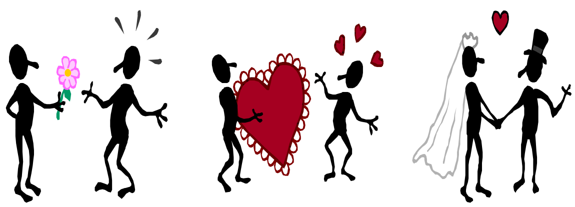
The Mathematics Of 1950's Dating: Who wins The Battle of The Sexes?

Lecture 9 (September 25, 2007)



WARNING: This lecture contains mathematical content that may be shocking to some students

Dating Scenario

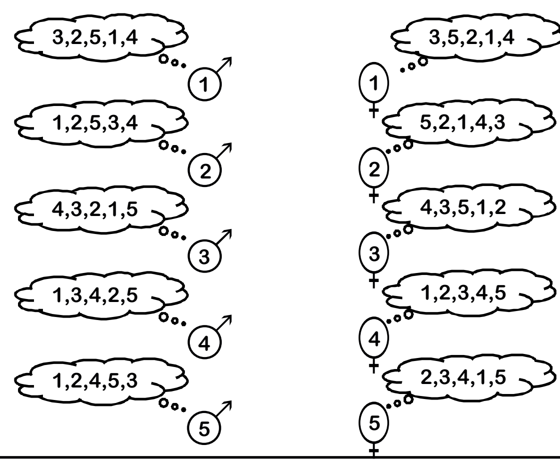
There are n boys and n girls

Each girl has her own ranked preference list of all the boys

Each boy has his own ranked preference list of the girls

The lists have no ties

Question: How do we pair them off?



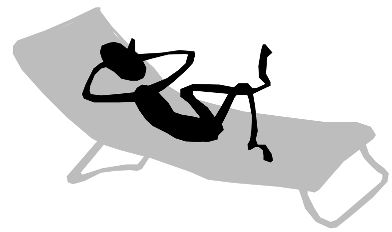
More Than One Notion of What Constitutes A "Good" Pairing

Maximizing total satisfaction
- Hong Kong and to an extent the USA

Maximizing the minimum satisfaction
- Western Europe

Minimizing maximum difference in mate ranks
- Sweden

Maximizing people who get their first choice
- Barbie and Ken Land



We will ignore the issue of what is "equitable"!

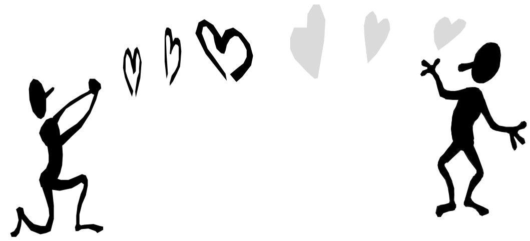
Rogue Couples

Suppose we pair off all the boys and girls

Now suppose that some boy and some girl prefer each other to the people to whom they are paired

They will be called a rogue couple

Why be with them when we can be with each other?



What use is fairness, if it is not stable?

Any list of criteria for a good pairing must include stability. (A pairing is doomed if it contains a rogue couple)

Stable Pairings

A pairing of boys and girls is called stable if it contains no rogue couples

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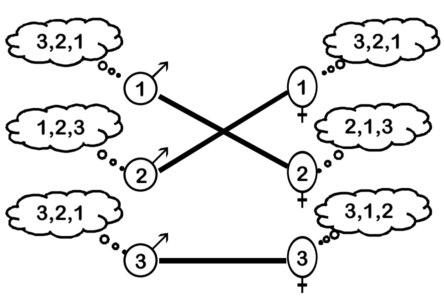
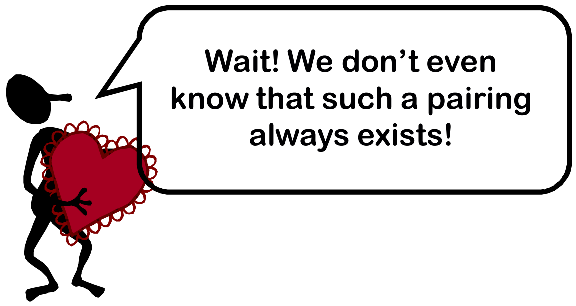


The study of stability will be the subject of the entire lecture

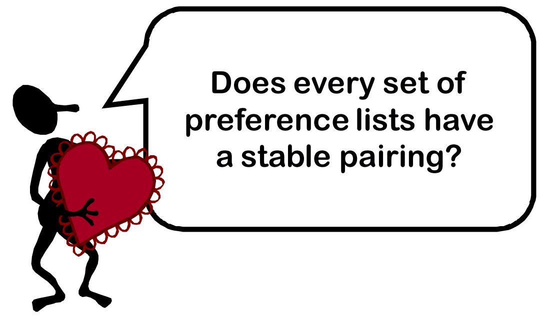
We will:

- Analyze various mathematical properties of an algorithm that looks a lot like 1950's dating
- Discover the naked mathematical truth about which sex has the romantic edge
- Learn how the world's largest, most successful dating service operates

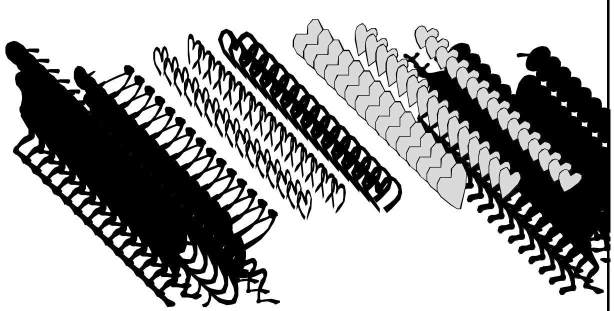
Given a set of preference lists, how do we find a stable pairing?

