15-251

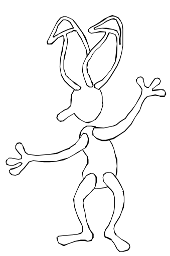
Great Theoretical Ideas in Computer Science

Fibonacci Numbers and the Golden Ratio

Lecture 15 (October 16, 2007)

Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations



Rabbit Reproduction

A rabbit lives forever

The population starts as single newborn pair

Every month, each productive pair begets a new pair which will become productive after 2 months old

F_n= # of rabbit pairs at the beginning of the n^{th} month

month	1	2	3	4	5	6	7
rabbits	1	1	2	3	5	8	13

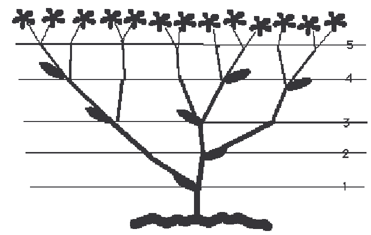
Fibonacci Numbers

month	1	2	3	4	5	6	7
rabbits	1	1	2	3	5	8	13

Stage 0, Initial Condition, or Base Case:
Fib(1) = 1; Fib (2) = 1

Inductive Rule:
For n>3, Fib(n) = Fib(n-1) + Fib(n-2)

Sneezwort (Achilleaptarmica)



Each time the plant starts a new shoot it takes two months before it is strong enough to support branching.

Counting Petals

5 petals: buttercup, wild rose, larkspur, columbine (aquilegia)
8 petals: delphiniums
13 petals: ragwort, corn marigold, cineraria, some daisies
21 petals: aster, black-eyed susan, chicory
34 petals: plantain, pyrethrum
55, 89 petals: michaelmas daisies, the asteraceae family.

Definition of ϕ (Euclid)

Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

$$\phi = \frac{AC}{AB} = \frac{AB}{BC}$$

A B C

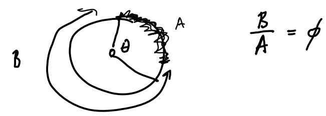
$$\phi^2 = \frac{AC}{BC}$$

$$\phi^2 - \phi = \frac{AC}{BC} - \frac{AB}{BC} = \frac{BC}{BC} = 1$$

$$\phi^2 - \phi - 1 = 0$$

$$\phi = \frac{1 + \sqrt{5}}{2}$$

Fig. 12.4. Phyllotaxis



Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

$$= 1 + \frac{1}{1 + \frac{1}{\phi}}$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}}$$

Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

1,1,2,3,5,8,13,21,34,55,....

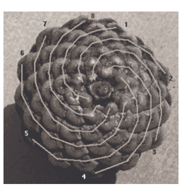
2/1	=	2
3/2	=	1.5
5/3	=	1.666…
8/5	=	1.6
13/8	=	1.625
21/13	=	1.6153846…
34/21	=	1.61904…

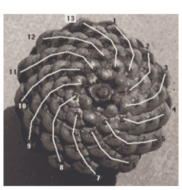
ϕ = 1.6180339887498948482045

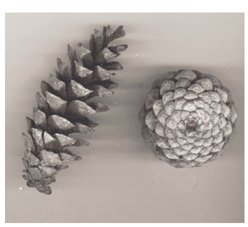
Pineapple whorls

Church and Turing were both interested in the number of whorls in each ring of the spiral.

The ratio of consecutive ring lengths approaches the Golden Ratio.







Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$f_1 = 1$ 0 = the empty sum

$f_2 = 1$ 1 = 1

$f_3 = 2$ 2 = 1 + 1

2

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

4 = 2 + 2
2 + 1 + 1
1 + 2 + 1
1 + 1 + 2
1 + 1 + 1 + 1

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

# of sequences beginning with a 1

# of sequences beginning with a 2

Fibonacci Numbers Again

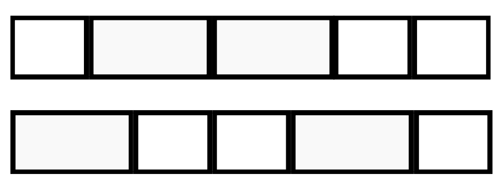
Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

$$f_1 = 1 \qquad f_2 = 1$$

Visual Representation: Tiling

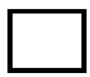
Let f_{n+1} be the number of different ways to tile a 1 × n strip with squares and dominoes.



