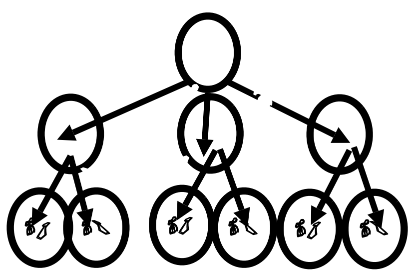
15-251

Great Theoretical Ideas in Computer Science

Counting I: One-To-One Correspondence and Choice Trees

Lecture 6, September 11, 2008



Addition Rule

Let A and B be two disjoint finite sets

The size of $(A \cup B)$ is the sum of the size of A and the size of B

$$|A \cup B| = |A| + |B|$$

Addition Rule
(2 possibly overlapping sets)

Let A and B be two finite sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Addition of multiple disjoint sets:

Let $A_1, A_2, A_3, \ldots, A_n$ be disjoint, finite sets.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Partition Method

To count the elements of a finite set S, partition the elements into non-overlapping subsets $A_1, A_2, A_3, \ldots, A_n$.

$|s|$ =

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Partition Method

S = all possible outcomes of one white die and one black die.

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Partition S into 6 sets:

A_1 = the set of outcomes where the white die is 1.
A_2 = the set of outcomes where the white die is 2.
A_3 = the set of outcomes where the white die is 3.
A_4 = the set of outcomes where the white die is 4.
A_5 = the set of outcomes where the white die is 5.
A_6 = the set of outcomes where the white die is 6.

Each of 6 disjoint sets has size 6 = 36 outcomes

Partition Method

S = all possible outcomes where the white die and the black die have different values

$S \equiv$ Set of all outcomes where the dice show different values. $|S| = ?$

$A_i \equiv$ set of outcomes where black die says i and the white die says something else.

$$|S| = \left|\bigcup_{i=1}^{6} A_i\right| = \sum_{i=1}^{6} |A_i| = \sum_{i=1}^{6} 5 = 30$$

$S \equiv$ Set of all outcomes where the dice show different values. $|S| = ?$

$T \equiv$ set of outcomes where dice agree.
= { <1,1>, <2,2>, <3,3>,<4,4>,<5,5>,<6,6>}

$| S \cup T | = \#$ of outcomes $= 36$

$|S| + |T| = 36$

$|T| = 6$

$|S| = 36 – 6 = 30$

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$A_i \equiv$ set of outcomes where the black die says i and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

$$|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15$$

S ≡ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

L ≡ set of all outcomes where the black die shows a larger number than the white die.

$$|S| + |L| = 30$$

It is clear by symmetry that $|S| = |L|$.

Therefore $|S| = 15$

"It is clear by symmetry that |S| = |L|?"

Pinning Down the Idea of Symmetry by Exhibiting a Correspondence

Put each outcome in S in correspondence with an outcome in L by swapping color of the dice.

$$S \longleftrightarrow L$$

Each outcome in S gets matched with exactly one outcome in L, with none left over.

