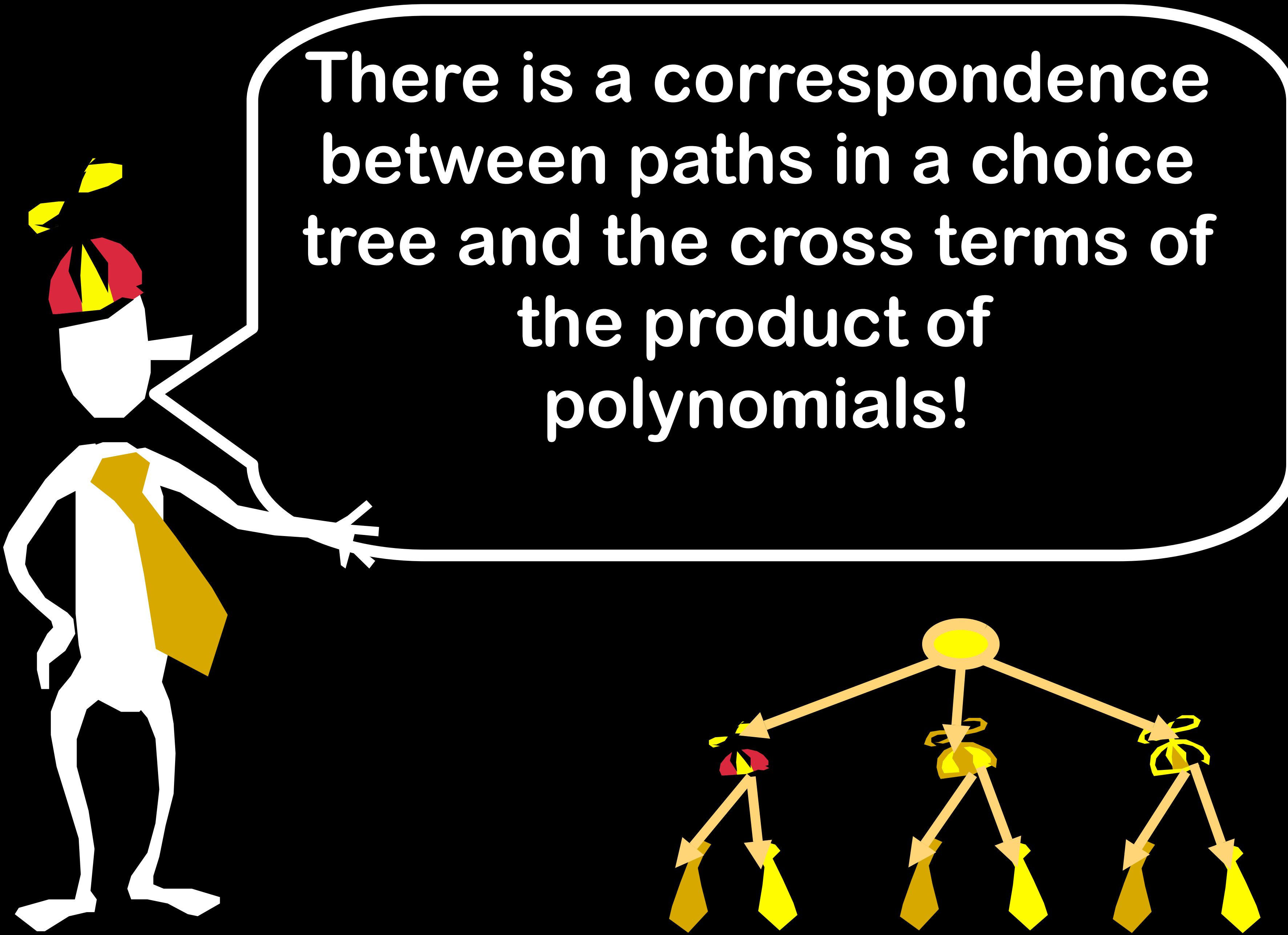
t^1
t^2
t^1
t^2
t^1
t^2



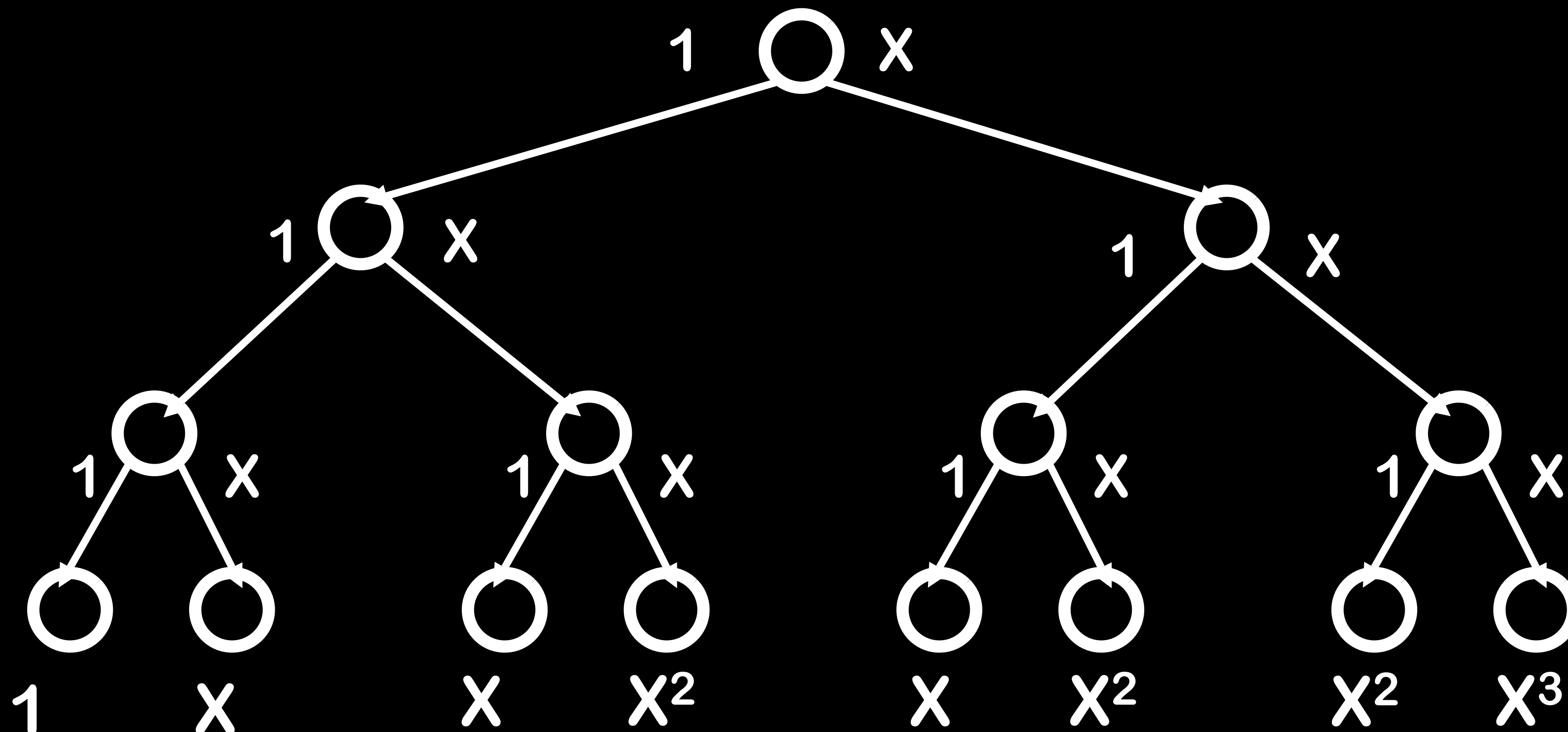
b^1
b^2
b^3
t^1
t^2
t^1
t^2
t^1
t^2
b^1t^1
b^1t^2
b^2t^1
b^2t^2
b^3t^1
b^3t^2



$$(b^1+b^2+b^3)(t^1+t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2$$


There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

Choice Tree for Terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

What is a Closed Form Expression For c_k?

$$(1+X)^n = c_0 + c_1X + c_2X^2 + \ldots + c_nX^n$$

$$(1+X)(1+X)(1+X)(1+X)\ldots(1+X)$$

After multiplying things out, but before combining like terms, we get 2^n cross terms, each corresponding to a path in the choice tree

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$$c_k = \binom{n}{k}$$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} X^0 + \binom{n}{1} X^1 + \ldots + \binom{n}{n} X^n$$

