15-251

Great Theoretical Ideas in Computer Science

Ancient Wisdom: On Raising A Number To A Power

Lecture 13 (October 9, 2007)

Egyptian Multiplication

The Egyptians used decimal numbers but multiplied and divided in binary

a x b By Repeated Doubling

b has n-bit representation: $b_{n-1}b_{n-2}\ldots b_1b_0$

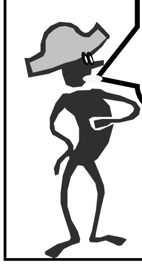
Starting with a,
repeatedly double largest number so far
to obtain: a, 2a, 4a, …., $2^{n-1}a$

Sum together all the $2^k a$ where $b_k = 1$

$$b = b_0 2^0 + b_1 2^1 + b_2 2^2 + \ldots + b_{n-1} 2^{n-1}$$

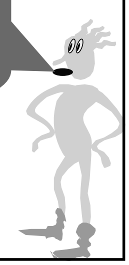
$$ab = b_0 2^0 a + b_1 2^1 a + b_2 2^2 a + \ldots + b_{n-1} 2^{n-1} a$$

$2^k a$ is in the sum if and only if $b_k = 1$



Wait!
How did the Egyptians do the part where they converted b to binary?

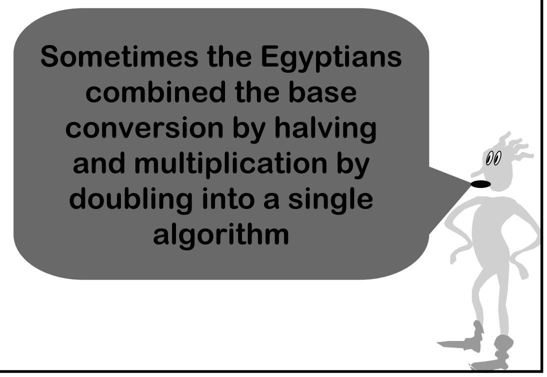
They used repeated halving to do base conversion!



Egyptian Base Conversion

Output stream will print right to left

```
Input X;
repeat {
   if (X is even)
      then print 0;
   else
      {X := X-1; print 1;}
   X := X/2 ;
} until X=0;
```



70 x 13

Rhind Papyrus [1650 BC]

Doubling	Halving	Odd?	Running Total
70	13	*	70
140	6		
280	3	*	350
560	1	*	910

Binary for 13 is 1101 = $2^3 + 2^2 + 2^0$

$70*13 = 70*2^3 + 70*2^2 + 70*2^0$

30 x 5

Doubling	Halving	Odd?	Running Total
5	30		
10	15	*	10
20	7	*	30
40	3	*	70
80	1	*	150

184 / 17

Rhind Papyrus [1650 BC]

Doubling	Powers of 2	Check
17	1	
34	2	*
68	4	
136	8	*

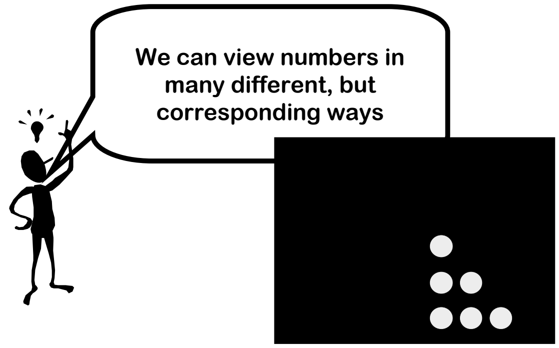
184 = 17*8 + 17*2 + 14

184/17 = 10 with remainder 14

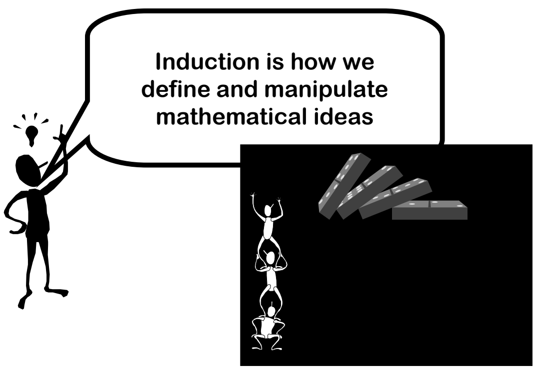
Standard Binary Multiplication = Egyptian Multiplication

```
        * * * * * * * *
  ×               1101
  --------------------
        * * * * * * * *
       0 0 0 0 0 0 0 0
     * * * * * * * *
   * * * * * * * *
-------------------------
* * * * * * * * * * * * * *
```

