15-251

Great Theoretical Ideas in Computer Science

Counting III

Lecture 8 (September 20, 2007)

Arrange n symbols: r_1 of type 1, r_2 of type 2, …, r_k of type k

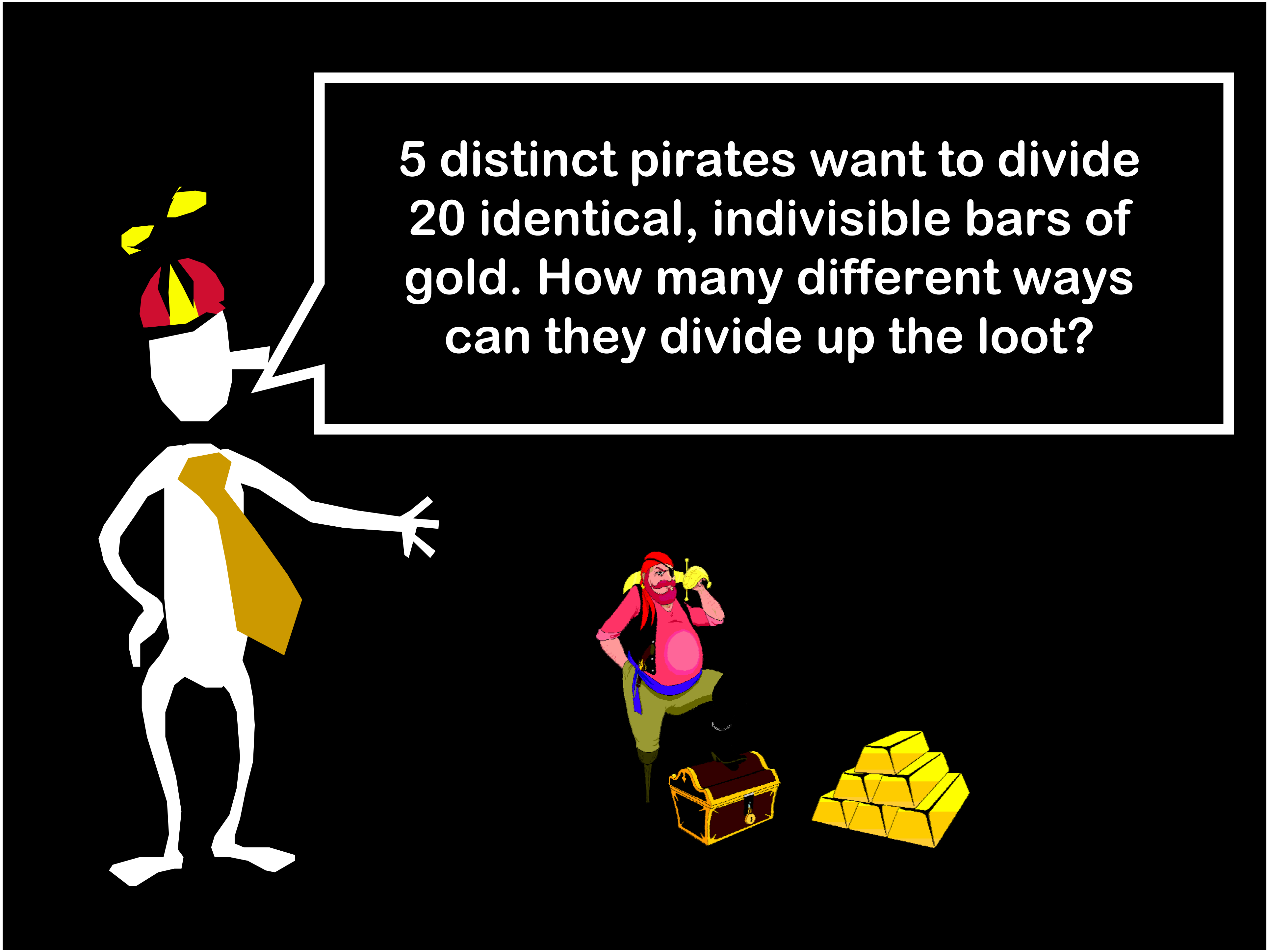
$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\ldots-r_{k-1}}{r_k}$$

$$= \frac{n!}{(n-r_1)!r_1!}\;\frac{(n-r_1)!}{(n-r_1-r_2)!r_2!}\;\cdots$$

$$= \frac{n!}{r_1!r_2!\ldots r_k!}$$

CARNEGIEMELLON

$$\frac{14!}{2!3!2!} = 3{,}632{,}428{,}800$$

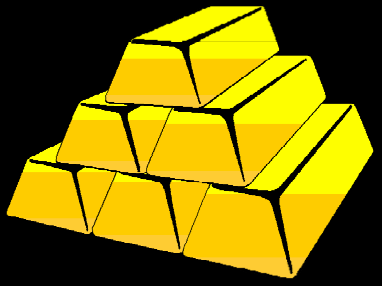

5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?