Binomial Coefficients

binomial
expression

The Binomial Formula

$$(1+X)^0 = 1$$

$$(1+X)^1 = 1 + 1X$$

$$(1+X)^2 = 1 + 2X + 1X^2$$

$$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$$

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$$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$$

$$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$$

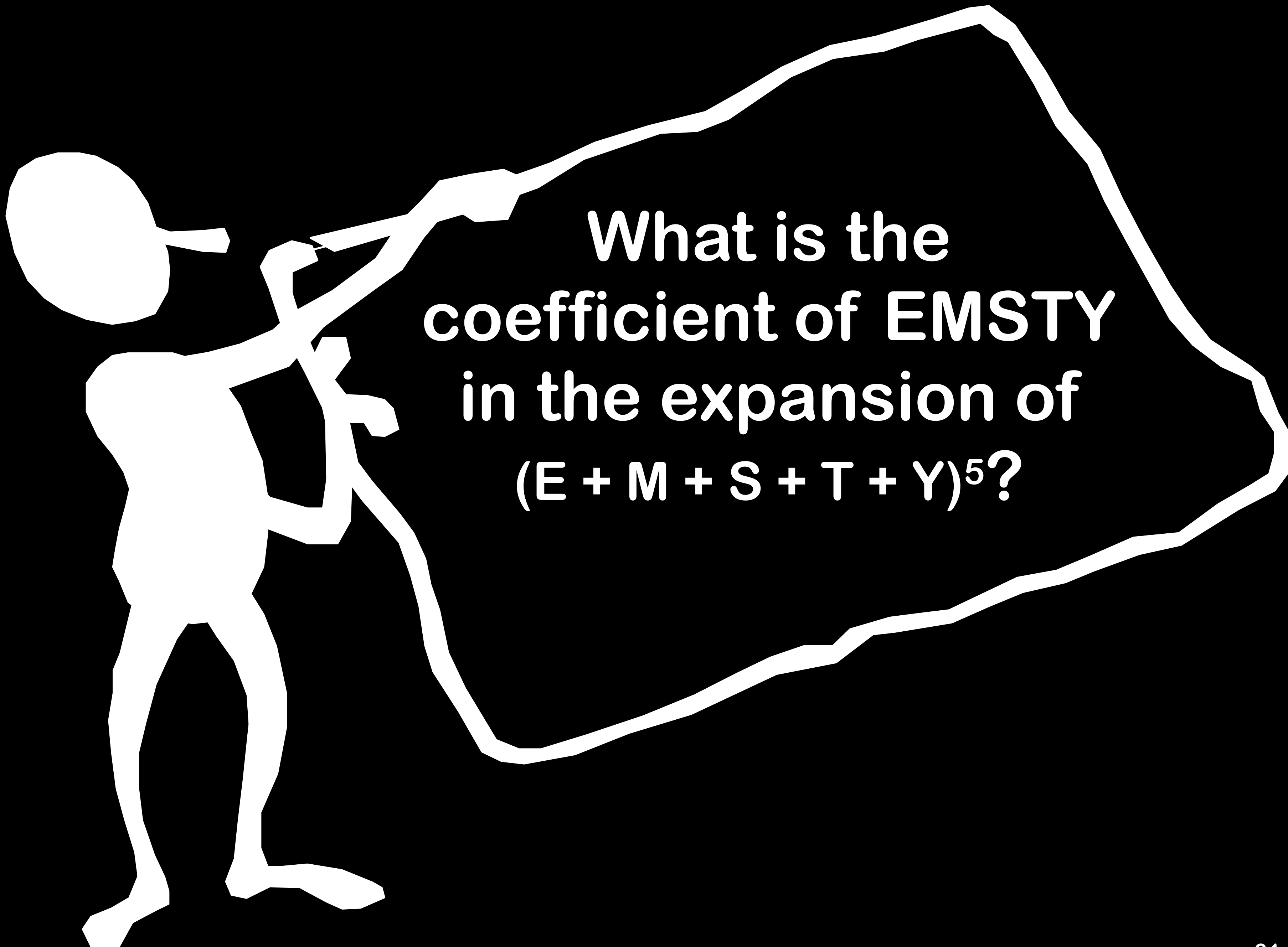