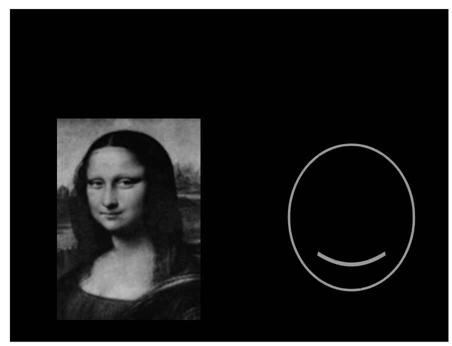
Our story so far…



Our story so far…



Let's Articulate New One:



$b:=a^8$

b:=a*a
b:=b*a
b:=b*a
b:=b*a
b:=b*a
b:=b*a
b:=b*a

> b:=a*a
> b:=b*b
> b:=b*b
>
> This method costs only 3 multiplications. The savings are significant if $b:=a^8$ is executed often

Powering By Repeated Multiplication

Input: a,n

Output: Sequence starting with a, ending with a^n, such that each entry other than the first is the product of two previous entries

Example

Input: a,5

Output: a, a^2, a^3, a^4, a^5

or

Output: a, a^2, a^3, a^5

or

Output: a, a^2, a^4, a^5

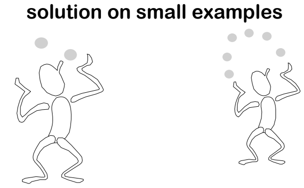
Given a constant n,
how do we implement
$b:=a^n$
with the fewest number
of multiplications?

Definition of M(n)

M(n) = Minimum number of multiplications required to produce a^n from a by repeated multiplication

What is M(n)? Can we calculate it exactly? Can we approximate it?

Exemplification:
Try out a problem or
solution on small examples



Very Small Examples

What is M(1)?

M(1) = 0 [a]

What is M(0)?

Not clear how to define M(0)

What is M(2)?

M(2) = 1 $[a,a^2]$

M(8) = ?

a, a^2, a^4, a^8 is one way to make a^8 in 3 multiplications

What does this tell us about the value of M(8)?

$M(8) \leq 3$ Upper Bound

? ≤ M(8) ≤ 3

$3 \leq M(8)$ by exhaustive search

There are only two sequences with 2 multiplications. Neither of them make 8:

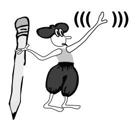
a, a^2, a^3 and a, a^2, a^4

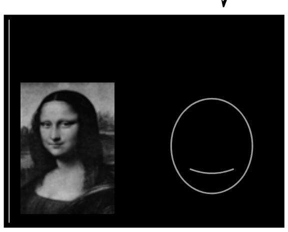
3 ≤ M(8) ≤ 3

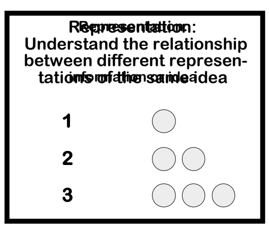
Lower Bound ↑ Upper Bound ↑

M(8) = 3

What is the more essential representation of M(n)?