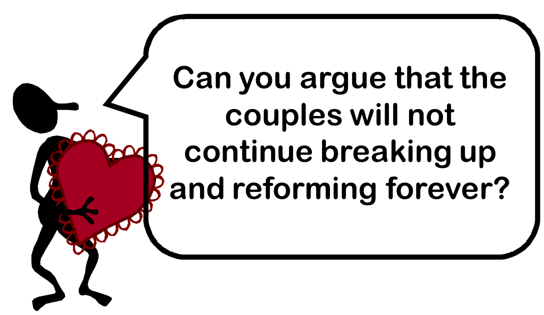


Better Question:

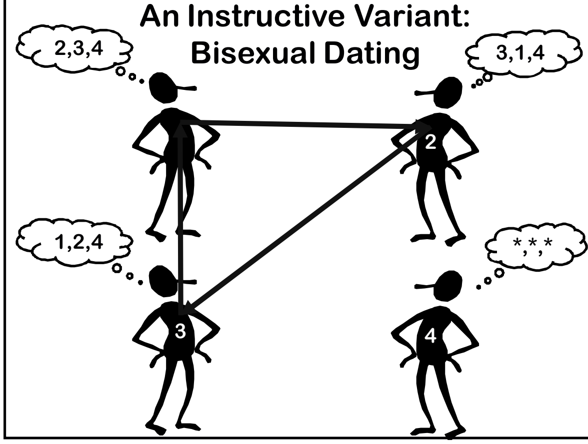


Idea: Allow the pairs to keep breaking up and reforming until they become stable





An Instructive Variant: Bisexual Dating

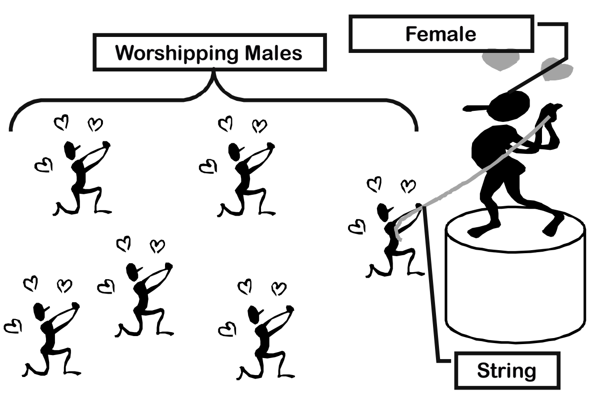


Insight

Any proof that heterosexual couples do not break up and re-form forever must contain a step that fails in the bisexual case

If you have a proof idea that works equally well in the hetero and bisexual versions, then your idea is not adequate to show the couples eventually stop

The Traditional Marriage Algorithm



The Traditional Marriage Algorithm

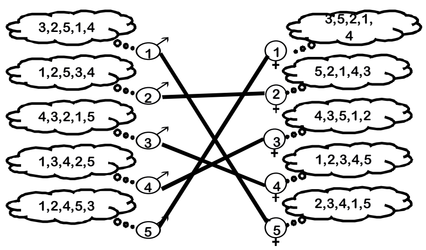
For each day that some boy gets a "No" do:

Morning
- Each girl stands on her balcony
- Each boy proposes to the best girl whom he has not yet crossed off

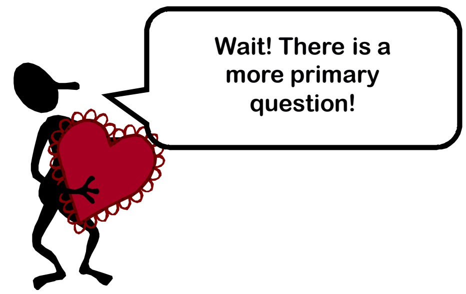
Afternoon (for girls with at least one suitor)
- To today's best: "Maybe, return tomorrow"
- To any others: "No, I will never marry you"

Evening
- Any rejected boy crosses the girl off his list

If no boys get a "No", each girl marries boy to whom she just said "maybe"



Does Traditional Marriage Algorithm always produce a stable pairing?



Does TMA Always Terminate?

