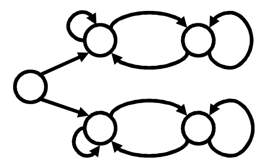
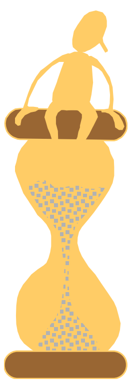
15-251

Great Theoretical Ideas in Computer Science

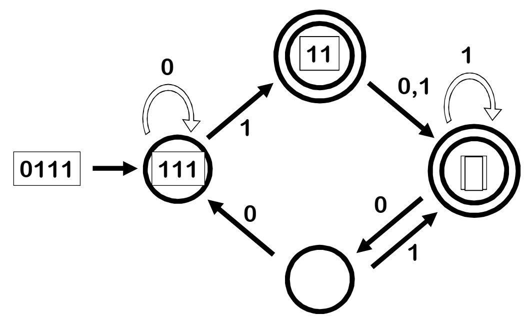
Deterministic Finite Automata

Lecture 19 (October 30, 2006)



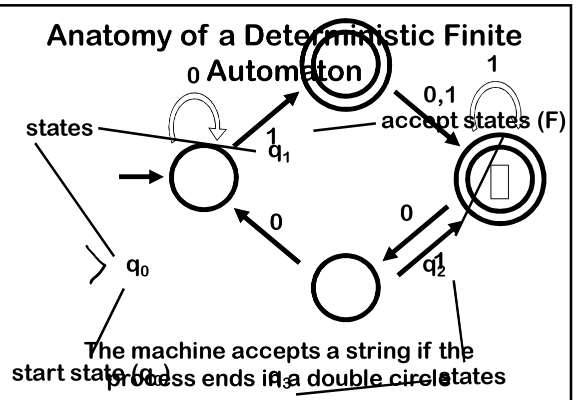


Let me show you a machine so simple that you can understand it in less than two minutes



The machine accepts a string if the process ends in a double circle

Anatomy of a Deterministic Finite Automaton



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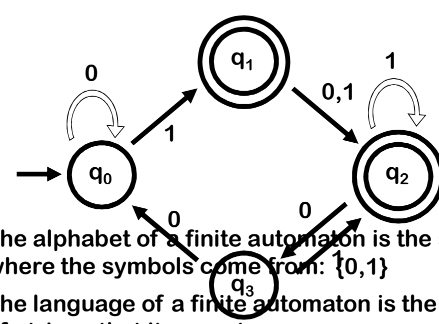
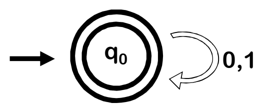
Anatomy of a Deterministic Finite Automaton



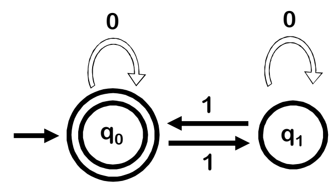
The alphabet of a finite automaton is the set where the symbols come from: {0,1}

The language of a finite automaton is the set of strings that it accepts



L(M) = All strings of 0s and 1s

The Language of Machine M



L(M) = { w | w has an even number of 1s}

Notation

An alphabet Σ is a finite set (e.g., $\Sigma = \{0,1\}$)

A string over Σ is a finite-length sequence of elements of Σ. The set of all strings over Σ is denoted by Σ^*.

For x a string, |x| is the length of x

The unique string of length 0 will be denoted by ε and will be called the empty or null string

A language over Σ is a set of strings over Σ

Language $\subseteq \Sigma^*$

A finite automaton is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

- Q is the set of states
- Σ is the alphabet
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

L(M) = the language of machine M
= set of all strings machine M accepts

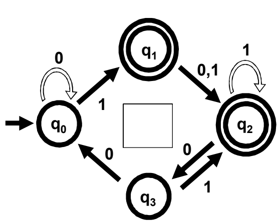
$M = (Q, \Sigma, \delta, q_0, F)$ where $Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{0,1\}$

$q_0 \in Q$ is start state

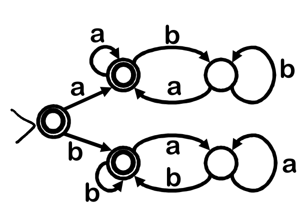
$F = \{q_1, q_2\} \subseteq Q$ accept states

$\delta : Q \times \Sigma \to Q$ transition function



δ	0	1
q_0	q_0	q_1
q_1	q_2	q_2
q_2	q_3	q_2
q_3	q_0	q_2

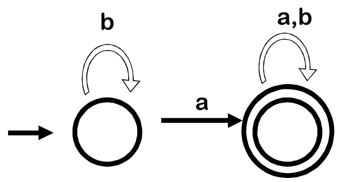
"ABA" The Automaton



Input String	Result
aba	Accept
aabb	Reject
aabba	Accept
ε	Accept

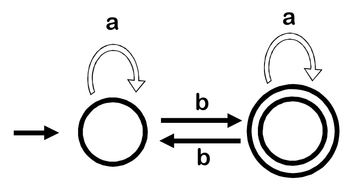
What machine accepts this language?