The Binomial Formula

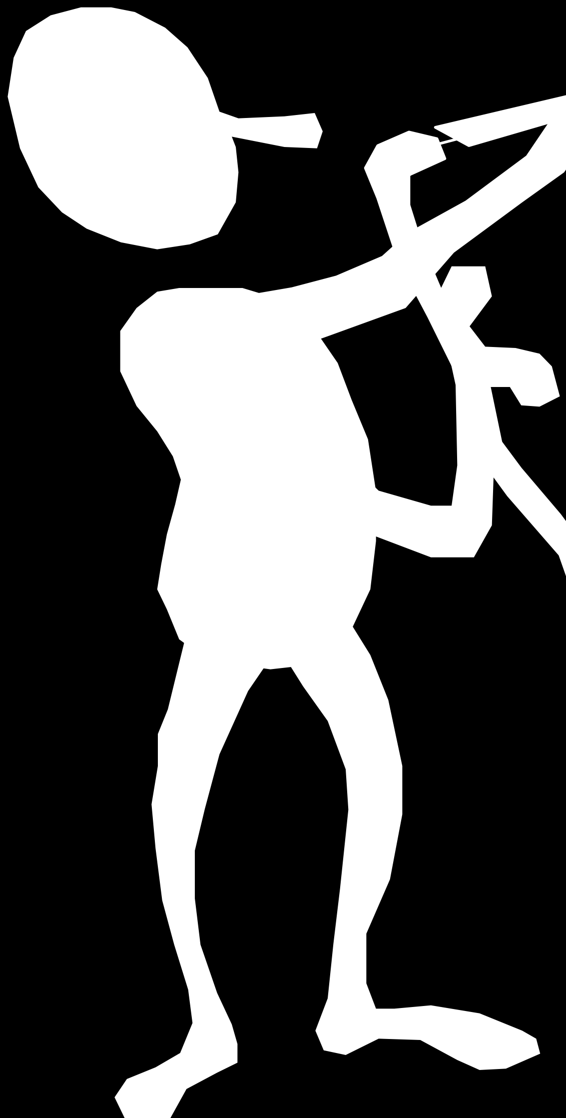
$$(X+Y)^n = \binom{n}{0} X^n Y^0 + \binom{n}{1} X^{n-1} Y^1$$

$$+ \ldots + \binom{n}{k} X^{n-k} Y^k + \ldots + \binom{n}{n} X^0 Y^n$$

The Binomial Formula

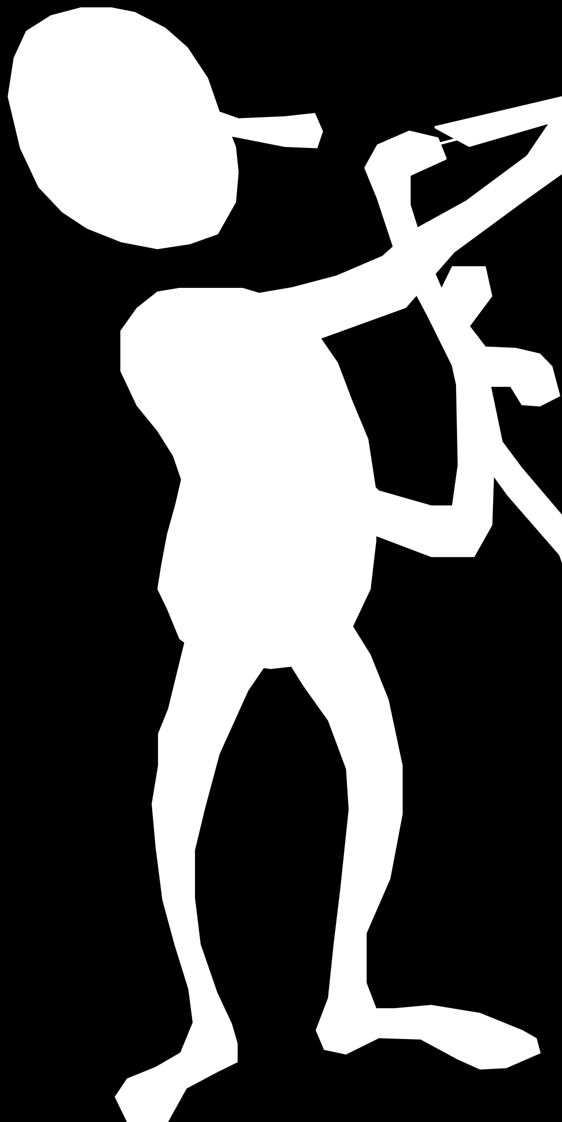
$$(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^{n-k} Y^k$$


What is the
coefficient of EMSTY
in the expansion of
$(E + M + S + T + Y)^5$?