It might encounter a situation where algorithm does not specify what to do next (e.g. "core dump error")

It might keep on going for an infinite number of days

Improvement Lemma:
If a girl has a boy on a string, then she will always have someone at least as good on a string (or for a husband)

She would only let go of him in order to "maybe" someone better

She would only let go of that guy for someone even better

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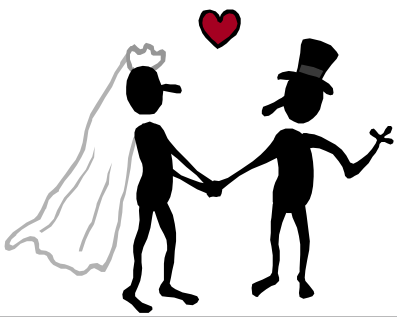
AND SO ON…

Corollary: Each girl will marry her absolute favorite of the boys who visit her during the TMA



Lemma: No boy can be rejected by all the girls

Proof (by contradiction):

Suppose boy b is rejected by all the girls

At that point:

- Each girl must have a suitor other than b
- (By Improvement Lemma, once a girl has a suitor she will always have at least one)
- The n girls have n suitors, and b is not among them. Thus, there are at least n+1 boys

Contradiction

Theorem: The TMA always terminates in at most n^2 days

A "master list" of all n of the boys lists starts with a total of $n \times n = n^2$ girls on it

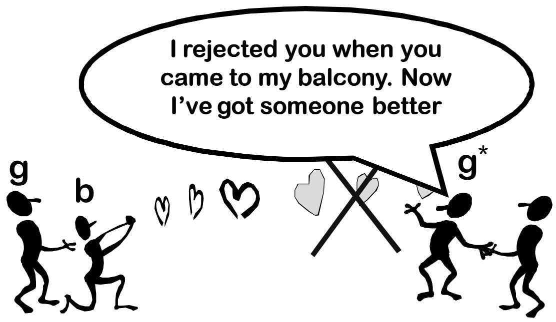
Each day that at least one boy gets a "No", so at least one girl gets crossed off the master list

Therefore, the number of days is bounded by the original size of the master list

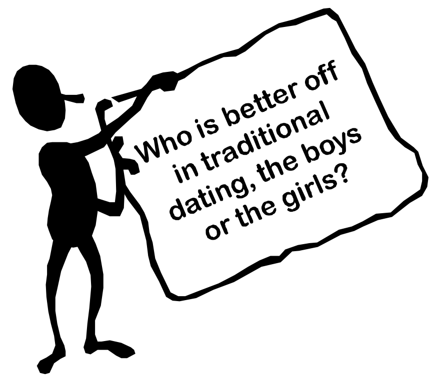
Great! We know that TMA will terminate and produce a pairing

But is it stable?

Theorem: The pairing T produced by TMA is stable



Opinion Poll



Forget TMA For a Moment…

How should we define what we mean when we say "the optimal girl for boy b"?

Flawed Attempt:
"The girl at the top of b's list"

The Optimal Girl

A boy's optimal girl is the highest ranked girl for whom there is some stable pairing in which the boy gets her

She is the best girl he can conceivably get in a stable world. Presumably, she might be better than the girl he gets in the stable pairing output by TMA

The Pessimal Girl

A boy’s pessimal girl is the lowest ranked girl for whom there is some stable pairing in which the boy gets her

She is the worst girl he can conceivably get in a stable world

Dating Heaven and Hell

A pairing is male-optimal if every boy gets his optimal mate. This is the best of all possible stable worlds for every boy simultaneously

A pairing is male-pessimal if every boy gets his pessimal mate. This is the worst of all possible stable worlds for every boy simultaneously

Dating Heaven and Hell

A pairing is female-optimal if every girl gets her optimal mate. This is the best of all possible stable worlds for every girl simultaneously

A pairing is female-pessimal if every girl gets her pessimal mate. This is the worst of all possible stable worlds for every girl simultaneously

The Naked Mathematical Truth!

The Traditional Marriage Algorithm always produces a male-optimal, female-pessimal pairing

Theorem: TMA produces a male-optimal pairing

Suppose, for a contradiction, that some boy gets rejected by his optimal girl during TMA

Let t be the earliest time at which this happened

At time t, boy b got rejected by his optimal girl g because she said “maybe” to a preferred b*

By the definition of t, b* had not yet been rejected by his optimal girl

Therefore, b* likes g at least as much as his optimal

Some boy b got rejected by his optimal girl g because she said “maybe” to a preferred b*. b* likes g at least as much as his optimal girl

There must exist a stable pairing S in which b and g are married

b* wants g more than his wife in S: g is at least as good as his best and he does not have her in stable pairing S

g wants b* more than her husband in S: b is her husband in S and she rejects him for b* in TMA

Contradiction

Theorem: The TMA pairing, T, is female-pessimal

We know it is male-optimal. Suppose there is a stable pairing S where some girl g does worse than in T

Let b be her mate in T

Let b* be her mate in S

By assumption, g likes b better than her mate in S

b likes g better than his mate in S (we already know that g is his optimal girl)

Therefore, S is

Contradiction

The largest, most successful dating service in the world uses a computer to run TMA!

Here's What You Need to Know…

Definition of:

- Stable Pairing
- Traditional Marriage Algorithm

Proof that:

- TMA Produces a Stable Pairing
- TMA Produces a Male-Optimal, Female-Pessimal Pairing