Thus: $|S| = |L|$

Let $f : A \rightarrow B$ Be a Function From a Set A to a Set B

f is 1-1 if and only if

$$\forall x,y \in A,\ x \neq y \Rightarrow f(x) \neq f(y)$$

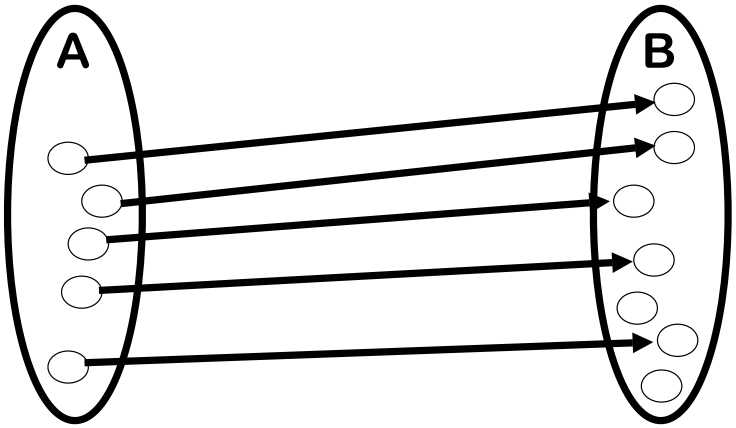
f is onto if and only if

$$\forall z \in B\ \ \exists x \in A\ \ f(x) = z$$

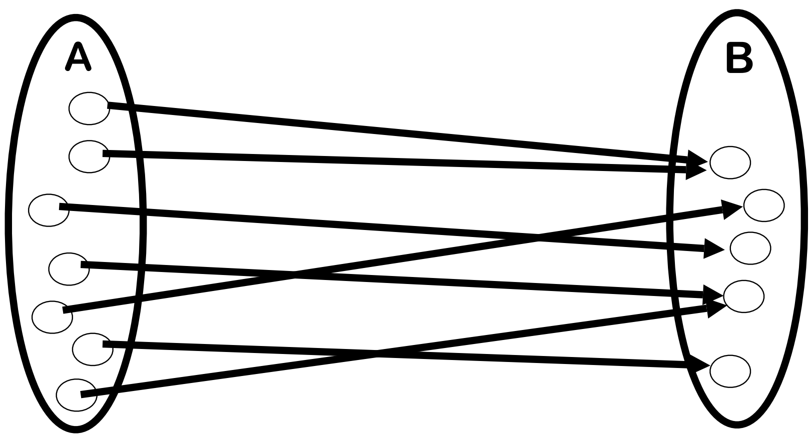
For Every

There Exists

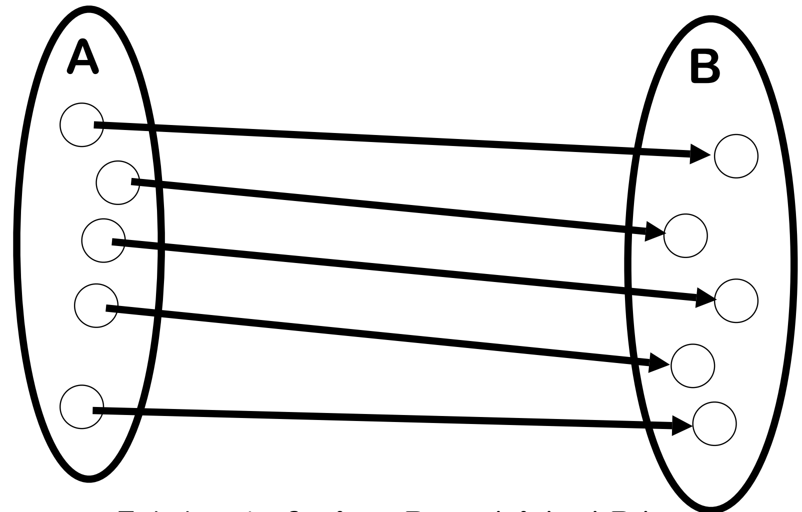
Let's Restrict Our Attention to <u>Finite</u> Sets



$\exists$ 1-1 $f : A \rightarrow B \Rightarrow |A| \leq |B|$

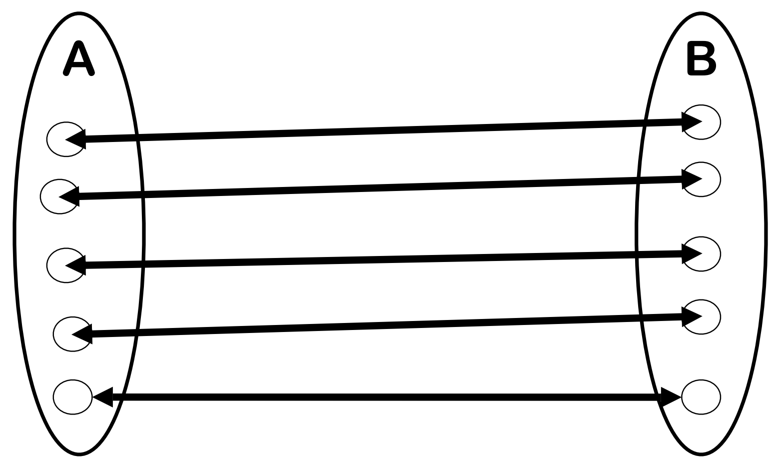


$\exists$ onto $f : A \rightarrow B \Rightarrow |A| \geq |B|$



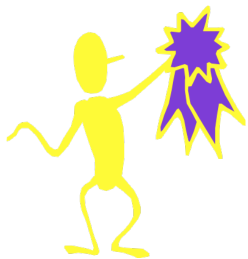
$\exists$ 1-1 onto $f : A \rightarrow B \Rightarrow |A| = |B|$

f being 1-1 onto means f^{-1} is well defined and unique



Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size



It's one of the most important mathematical ideas of all time!

Question: How many n-bit sequences are there?