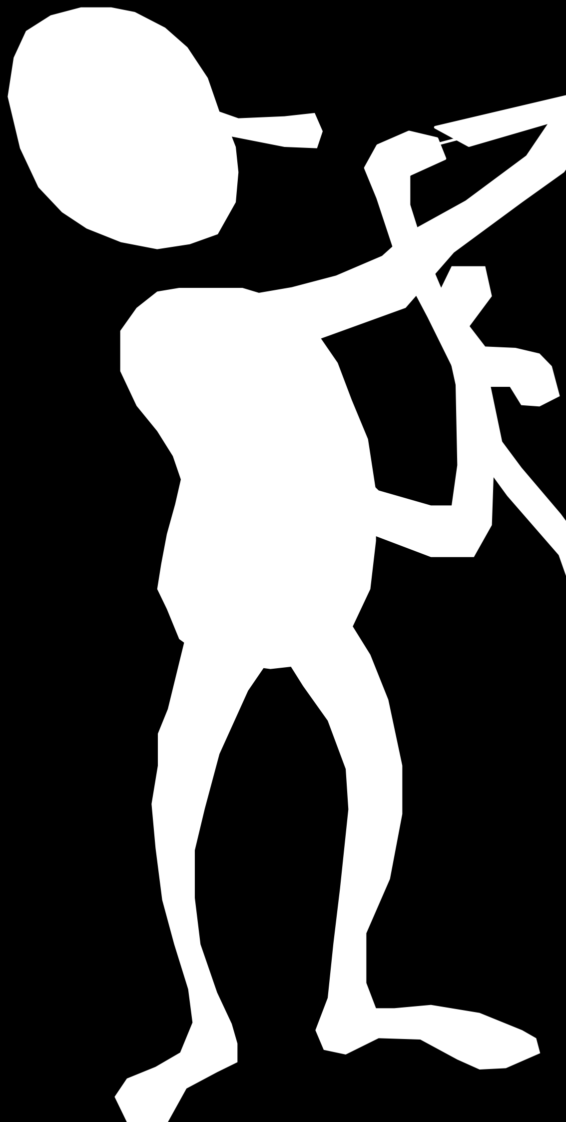


What is the coefficient of EMSTY in the expansion of $(E + M + S + T + Y)^5$?

5!

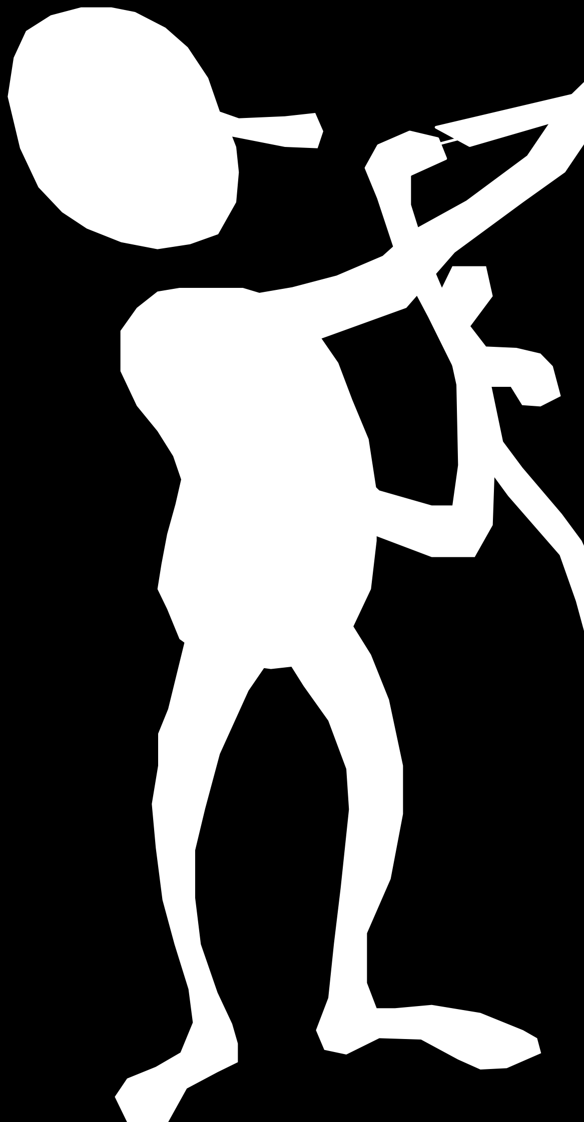


What is the coefficient of EMS^3TY in the expansion of $(E + M + S + T + Y)^7$?