Visual Representation: Tiling

1 way to tile a strip of length 0

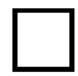
1 way to tile a strip of length 1:



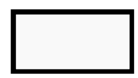
2 ways to tile a strip of length 2:

$f_{n+1} = f_n + f_{n-1}$

f_{n+1} is number of ways to tile length n.



f_n tilings that start with a square.



f_{n-1} tilings that start with a domino.

Fibonacci Identities

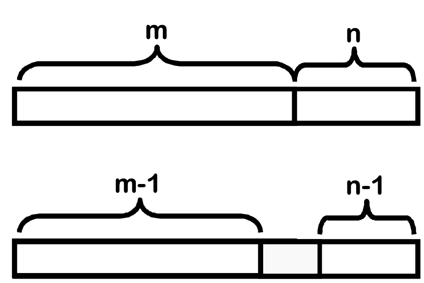
Some examples:

$F_{2n} = F_1 + F_3 + F_5 + \dots + F_{2n-1}$

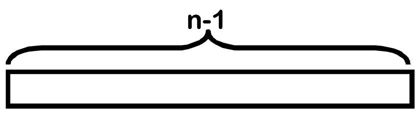
$F_{m+n+1} = F_{m+1}\, F_{n+1} + F_m\, F_n$

$(F_n)^2 \quad = \ F_{n-1}\, F_{n+1} + (-1)^n$

$F_{m+n+1} \quad = \ F_{m+1}\, F_{n+1} \quad + \quad F_m\, F_n$

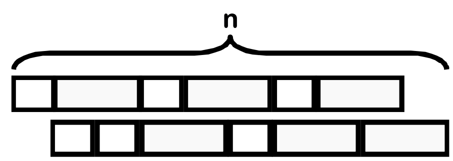


$(F_n)^2 \;=\; F_{n-1}\, F_{n+1} \;+\; (-1)^n$



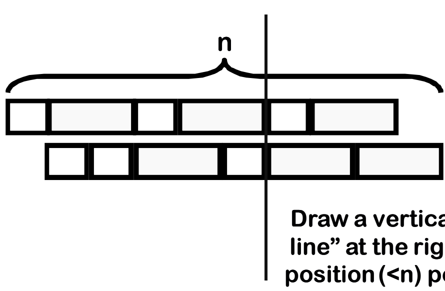
F_n tilings of a strip of length n-1

$(F_n)^2 \;=\; F_{n-1}\, F_{n+1} \;+\; (-1)^n$



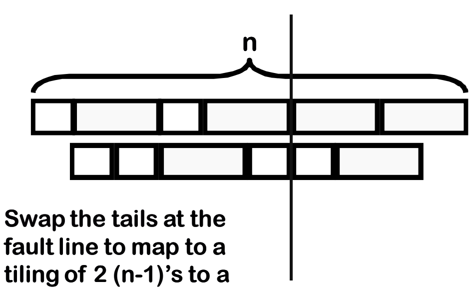
$(F_n)^2$ tilings of two strips of size n-1

$(F_n)^2 \;=\; F_{n-1}\, F_{n+1} \;+\; (-1)^n$



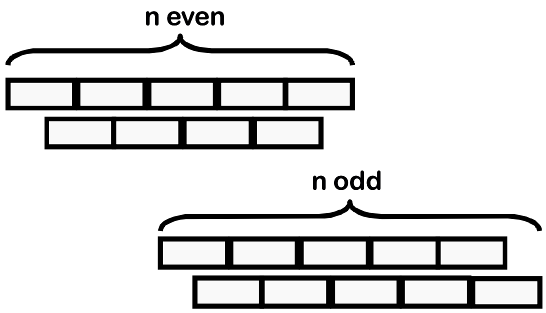
Draw a vertical “fault line” at the rightmost position (<n) possible without cutting any dominoes

$(F_n)^2 \;=\; F_{n-1}\, F_{n+1} \;+\; (-1)^n$



Swap the tails at the fault line to map to a tiling of 2 (n-1)’s to a tiling of an n-2 and an n.

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^{n-1}$$



The Fibonacci Quarterly