15-251

Great Theoretical Ideas in Computer Science

Ancient Wisdom: Unary and Binary

Lectur [illegible] 2008)

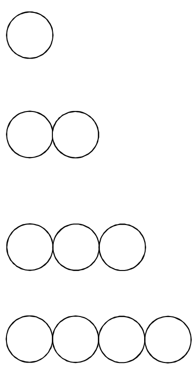
Prehistoric Unary

1

2

3

4



Consider the problem of finding a formula for the sum of the first n numbers

You already used induction to verify that the answer is ½n(n+1)

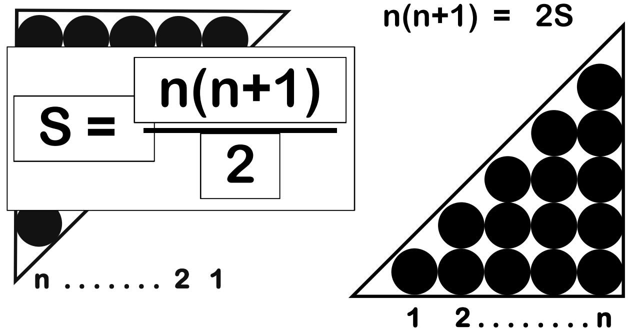
$$
\begin{array}{ccccccccccccc}
1 & + & 2 & + & 3 & + & \dots & + & n-1 & + & n & = & S \\
n & + & n-1 & + & n-2 & + & \dots & + & 2 & + & 1 & = & S \\
\hline
n+1 & + & n+1 & + & n+1 & + & \dots & + & n+1 & + & n+1 & = & 2S
\end{array}
$$

$$n(n+1) = 2S$$

$$S = \frac{n(n+1)}{2}$$

$$
\begin{array}{ccccccccccccc}
1 & + & 2 & + & 3 & + & \dots & + & n-1 & + & n & = & S \\
n & + & n-1 & + & n-2 & + & \dots & + & 2 & + & 1 & = & S \\
\hline
\end{array}
$$