L = all strings in $\{a,b\}^*$ that contain at least one a

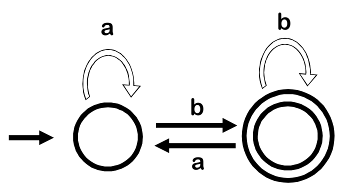


What machine accepts this language?

***L* =** strings with an odd number of *b*'s and any number of *a*'s

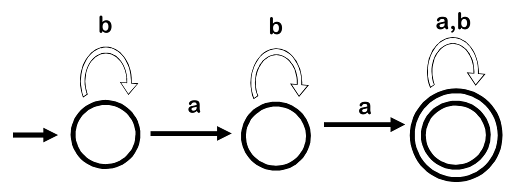


What is the language accepted by this machine?



L* =** any string ending with a ***b

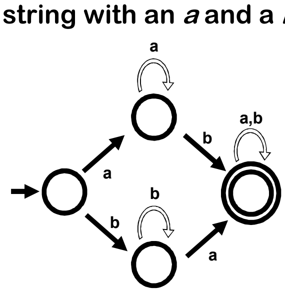
What is the language accepted by this machine?



***L(M)* =** any string with at least two *a*'s

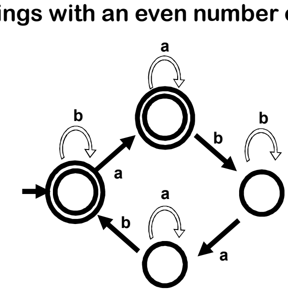
What machine accepts this language?

L* =** any string with an *a* and a ***b

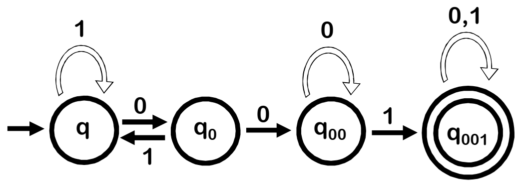


What machine accepts this language?

***L* =** strings with an even number of *ab* pairs

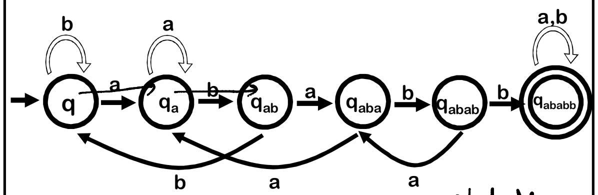


Build an automaton that accepts all and only those strings that contain 001



Steven Rudich:

L = all strings containing *ababb* as a consecutive substring



abababb

Invariant:
I am state s exactly when s is the longest suffix of the input (so far) forming a prefix of *ababb.*

The "Grep" Problem

Input: Text T of length t, string S of length n

Problem: Does string S appear inside text T?

Naïve method:

$\boxed{a_1, a_2, a_3,} a_4, a_5, \ldots, a_t$

Cost: Roughly nt comparisons

Automata Solution

Build a machine M that accepts any string with S as a consecutive substring

Feed the text to M

Cost: t comparisons + time to build M

As luck would have it, the Knuth, Morris, Pratt algorithm builds M quickly

Real-life Uses of DFAs

automata
finite
deterministic

Grep

Coke Machines

Thermostats (fridge)

Elevators

Train Track Switches

Lexical Analyzers for Parsers

A language is regular if it is recognized by a deterministic finite automaton

L = { w | w contains 001} is regular

L = { w | w has an even number of 1s} is regular

Union Theorem

Given two languages, L_1 and L_2, define the union of L_1 and L_2 as

$$L_1 \cup L_2 = \{ w \mid w \in L_1 \text{ or } w \in L_2 \}$$

Theorem: The union of two regular languages is also a regular language

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Theorem: The union of two regular languages is also a regular language

Proof Sketch: Let

$M_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$ be finite automaton for L_1

and

$M_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$ be finite automaton for L_2

We want to construct a finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $L = L_1 \cup L_2$

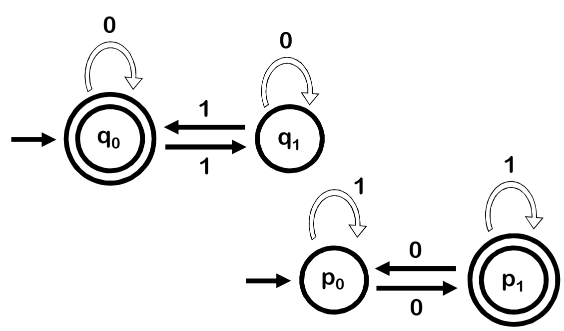
Idea: Run both M_1 and M_2 at the same time!