How many different ways to divide up the loot?

Sequences with 20 G's and 4 /'s

$$\binom{24}{4}$$

How many different ways can n distinct pirates divide k identical, indivisible bars of gold?



$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer solutions to the following equations?

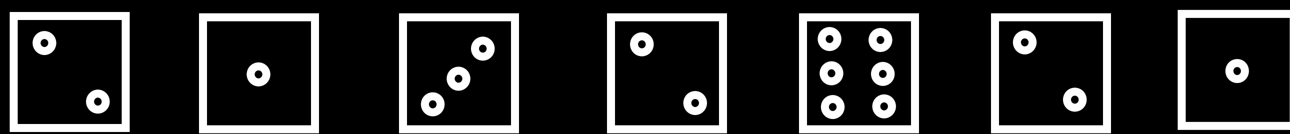
$$x_1 + x_2 + x_3 + \ldots + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

$$\binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k}$$

Identical/Distinct Dice

Suppose that we roll seven dice



How many different outcomes are there, if order matters? 6^7

What if order doesn't matter? (E.g., Yahtzee) $\binom{12}{7}$

(Corresponds to 6 pirates and 7 bars of gold)

Identical/Distinct Objects

If we are putting k objects into
n distinct bins.

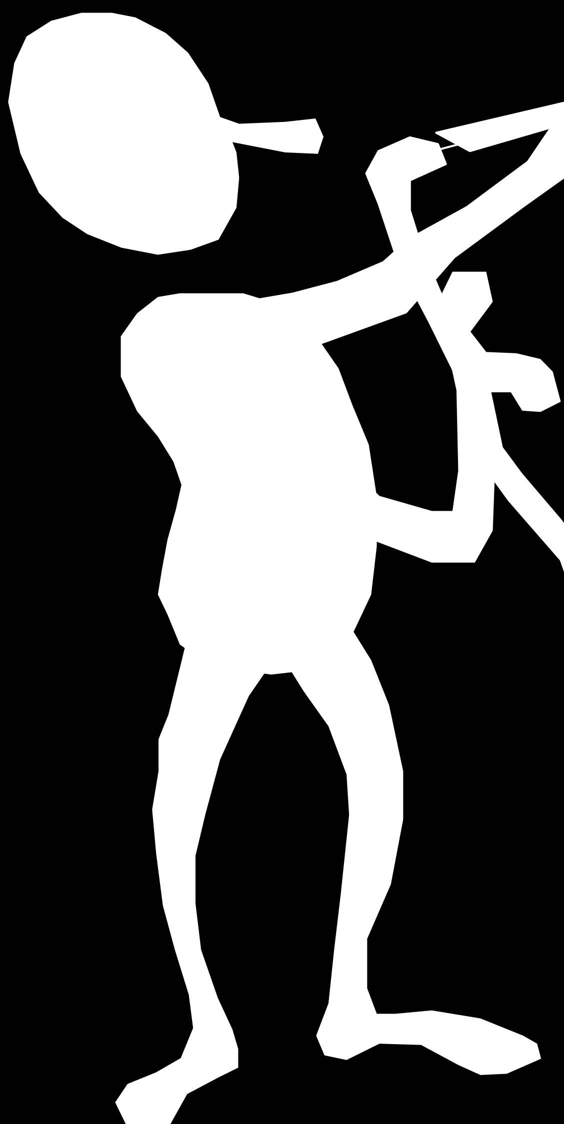
Objects are distinguishable	n^k
Objects are indistinguishable	$\binom{k+n-1}{k}$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} X^0 + \binom{n}{1} X^1 + \ldots + \binom{n}{n} X^n$$

Binomial Coefficients

binomial expression



What is the coefficient of $(X_1^{r_1} X_2^{r_2} \ldots X_k^{r_k})$ in the expansion of $(X_1+X_2+X_3+\ldots+X_k)^n$?