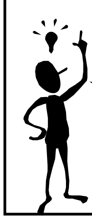






The "a" is a red herring

$a^x a^y$ is a^{x+y}



Everything besides the exponent is inessential. This should be viewed as a problem of repeated addition, rather than repeated multiplication

Addition Chains

M(n) = Number of stages required to make n, where we start at 1 and in each stage we add two previously constructed numbers

Examples

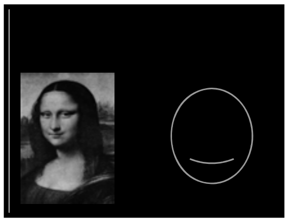
Addition Chain for 8:

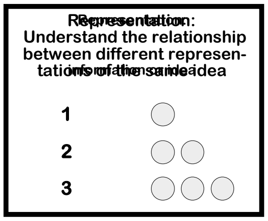
1 2 3 5 8

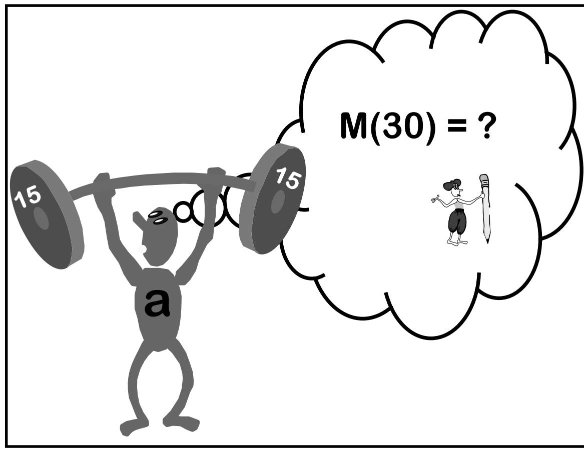
Minimal Addition Chain for 8:

1 2 4 8

Addition Chains Are a Simpler Way To Represent The Original Problem

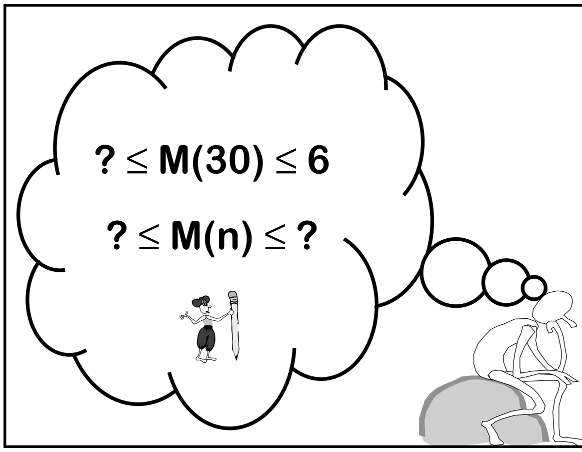






Addition Chains For 30

1	2	4	8	16	24	28	30
1	2	4	5	10	20	30	
1	2	3	5	10	15	30	
1	2	4	8	10	20	30	



Binary Representation

Let B_n be the number of 1s in the binary representation of n

E.g.: $B_5 = 2$ since $5 = (101)_2$

Proposition: $B_n \leq \lfloor \log_2 (n) \rfloor + 1$
(It is at most the number of bits in the binary representation of n)

Binary Method

(Repeated Doubling Method)

Phase I (Repeated Doubling)

For $\lfloor \log_2 (n) \rfloor$ stages:
Add largest so far to itself
(1, 2, 4, 8, 16, . . .)

Phase II (Make n from bits and pieces)

Expand n in binary to see how n is the sum of B_n powers of 2. Use B_n - 1 stages to make n from the powers of 2 created in phase I

Total cost: $\lfloor \log_2 n \rfloor + B_n - 1$

Binary Method

Applied To 30

Phase I

1, 2, 4, 8, 16 — Cost: 4 additions

Phase II

$30 = (11110)_2$

2 + 4 = 6

6 + 8 = 14

14 + 16 = 30 — Cost: 3 additions

$M(n) \le \lfloor \log_2 n \rfloor + B_n - 1 \le 2\lfloor$ [redacted] $\rfloor$

Rhind Papyrus [1650 BC]

What is 30 x 5?