000000	↔	0
000001	↔	1
000010	↔	2
000011	↔	3
:	:	:
111111	↔	2^n-1

Each sequence corresponds to a unique number from 0 to 2^n-1. Hence 2^n sequences.

A = { a,b,c,d,e } Has Many Subsets

{a}, {a,b}, {a,d,e}, {a,b,c,d,e}, {e}, Ø, …

The entire set and the empty set are subsets with all the rights and privileges pertaining thereto

Question: How Many Subsets Can Be Made From The Elements of a 5-Element Set?

a	b	c	d	e
0	1	1	0	1

{ b c e }

1 means "TAKE IT"
0 means "LEAVE IT"

Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

$A = \{a_1, a_2, a_3, \ldots, a_n\}$
B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b = b_1b_2b_3\ldots b_n$, let $f(b) = \{ a_i \mid b_i=1\}$

Claim: f is 1-1

Any two distinct binary sequences b and b′ have a position i at which they differ

Hence, f(b) is not equal to f(b′) because they disagree on element a_i

$A = \{a_1, a_2, a_3, \ldots, a_n\}$
B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b = b_1b_2b_3\ldots b_n$, let $f(b) = \{ a_i \mid b_i=1\}$

Claim: f is onto

Let S be a subset of $\{a_1,\ldots,a_n\}$.
Define $b_k = 1$ if a_k in S and $b_k = 0$ otherwise.
Note that $f(b_1b_2\ldots b_n) = S$.

The number of subsets of an n-element set is 2^n

Let $f : A \to B$ Be a Function From Set A to Set B

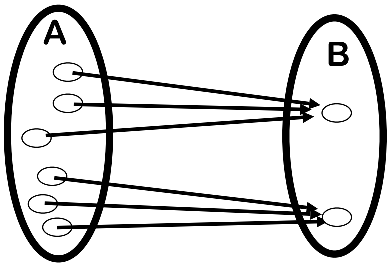
f is 1-1 if and only if
$\forall x,y \in A,\ x \neq y \Rightarrow f(x) \neq f(y)$

f is onto if and only if
$\forall z \in B\ \exists x \in A$ such that $f(x) = z$

Let f : A → B Be a Function From Set A to Set B

f is a 1-to-1 correspondence iff
$\forall z \in B\ \exists$ exactly one $x \in A$ such that $f(x) = z$

f is a k-to-1 correspondence iff
$\forall z \in B\ \exists$ exactly k $x \in A$ such that $f(x) = z$

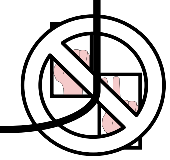


3 to 1 function

To count the number of horses in a barn, we can count the number of hoofs and then divide by 4

If a finite set A has a k-to-1 correspondence to finite set B, then |B| = |A|/k

How many seats in this auditorium?

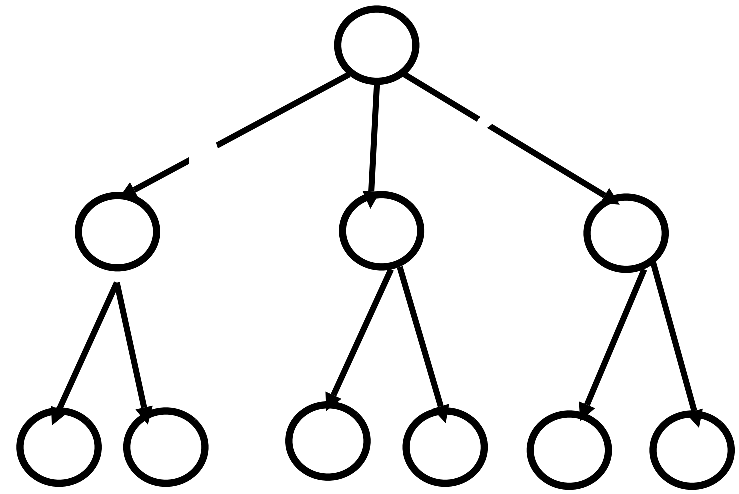


Count without Counting:
The auditorium can be partitioned into n rows with k seats each

Thus, we have nk seats in the room

Choice Trees

I own 3 beanies and 2 ties. How many different ways can I dress up in a beanie and a tie?



A Restaurant Has a Menu With 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts

How many items on the menu?

$5 + 6 + 3 + 7 = 21$

How many ways to choose a complete meal?

$5 \times 6 \times 3 \times 7 = 630$

How many ways to order a meal if I am allowed to skip some (or all) of the courses?