$$n(n+1) = 2S$$

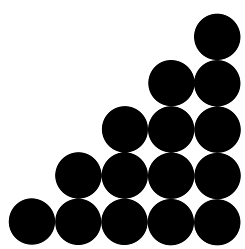


$$S = \frac{n(n+1)}{2}$$

n^{th} Triangular Number

$\Delta_n = 1 + 2 + 3 + \dots + n-1 + n$

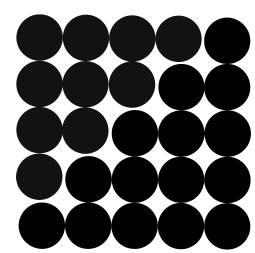
$= n(n+1)/2$

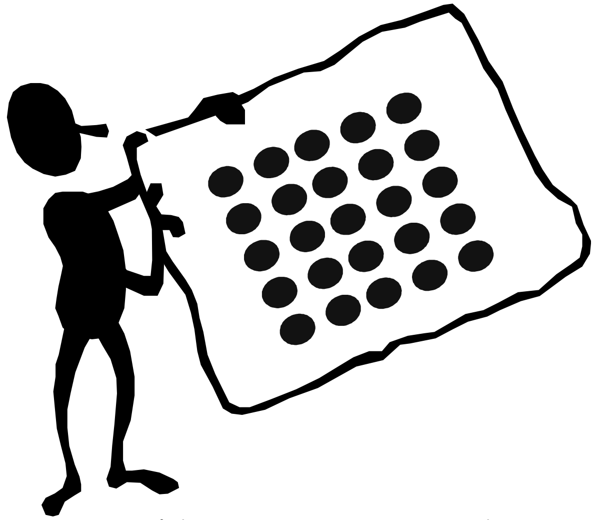


n^{th} Square Number

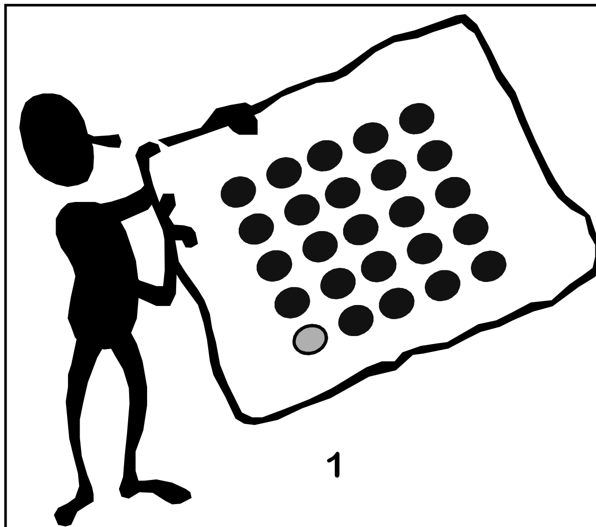
$\square_n = n^2$

$= \Delta_n + \Delta_{n-1}$





Breaking a square up in a new way



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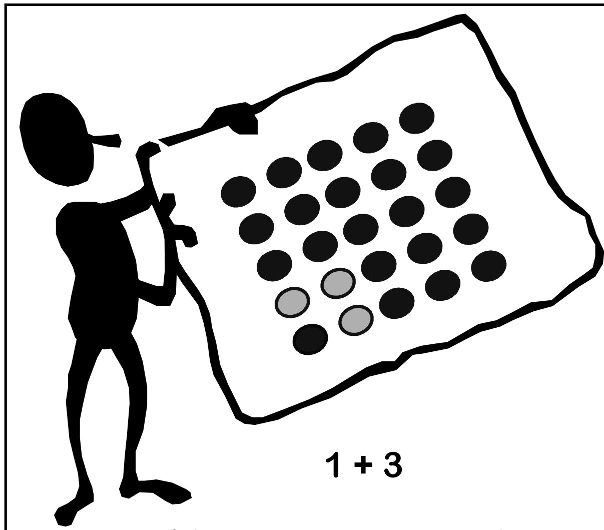

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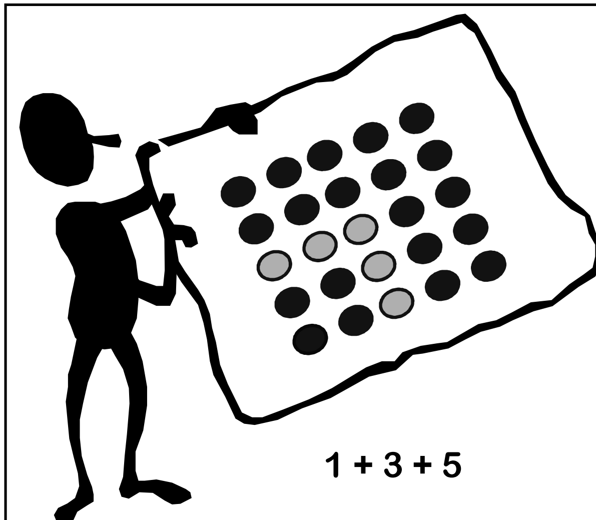

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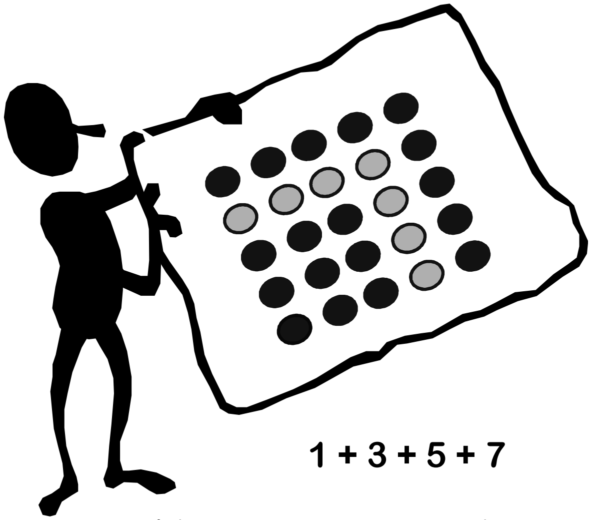
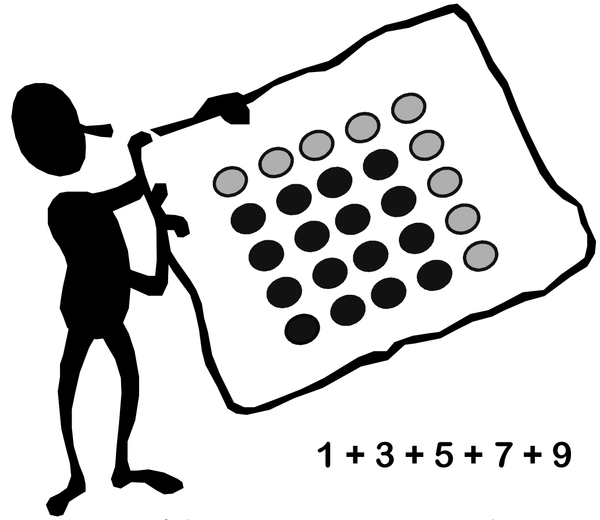