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

Power Series Representation

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} X^k$$

For k>n, $\binom{n}{k} = 0$

"Product form" or "Generating form"

"Power Series" or "Taylor Series" Expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Let x = 1, $\quad 2^n = \sum_{k=0}^{n} \binom{n}{k}$

The number of subsets of an n-element set

By varying x, we can discover new identities:

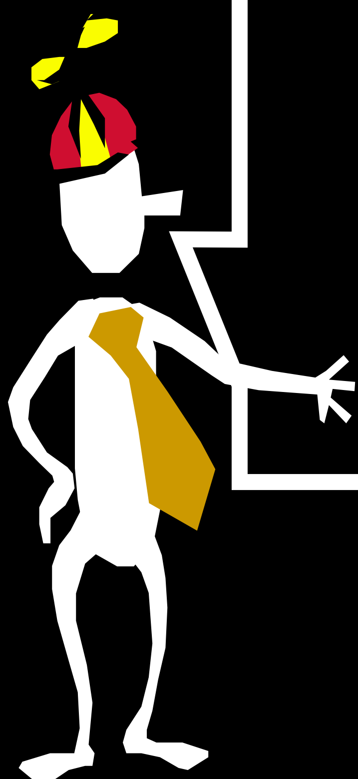
$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Let x = -1, $$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Equivalently, $$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

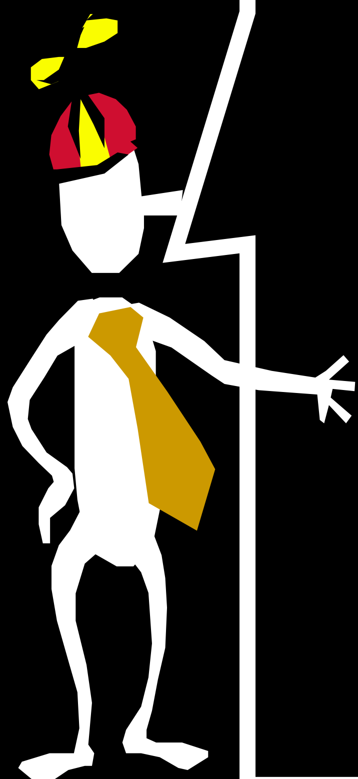
The number of subsets with even size is the same as the number of subsets with odd size

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$



Proofs that work by manipulating algebraic forms are called "algebraic" arguments.
Proofs that build a bijection are called "combinatorial" arguments

$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We just saw an algebraic proof that

$$|O_n| = |E_n|$$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones

Let E_n be the set of binary strings of length n with an even number of ones

A combinatorial proof must construct a bijection between O_n and E_n

An Attempt at a Bijection

Let f_n be the function that takes an n-bit string and flips all its bits

f_n is clearly a one-to-one and onto function

for odd n. E.g. in f_7 we have:

0010011 → 1101100
1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100
101010 → 010101

Uh oh. Complementing maps evens to evens!

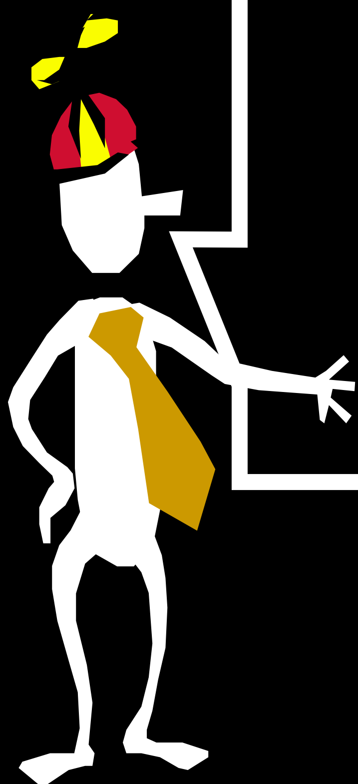
A Correspondence That Works for all n

Let f_n be the function that takes an n-bit string and flips only the first bit. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Set of all k-subsets of {1..n}

Either we **do not** pick n: then we have to pick k elements out of the remaining n-1.

Or we **do** pick n: then we have to pick k-1 elts. out of the remaining n-1.