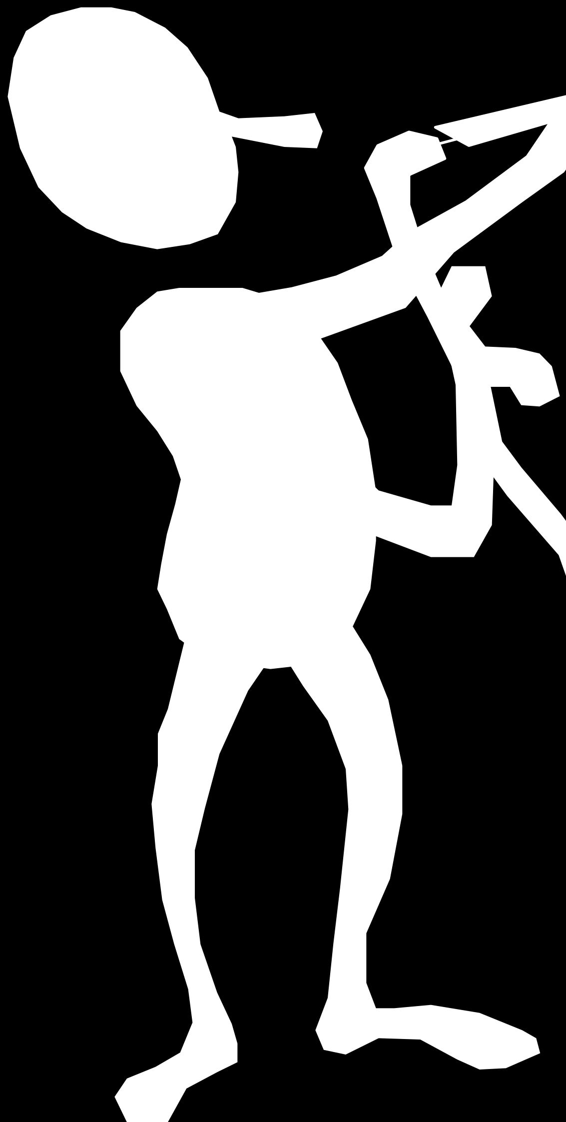


What is the coefficient of EMS^3TY in the expansion of $(E + M + S + T + Y)^7$?

The number of ways to rearrange the letters in the word SYSTEMS

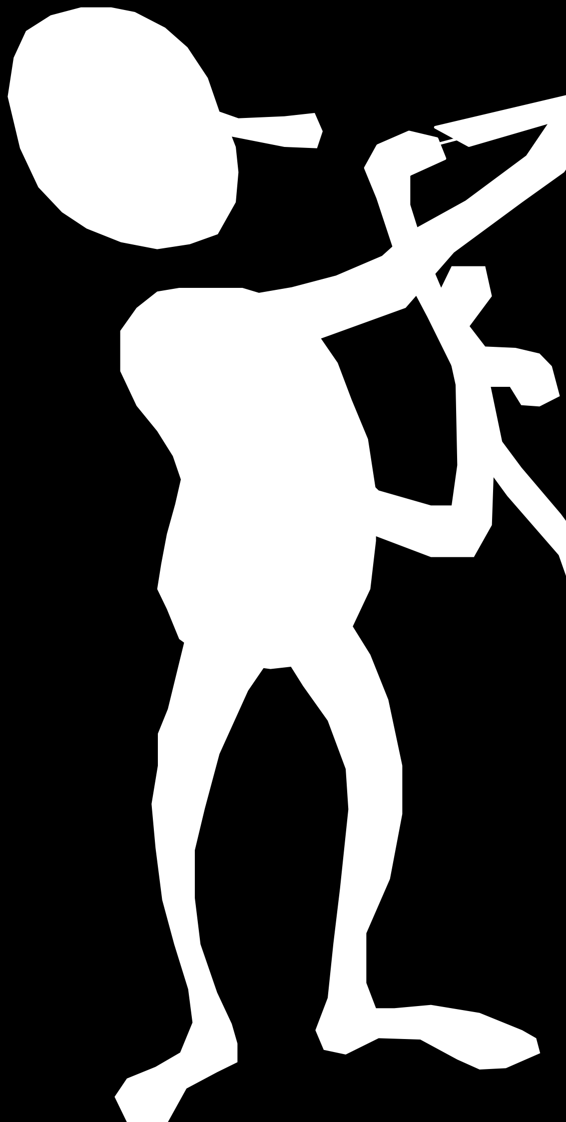


What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?