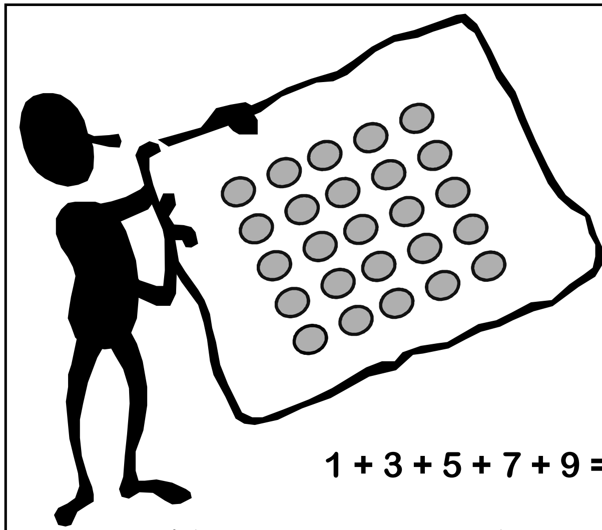
$1 + 3 + 5 + 7$

Breaking a square up in a new way



$1 + 3 + 5 + 7 + 9$

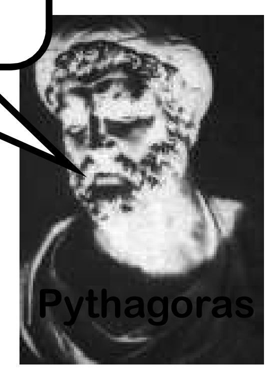
Breaking a square up in a new way



$1 + 3 + 5 + 7 + 9 = 5^2$

Breaking a square up in a new way

The sum of the first n odd numbers is n^2

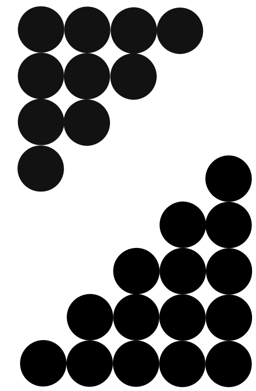


Here is an alternative dot proof of the same sum….

n^{th} Square Number

$\square_n = \Delta_n + \Delta_{n-1}$

$= n^2$



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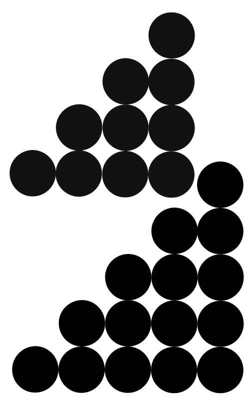

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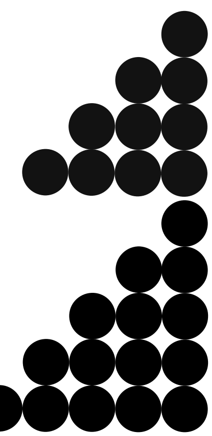
$\square_n = \Delta_n + \Delta_{n-1}$



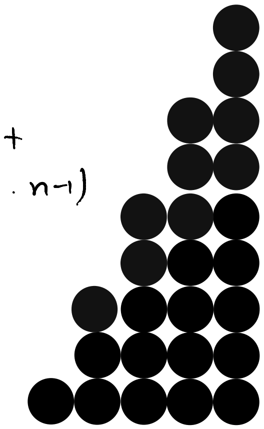
n^{th} Square Number

$\square_n = \Delta_n + \Delta_{n-1}$

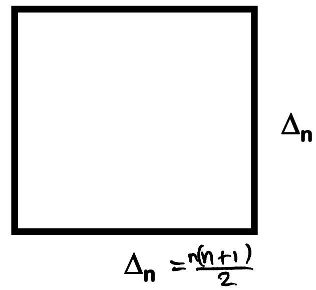
$= \cancel{1+3+} (1+2+3 \dots n) + (1+2+3 \dots n-1)$

= Sum of first n odd numbers

$$\begin{array}{r} 1+2+3+4+ \cdot + n \\ 1+2+3+ \cdot\cdot + n-1 \\ \hline 1+3+5+7+ \cdot\cdot + (2n-1) \end{array}$$



Area of square $= (\Delta_n)^2 = \left(\frac{n(n+1)}{2}\right)^2$



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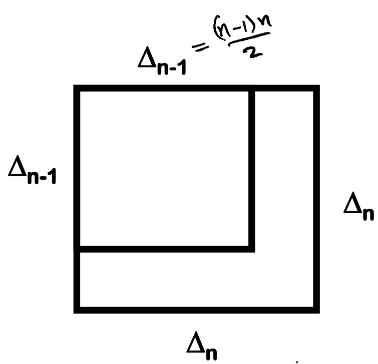