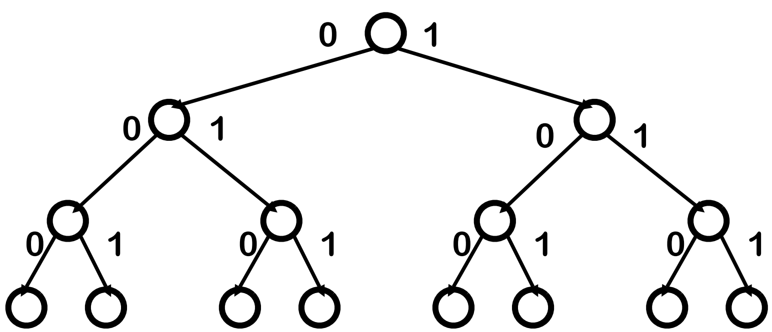
$6 \times 7 \times 4 \times 8 = 1344$

Hobson's Restaurant Has Only 1 Appetizer, 1 Entree, 1 Salad, and 1 Dessert

2^4 ways to order a meal if I might not have some of the courses

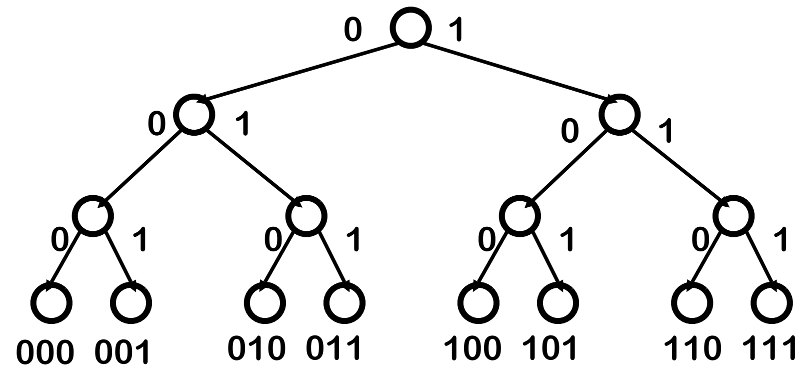
Same as number of subsets of the set {Appetizer, Entrée, Salad, Dessert}

Choice Tree For 2^n n-bit Sequences

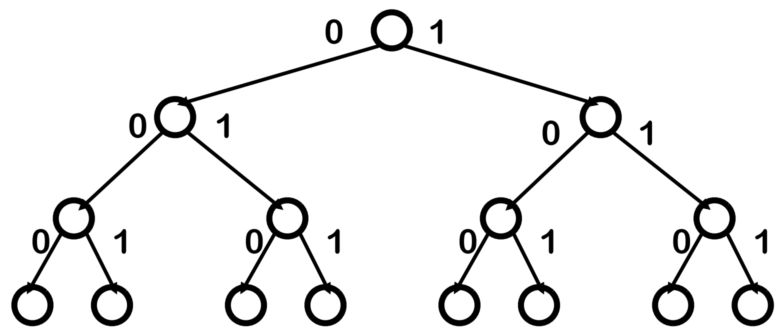


We can use a "choice tree" to represent the construction of objects of the desired type

Choice Tree For 2^n n-bit Sequences



Label each leaf with the object constructed by the choices along the path to the leaf



2 choices for first bit
× 2 choices for second bit
× 2 choices for third bit
: :
× 2 choices for the n^{th}

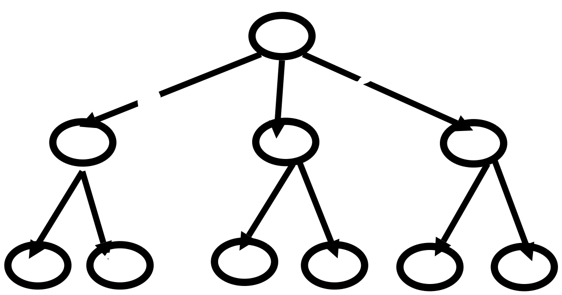
Leaf Counting Lemma

Let T be a depth-n tree when each node at depth $0 \le i \le n-1$ has P_{i+1} children

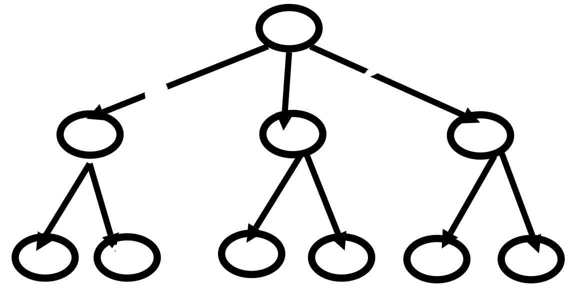
The number of leaves of T is given by:

$P_1 P_2 \ldots P_n$

Choice Tree

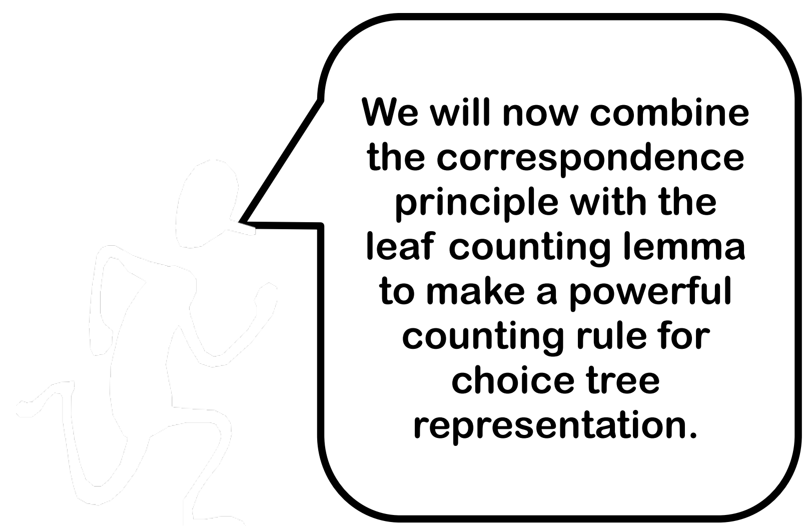


A choice tree is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf



A choice tree provides a "choice tree representation" of a set S, if

1. Each leaf label is in S, and each element of S is some leaf label
2. No two leaf labels are the same



Product Rule

IF set S has a choice tree representation with
P_1 possibilities for the first choice,
P_2 for the second, P_3 for the third,
and so on,
THEN
there are $P_1P_2P_3 \ldots P_n$ objects in S

Proof:

There are $P_1P_2P_3 \ldots P_n$ leaves of the choice tree which are in 1-1 onto correspondence with the elements of S.

Product Rule (Rephrased)

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

THEN

There are $P_1P_2P_3 \ldots P_n$ objects of type S

How Many Different Orderings of Deck With 52 Cards?

What object are we making? Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;
51 possible choices for the second card;
: :
1 possible choice for the 52nd card.

By product rule: $52 \times 51 \times 50 \times \ldots \times 2 \times 1 = 52!$

A permutation or arrangement of n objects is an ordering of the objects

The number of permutations of n distinct objects is n!

How many sequences of
7 letters are there?

$$26^7$$

(26 choices for each of the 7 positions)

How many sequences of
7 letters contain at least
two of the same letter?

$$26^7 - \underbrace{26\times25\times24\times23\times22\times21\times20}_{\text{number of sequences containing all different letters}}$$

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.

Helpful Advice:

In logic, it can be useful to represent a statement in the contra positive.

In counting, it can be useful to represent a set in terms of its complement.

If 10 horses race, how many orderings of the top three finishers are there?

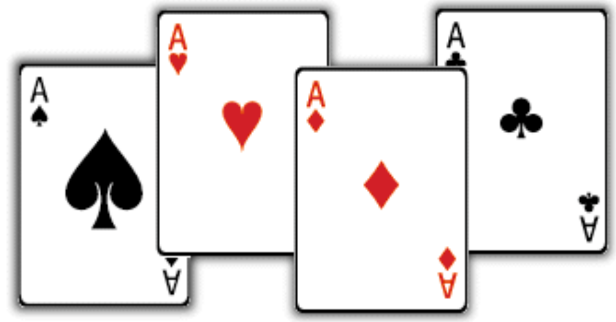
$$10 \times 9 \times 8 = 720$$

Number of ways of ordering, permuting, or arranging r out of n objects

n choices for first place, n-1 choices for second place, . . .

$$n \times (n-1) \times (n-2) \times \ldots \times (n-(r-1))$$

$$= \frac{n!}{(n-r)!}$$



Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52×51

How many unordered pairs?

$52 \times 51 / 2$ ← divide by overcount

Each unordered pair is listed twice on a list of the ordered pairs

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52×51

How many unordered pairs?