1	5
2	10
4	20
8	40
16	80
24	120
28	140
30	150

Addition chain for 30 →

← Start at 5 and perform same additions as chain for 30

Repeated doubling is the same as the Egyptian binary multiplication

Rhind Papyrus [1650 BC]

Actually used faster chain for 30*5

1	5
2	10
4	20
8	40
10	50
20	100
30	150

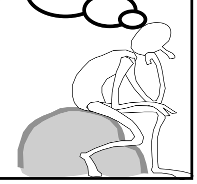
The Egyptian Connection

A shortest addition chain for n gives a shortest method for the Egyptian approach to multiplying by the number n

The fastest scribes would seek to know M(n) for commonly arising values of n

$? \le M(30) \le 6$

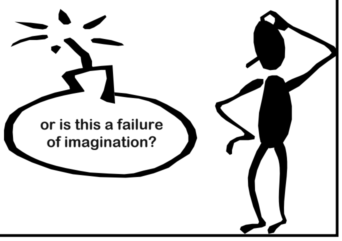
$? \le M(n) \le 2\lfloor$lo [redacted]



A Lower Bound Idea

You can't make any number bigger than 2^n in n steps

1 2 4 8 16 32 64 . . .



Let S_k be the statement that no k stage addition chain contains a number greater than 2^k

Base case: k=0. S_0 is true since no chain can exceed 2^0 after 0 stages

$\forall\, k \geqslant 0, \quad S_k \Rightarrow S_{k+1}$

At stage k+1 we add two numbers from the previous stage

From S_k we know that they both are bounded by 2^k

Hence, their sum is bounded by 2^{k+1}. No number greater than 2^{k+1} can be present by stage k+1

Change Of Variable

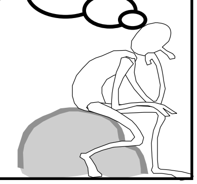
All numbers obtainable in m stages are bounded by 2^m. Let m = $\log_2(n)$

Thus, all numbers obtainable in $\log_2(n)$ stages are bounded by n

$$M(n) \geq \lceil \log_2 n \rceil$$

$? \leq M(30) \leq 6$

$\lceil \log_2 n \rceil \leq M(n) \leq 2 \lfloor \log$ [redacted]



Theorem: 2^i is the largest number that can be made in i stages, and can only be made by repeated doubling

Proof by Induction:

Base i = 0 is clear

To make anything as big as 2^i requires having some X as big as 2^{i-1} in i-1 stages

By I.H., we must have all the powers of 2 up to 2^{i-1} at stage i-1. Hence, we can only double 2^{i-1} at stage i

5 < M(30)

Suppose that M(30)=5

At the last stage, we added two numbers x_1 and x_2 to get 30

Without loss of generality (WLOG), we assume that $x_1 \geq x_2$

Thus, $x_1 \geq 15$

By doubling bound, $x_1 \leq 16$

But $x_1 \neq 16$ since there is only one way to make 16 in 4 stages and it does not make 14 along the way. Thus, x_1 = 15 and M(15)=4

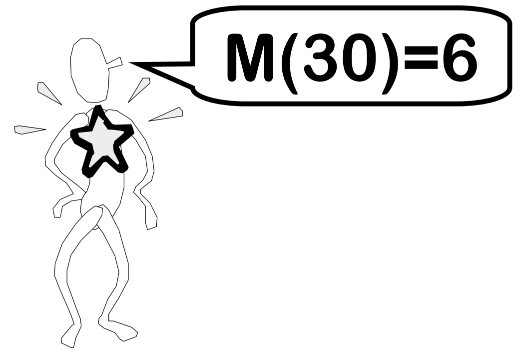
Suppose M(15) = 4

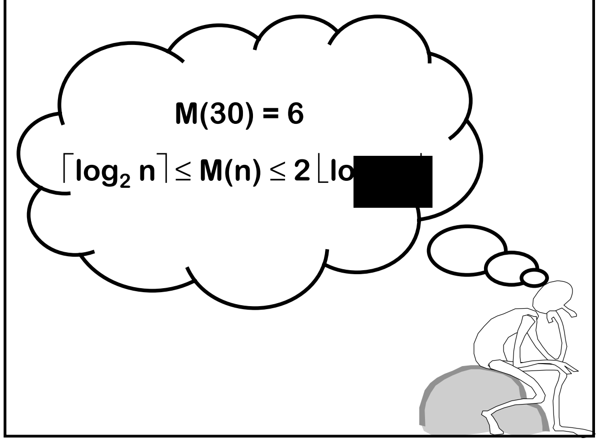
At stage 3, a number bigger than 7.5, but not more than 8 must have existed