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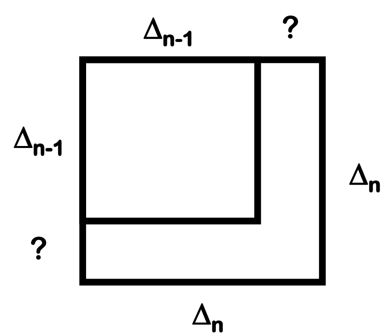
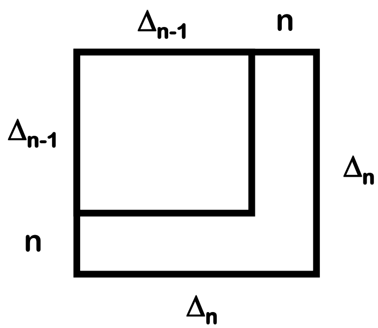
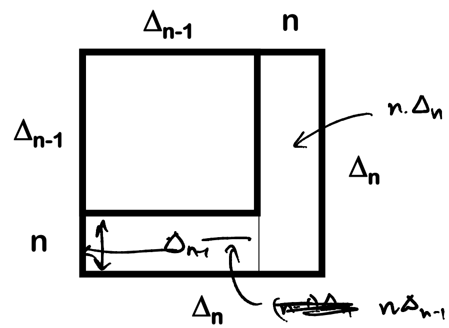


Area of square $= (\Delta_n)^2$

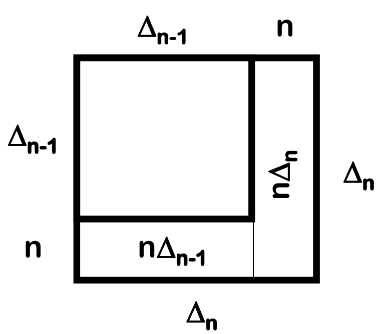


Area of square $= (\Delta_n)^2$



Area of square $= (\Delta_n)^2$

$$
\begin{aligned}
&= (\Delta_{n-1})^2 + n\Delta_{n-1} + n\Delta_n \\
&= (\Delta_{n-1})^2 + n(\Delta_{n-1} + \Delta_n) \\
&= (\Delta_{n-1})^2 + n(\square_n) \\
&= (\Delta_{n-1})^2 + n^3
\end{aligned}
$$

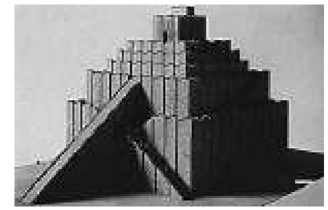


$$
\begin{aligned}
(\Delta_n)^2 &= n^3 + (\Delta_{n-1})^2 \\
&= n^3 + (n-1)^3 + (\Delta_{n-2})^2 \\
&= n^3 + (n-1)^3 + (n-2)^3 + (\Delta_{n-3})^2 \\
&= n^3 + (n-1)^3 + (n-2)^3 + \ldots + 1^3
\end{aligned}
$$

$$(\Delta_n)^2 = 1^3 + 2^3 + 3^3 + \ldots + n^3$$
$$= [\, n(n+1)/2 \,]^2$$

Can you find a formula for the sum of the first n squares?

$1^2+2^2+3^2+ \cdots +n^2 = \frac{n(n+1)(2n+1)}{6}$



Babylonians needed this sum to compute the number of blocks in their pyramids

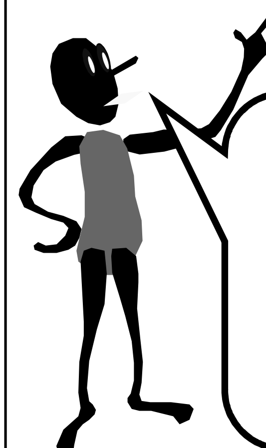
Rhind Papyrus

Scribe Ahmes was Martin Gardener of his day!

A man has 7 houses,
Each house contains 7 cats,
Each cat has killed 7 mice,
Each mouse had eaten 7 ears of spelt,
Each ear had 7 grains on it.
What is the total of all of these?