$52 \times 51 / 2$ ← divide by overcount

We have a 2-1 map from ordered pairs to unordered pairs.
Hence #unordered pairs = (#ordered pairs)/2

Ordered Versus Unordered

How many ordered 5 card sequences can be formed from a 52-card deck?

$52 \times 51 \times 50 \times 49 \times 48$

How many orderings of 5 cards?

$5!$

How many unordered 5 card hands?

$(52 \times 51 \times 50 \times 49 \times 48)/5! = 2{,}598{,}960$

A combination or choice of r out of n objects is an (unordered) set of r of the n objects

The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

n “choose” r

The number of subsets of size r that can be formed from an n-element set is:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Product Rule (Rephrased)

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

THEN

There are $P_1 P_2 P_3 \ldots P_n$ objects of type S

How Many 8-Bit Sequences Have 2 0's and 6 1's?

Tempting, but incorrect:
8 ways to place first 0, times
7 ways to place second 0

Violates condition 2 of product rule!

Choosing position i for the first 0 and then position j for the second 0 gives same sequence as choosing position j for the first 0 and position i for the second 0 } 2 ways of generating same object!

How Many 8-Bit Sequences Have 2 0's and 6 1's?

1. Choose the set of 2 positions to put the 0's. The 1's are forced.

$$\binom{8}{2}$$

2. Choose the set of 6 positions to put the 1's. The 0's are forced.

$$\binom{8}{6}$$

Symmetry In The Formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

"# of ways to pick r out of n elements"
=
"# of ways to choose the (n-r) elements to omit"

How Many Hands Have at Least 3 As?

How Many Hands Have at Least 3 As?

$\binom{4}{3}$ = 4 ways of picking 3 out of 4 aces

$\binom{49}{2}$ = 1176 ways of picking 2 cards out of the remaining 49 cards

$$4 \times 1176 = 4704$$

How Many Hands Have at Least 3 As?

How many hands have exactly 3 aces?

$\binom{4}{3}$ = 4 ways of picking 3 out of 4 aces

$\binom{48}{2}$ = 1128 ways of picking 2 cards out of the 48 non-ace cards

$$\begin{array}{r} 4 \\ \times\ 1128 \\ \hline 4512 \end{array}$$

How many hands have exactly 4 aces?

$\binom{4}{4}$ = 1 way of picking 4 out of 4 aces

$\binom{48}{1}$ = 48 ways of picking 1 cards out of the 48 non-ace cards

$$\begin{array}{r} 4512 \\ +\ 48 \\ \hline 4560 \end{array}$$

$$4704 \neq 4560$$

At least one of the two counting arguments is not correct!

Four Different Sequences of Choices Produce the Same Hand

$\binom{4}{3}$ = 4 ways of picking 3 out of 4 aces

$\binom{49}{2}$ = 1176 ways of picking 2 cards out of the remaining 49 cards

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

Is the other argument correct? How do I avoid fallacious reasoning?

REVERSIBILTY CHECK:
For each object can I reverse engineer the unique sequence of choices that constructed it?

Scheme I
1. Choose 3 of 4 aces
2. Choose 2 of the remaining cards

A♣ A♦ A♥A♠ K♦

For this hand – you can’t reverse to a unique choice sequence.

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A ♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

Is the other argument correct? How do I avoid fallacious reasoning?

Scheme II

1. Choose 3 out of 4 aces
2. Choose 2 out of 48 non-ace cards

A♣ A♦ Q♦ A♠ K♦

REVERSE TEST: Aces came from choices in (1) and others came from choices in (2)

Scheme II

1. Choose 4 out of 4 aces
2. Choose 1 out of 48 non-ace cards

A♣ A♦ A♥ A♠ K♦

REVERSE TEST: Aces came from choices in (1) and others came from choices in (2)

Product Rule (Rephrased)

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

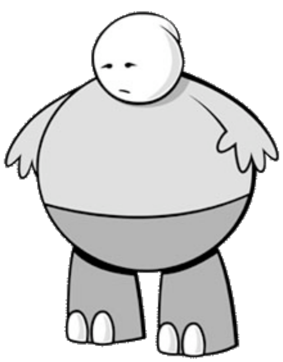
THEN

There are $P_1P_2P_3\ldots P_n$ objects of type S

DEFENSIVE THINKING
ask yourself:

Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?



Here's What You Need to Know…

Correspondence Principle
If two finite sets can be placed into 1-1 onto correspondence, then they have the same size

Choice Tree

Product Rule
two conditions

Reverse Test

Counting by complementing

Binomial coefficient