Combinatorial Games
Lecture 3 (September 2, 2008)

A Take-Away Game
Two Players: I and II
A move consists of removing one, two, or three chips from the pile
Players alternate moves, with Player I starting
Player that removes the last chip wins

Which player would you rather be?
Try Small Examples!

If there are 1, 2, or 3 only, player who moves next wins

If there are 4 chips left, player who moves next must leave 1, 2 or 3 chips, and his opponent will win

With 5, 6 or 7 chips left, the player who moves next can win by leaving 4 chips

What if the last player to move loses?

If there is 1 chip, the player who moves next loses

If there are 2, 3, or 4 chips left, the player who moves next can win by leaving only 1

In this case, 1, 5, 9, 13, ... are a win for the second player

Combinatorial Games

There are two players
There is a finite set of possible positions
The rules of the game specify for both players and each position which moves to other positions are legal moves
The players alternate moving
The game ends in a finite number of moves (no draws!)

Normal Versus Misère

Normal Play Rule: The last player to move wins
Misère Play Rule: The last player to move loses

A Terminal Position is one where neither player can move anymore

What is Omitted

No random moves
No hidden moves
No draws in a finite number of moves
This rules out games like poker
This rules out games like battleship
This rules out tic-tac-toe
P-Positions and N-Positions

P-Position: Positions that are winning for the Previous player (the player who just moved)

N-Position: Positions that are winning for the Next player (the player who is about to move)

0, 4, 8, 12, 16, … are P-positions; if a player moves to that position, they can win the game

21 chips is an N-position

What’s a P-Position?

“Positions that are winning for the Previous player (the player who just moved)”

That means:

For any move that N makes

There exists a move for P such that

For any move that N makes

There exists a move for P such that

There are no possible moves for N

P-positions and N-positions can be defined recursively by the following:

1) All terminal positions are P-positions

2) From every N-position, there is at least one move to a P-position

3) From every P-position, every move is to an N-position

Chomp!

Two-player game, where each move consists of taking a square and removing it and all squares to the right and above.

Player who takes position (0,0) loses

Show That This is a P-position
Show That This is an N-position

Let's Play! I'm player I

No matter what you do, I can mirror it!

Mirroring is an extremely important strategy in combinatorial games!

Theorem: Player I can win in any square starting position of Chomp

Proof:

The winning strategy for player I is to chomp on (1,1), leaving only an “L” shaped position

Then, for any move that Player II takes, Player I can simply mirror it on the flip side of the “L”

What about rectangular boards?
Theorem: Player I can win in any rectangular starting position
Proof:
Look at this first move:

If this is a P-position, then player 1 wins
Otherwise, there exists a P-position that can be obtained from this position
And player I could have just taken that move originally

Move-the-Token
Two-player game, where each move consists of taking the token and moving it either downwards or to the left (but not both).
Player who makes the last move (to (0,0)) wins

The Game of Nim
Two players take turns moving
Winner is the last player to remove chips
A move consists of selecting a pile and removing chips from it (you can take as many as you want, but you have to at least take one)
In one move, you cannot remove chips from more than one pile

Analyzing Simple Positions
We use (x,y,z) to denote this position
(0,0,0) is a: P-position
One-Pile Nim
What happens in positions of the form \((x, 0, 0)\)?
The first player can just take the entire pile, so \((x, 0, 0)\) is an N-position.

Two-Pile Nim
P-positions are those for which the two piles have an equal number of chips.
If it is the opponent’s turn to move from such a position, he must change to a position in which the two piles have different number of chips.
From a position with an unequal number of chips, you can easily go to one with an equal number of chips (perhaps the terminal position).

3-Pile Nim
Two players take turns moving.
Winner is the last player to remove chips.

Nim-Sum
The nim-sum of two non-negative integers is their addition (without carry) in base 2.
We will use \(\oplus\) to denote the nim-sum:

\[
\begin{align*}
2 \oplus 3 &= (10)_2 \oplus (11)_2 = (01)_2 = 1 \\
5 \oplus 3 &= (101)_2 \oplus (011)_2 = (110)_2 = 6 \\
7 \oplus 4 &= (111)_2 \oplus (100)_2 = (011)_2 = 3 \\
\end{align*}
\]

\(\oplus\) is associative: \((a \oplus b) \oplus c = a \oplus (b \oplus c)\)
\(\oplus\) is commutative: \(a \oplus b = b \oplus a\)

For any non-negative integer \(x\),

\[
x \oplus x = 0
\]
Cancellation Property Holds

If \( x \oplus y = x \oplus z \)
Then \( x \oplus x \oplus y = x \oplus x \oplus z \)
So \( y = z \)

Bouton’s Theorem: A position \((x,y,z)\) in Nim is a P-position if and only if \( x \oplus y \oplus z = 0 \)

Proof:
Let \( Z \) denote the set of Nim positions with nim-sum zero
Let \( NZ \) denote the set of Nim positions with non-zero nim-sum

We prove the theorem by proving that \( Z \) and \( NZ \) satisfy the three conditions of P-positions and N-positions

(1) All terminal positions are in \( Z \)
The only terminal position is \((0,0,0)\)

(2) From each position in \( NZ \), there is a move to a position in \( Z \)

(3) Every move from a position in \( Z \) is to a position in \( NZ \)
If \((x,y,z)\) is in \( Z \), and \( x \) is changed to \( x' < x \), then we cannot have
\[
\begin{align*}
\text{Because then } x &= x' \\
\end{align*}
\]

K-Pile Nim

Combinatorial Games

- P-positions versus N-positions
- When there are no draws, every position is either P or N

Nim

- Definition of the game
- Nim-sum
- Bouton’s Theorem

Here’s What You Need to Know…