Sum of powers of 7

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$



We'll use this fundamental sum again and again:

The Geometric Series

A Frequently Arising Calculation

$(X-1)\,(\,1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1}\,)$

$$\cancel{x} + \cancel{x^2} + \cancel{x^3} + \cdots + x^n$$
$$-1 - \cancel{x} - \cancel{x^2} - \cancel{x^3} \cdots - \cancel{x^{n-1}}$$
$$-1 \qquad\qquad x^n$$

$$\frac{x^n - 1}{x-1} = \frac{\cancel{(x-1)}(1 + x + x^2 + \cdots + x^{n-1})}{\cancel{(x-1)}}$$

A Frequently Arising Calculation

$(X-1)\,(\,1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1}\,)$

$= \quad X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n$

$- \;1 - X^1 - X^2 - X^3 - \ldots - X^{n-2} - X^{n-1}$

$= \; X^n - 1$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

(when $x \neq 1$)

Geometric Series for X=2

$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^{n-1} \;=\; 2^n - 1$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

(when $x \neq 1$)

Geometric Series for X=1/2

$$1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \cdots + \left(\tfrac{1}{2}\right)^{n-1} = \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} = 2\left(1 - \left(\tfrac{1}{2}\right)^n\right)$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

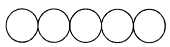
(when $x \neq 1$)

BASE X Representation

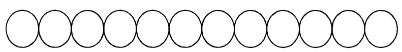
$S = a_{n-1}\, a_{n-2} \ldots a_1\, a_0$ represents the number:

$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$

Base 2 [Binary Notation]
101 represents: $1\,(2)^2 + 0\,(2^1) + 1\,(2^0)$

= 

Base 7
015 represents: $0\,(7)^2 + 1\,(7^1) + 5\,(7^0)$

= 

Bases In Different Cultures

Sumerian-Babylonian: 10, 60, 360
Egyptians: 3, 7, 10, 60
Maya: 20
Africans: 5, 10
French: 10, 20
English: 10, 12, 20

BASE X Representation

$S = (a_{n-1}\, a_{n-2} \ldots a_1\, a_0)_X$ represents the number:

$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$

Largest number representable in base-X with n "digits"

$= (X\text{-}1\ X\text{-}1\ X\text{-}1\ X\text{-}1\ X\text{-}1 \ldots X\text{-}1)_X$

$= (X-1)(X^{n-1} + X^{n-2} + \ldots + X^0)$

$= (X^n - 1)$

Fundamental Theorem For Binary

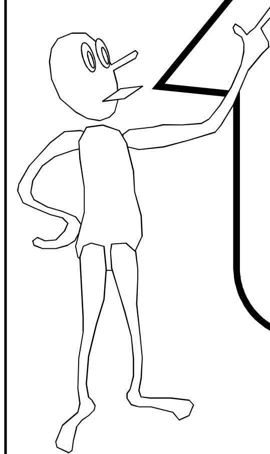
Each of the numbers from 0 to $\cancel{2^{n-1}}$ 2^n-1 **is uniquely represented by an n-bit number in binary**

$\boxed{k \text{ uses } \lfloor \log_2 k \rfloor + 1 \text{ digits in base 2}}$

Fundamental Theorem For Base-X

Each of the numbers from 0 to X^{n-1} is uniquely represented by an n-“digit” number in base X

k uses $\lfloor \log_X k \rfloor + 1$ digits in base X



n has length n in unary, but has length $\lfloor \log_2 n \rfloor + 1$ in binary

Unary is exponentially longer than binary

Other Representations: Egyptian Base 3

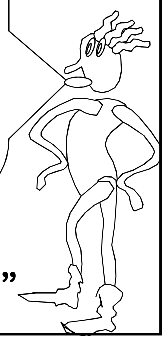
Conventional Base 3:
Each digit can be 0, 1, or 2

Here is a strange new one:
Egyptian Base 3 uses -1, 0, 1

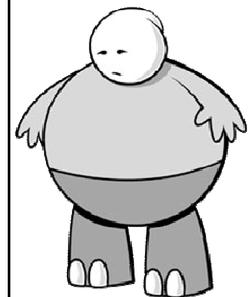
Example: 1 -1 -1 = 9 - 3 - 1 = 5

We can prove a unique representation theorem

How could this be Egyptian? Historically, negative numbers first appear in the writings of the Hindu mathematician Brahmagupta (628 AD)



One weight for each power of 3
Left = “negative”. Right = “positive”



Here’s What You Need to Know…

Unary and Binary

- **Triangular Numbers**
- **Dot proofs**

$(1+x+x^2 + \ldots + x^{n-1}) = (x^n - 1)/(x-1)$

Base-X representations

k uses $\lfloor \log_2 k \rfloor + 1 = \lceil \log_2 (k+1) \rceil$ **digits in base 2**