Q = pairs of states, one from M_1 and one from M_2

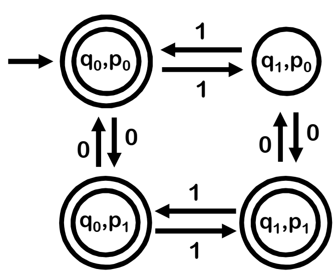
$= \{ (q_1, q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \}$

$= Q_1 \times Q_2$

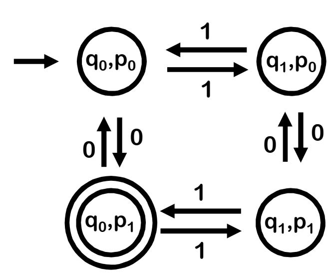
Theorem: The union of two regular languages is also a regular language

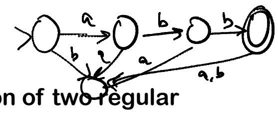


Automaton for Union



Automaton for Intersection





Theorem: The union of two regular languages is also a regular language

Corollary: Any finite language is regular

The Regular Operations

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$

Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$

Negation: $\neg A = \{ w \mid w \notin A \}$

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

Star: $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

Regular Languages Are Closed Under The Regular Operations

We have seen part of the proof for Union. The proof for intersection is very similar. The proof for negation is easy.

Are all languages regular?

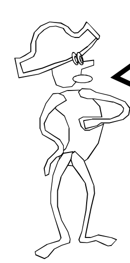
{ab, aabb, aaabbb, . . . $a^i b^i$ . . . $(a^n b^n)$ . .

Consider the language $L = \{ a^n b^n \mid n > 0 \}$

i.e., a bunch of a's followed by an equal number of b's

No finite automaton accepts this language

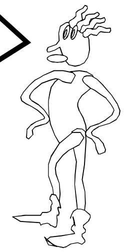
Can you prove this?



$a^n b^n$ is not regular. No machine has enough states to keep track of the number of a's it might encounter

That is a fairly weak argument

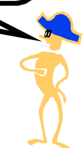
Consider the following example…

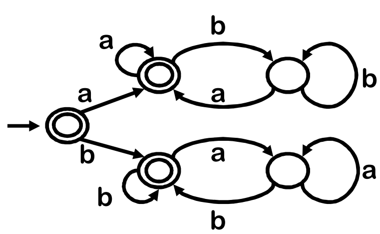


L = strings where the # of occurrences of the pattern ab is equal to the number of occurrences of the pattern ba

abab ×

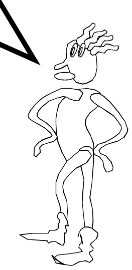
Can't be regular. No machine has enough states to keep track of the number of occurrences of ab

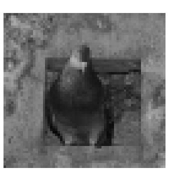




M accepts only the strings with an equal number of ab's and ba's!

Let me show you a professional strength proof that a^nb^n is not regular…





Pigeonhole principle:

Given n boxes and m > n objects, at least one box must contain more than one object

Letterbox principle:

If the average number of letters per box is x, then some box will have at least x letters (similarly, some box has at most x)

Theorem: L= $\{a^nb^n \mid n > 0\}$ is not regular

Proof (by contradiction):

Assume that L is regular

Then there exists a machine M with k states that accepts L

For each $0 \leq i \leq k$, let S_i be the state M is in after reading a^i

$\exists i,j \leq k$ such that $S_i = S_j$, but $i \neq j$

M will do the same thing on a^ib^i and a^jb^i

But a valid M must reject a^jb^i and accept a^ib^i

ε~d
a
aa
aaa
.
.
.
a^k

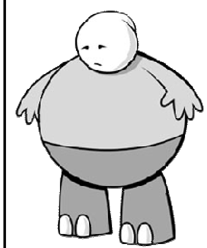
Advertisement

You can learn much more about these creatures in the FLAC course.

Formal Languages, Automata, and Computation

• There is a unique smallest automaton for any regular language

• It can be found by a fast algorithm.



Here's What You Need to Know…

Deterministic Finite Automata

• Definition
• Testing if they accept a string
• Building automata

Regular Languages

• Definition
• Closed Under Union, Intersection, Negation
• Using Pigeonhole Principle to show language ain't regular