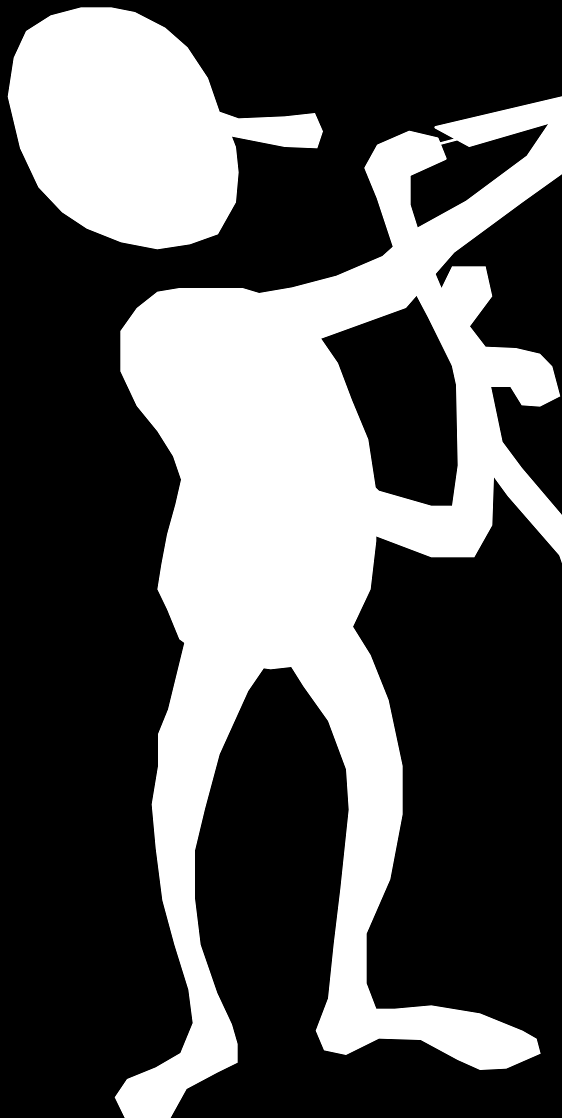


What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?