There is only one sequence that gets 8 in 3 additions: 1 2 4 8

That sequence does not make 7 along the way and hence there is nothing to add to 8 to make 15 at the next stage

Thus, M(15) > 4   CONTRADICTION





Factoring Bound

M(a × b) ≤ M(a) + M(b)

Proof:

Construct a in M(a) additions

Using a as a unit follow a construction method for b using M(b) additions.
In other words, each time the construction of b refers to a number y, use the number ay instead

Example

45 = 5 x 9

M(5)=3	[1 2 4 5]
M(9)=4	[1 2 4 8 9]
M(45) ≤ 3 + 4	[1 2 4 5 10 20 40 45]

Corollary (Using Induction)

$M(a_1a_2a_3\ldots a_n) \le M(a_1)+M(a_2)+\ldots+M(a_n)$

Proof:

For n = 1 the bound clearly holds

Assume it has been shown for up to n-1

Now apply previous theorem using
$A = a_1a_2a_3\ldots a_{n-1}$ and $b = a_n$ to obtain:
$M(a_1a_2a_3\ldots a_n) \le M(a_1a_2a_3\ldots a_{n-1}) + M(a_n)$

By inductive assumption,
$M(a_1a_2a_3\ldots a_{n-1}) \le M(a_1) + M(a_2) + \ldots + M(a_{n-1})$

More Corollaries

Corollary: $M(a^k) \leq kM(a)$

Corollary: $M(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}) \leq$
$\alpha_1 M(p_1) + \alpha_2 M(p_2) + \ldots + \alpha_n M(p_n)$

Does equality hold
for $M(a \times b) \leq M(a) + M(b)$

M(33) < M(3) + M(11)

M(3) = 2 [1 2 3]

M(11)= 5 [1 2 3 5 10 11]

M(3) + M(11) = 7

M(33) = 6 [1 2 4 8 16 32 33]

The conjecture of equality fails!

Conjecture: M(2n) = M(n) +1

(A. Goulard)

A fastest way to an even number is to make half that number and then double it

Clearly, $M(2n) \leq M(n) + 1$
but is this inequality tight?

Conjecture: M(2n) = M(n) +1

(A. Goulard)

A fastest way to an even number is to make half that number and then double it

Proof given in 1895 by E. de Jonquieres
in L'In… …s,
vo…

FALSE! M(191)=M(382)=11
Furthermore, there are infinitely many such examples

Open Problem

Is there an n such that:
M(2n) < M(n)

Conjecture

Each stage might as well consist of adding the largest number so far to one of the other numbers

First Counter-example: 12,509
[1 2 4 8 16 17 32 64 128 256 512 1024 1041 2082 4164 8328 8345 12509]

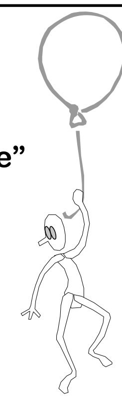
Open Problem

Prove or disprove the Scholz-Brauer Conjecture:

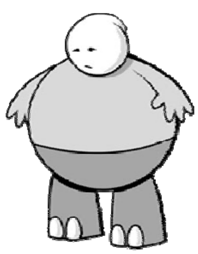
$M(2^n-1) \leq n - 1 + B_n$

(The bound that follows from this lecture is too weak: $M(2^n-1) \leq 2n - 1$)

High Level Point



Don't underestimate "simple" problems. Some "simple" mysteries have endured for thousand of years



Here's What You Need to Know…

Egyptian Multiplication

Raising To A Power
Minimal Addition Chain
Lower and Upper Bounds

Repeated doubling method