The number of ways to rearrange the letters in the word BANANA

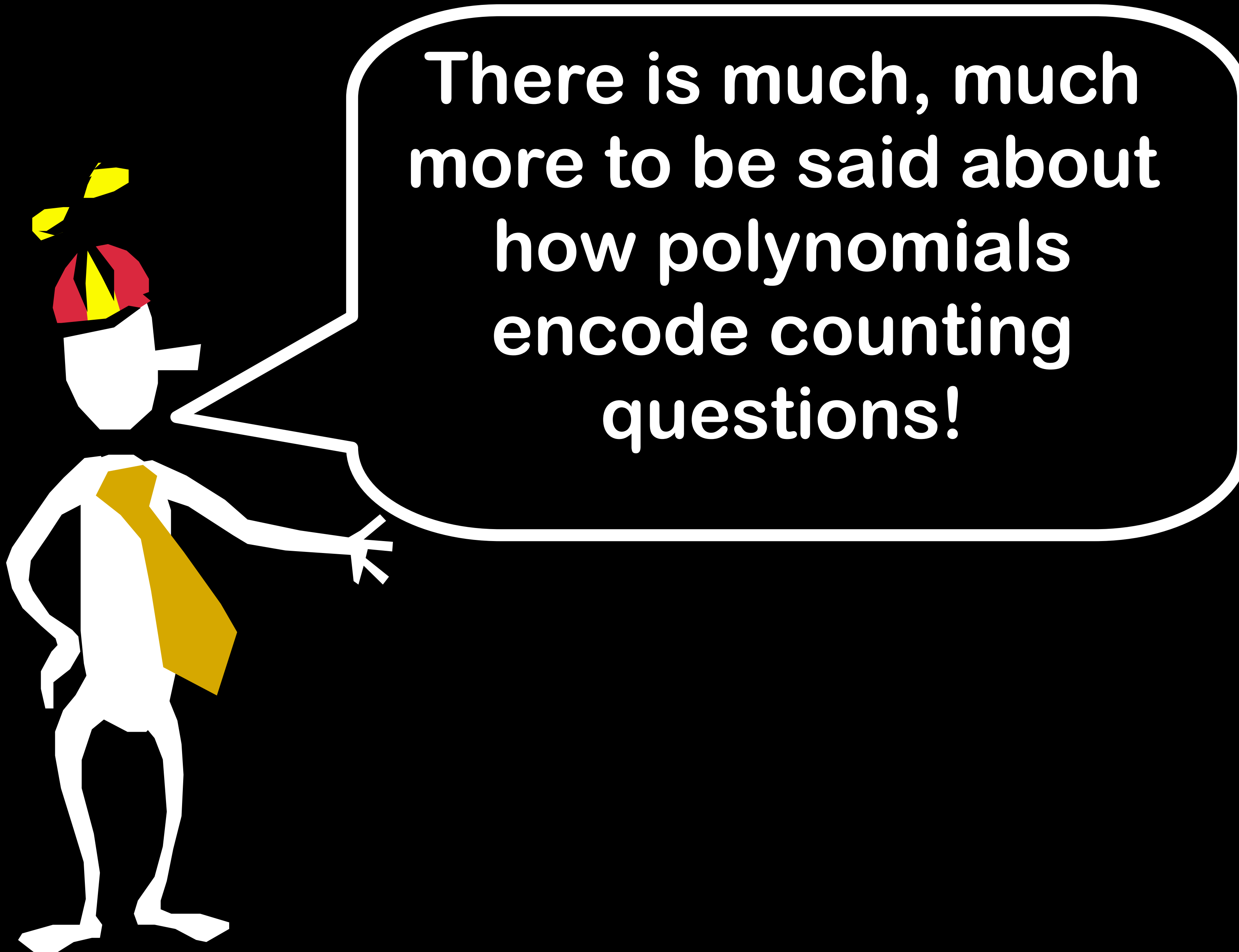


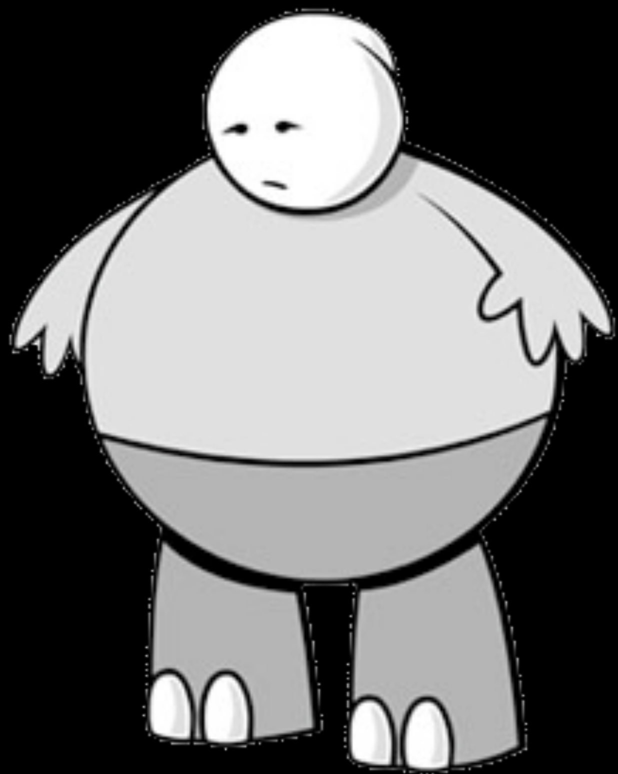
What is the coefficient of $(X_1^{r_1}X_2^{r_2}\ldots X_k^{r_k})$ in the expansion of $(X_1+X_2+X_3+\ldots+X_k)^n$?



What is the coefficient of $(X_1^{r_1}X_2^{r_2}\ldots X_k^{r_k})$ in the expansion of $(X_1+X_2+X_3+\ldots+X_k)^n$?

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$


There is much, much more to be said about how polynomials encode counting questions!



Here's What You Need to Know…

Inclusion-Exclusion

Counting Poker Hands

Number of rearrangements (multinomial coefficients)

Pirates and Gold

Counting Multisets

Binomial Formula