The Binomial Formula

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle: k^{th} row are coefficients of $(1+X)^k$

Inductive definition of k^{th} entry of n^{th} row:
Pascal(n,0) = Pascal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

“Pascal’s Triangle”

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$$

- Al-Karaji, Baghdad 953-1029
- Chu Shin-Chieh 1303
- Blaise Pascal 1654

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand"

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1 = 1

1 + 1 = 2

1 + 2 + 1 = 4

1 + 3 + 3 + 1 = 8

1 + 4 + 6 + 4 + 1 = 16

1 + 5 + 10 + 10 + 5 + 1 = 32

1 + 6 + 15 + 20 + 15 + 6 + 1 = 64

Odds and Evens

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 + 15 + 15 + 1 = 6 + 20 + 6

Summing on 1st Avenue

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} \binom{i}{1} = \binom{n+1}{2}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Summing on kth Avenue

1

1 1

1 2 1

1 3 3 1

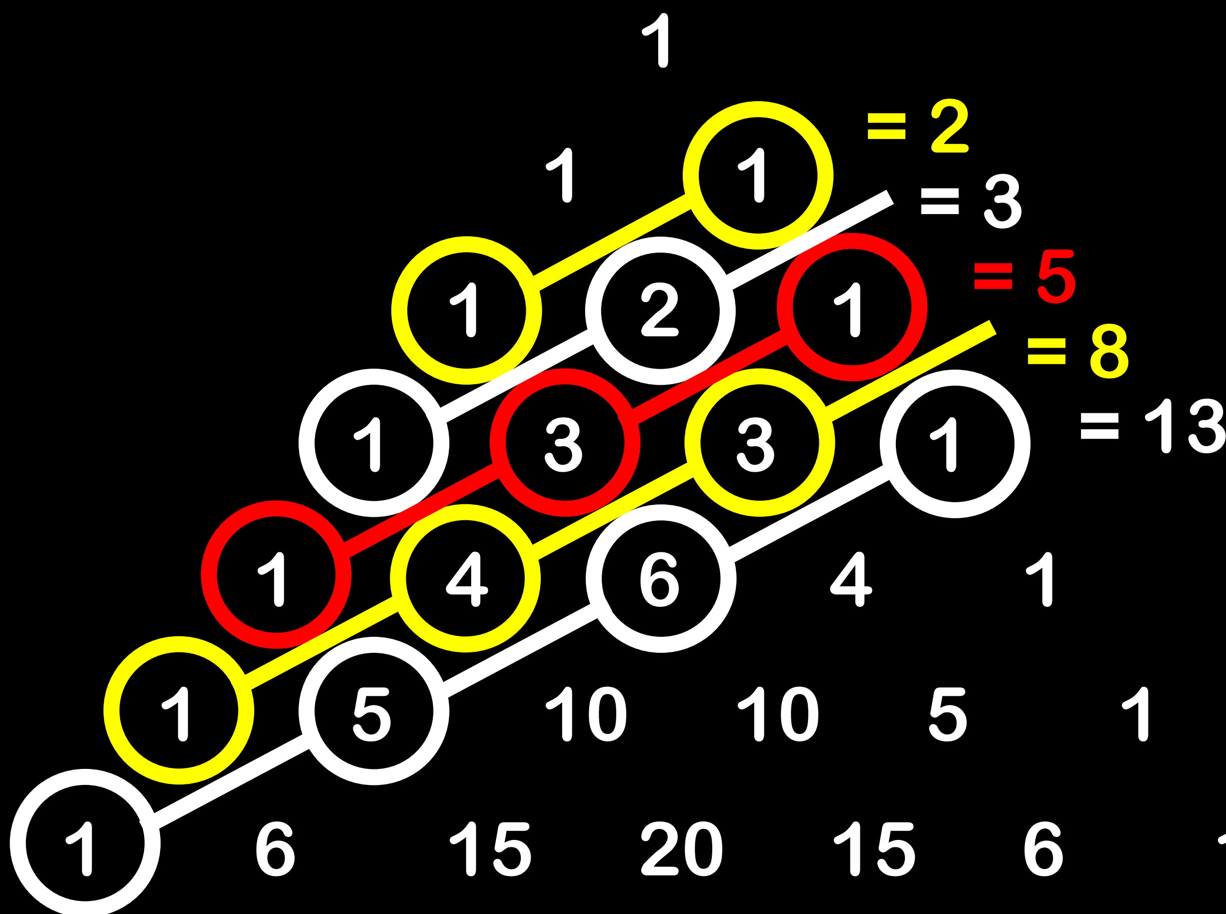
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Fibonacci Numbers



Sums of Squares

1

1 1

1^2 2^2 1^2

1^2 3^2 3^2 1^2

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Al-Karaji Squares

1

1 1 = 1

1 2 $+2\cdot$1 = 4

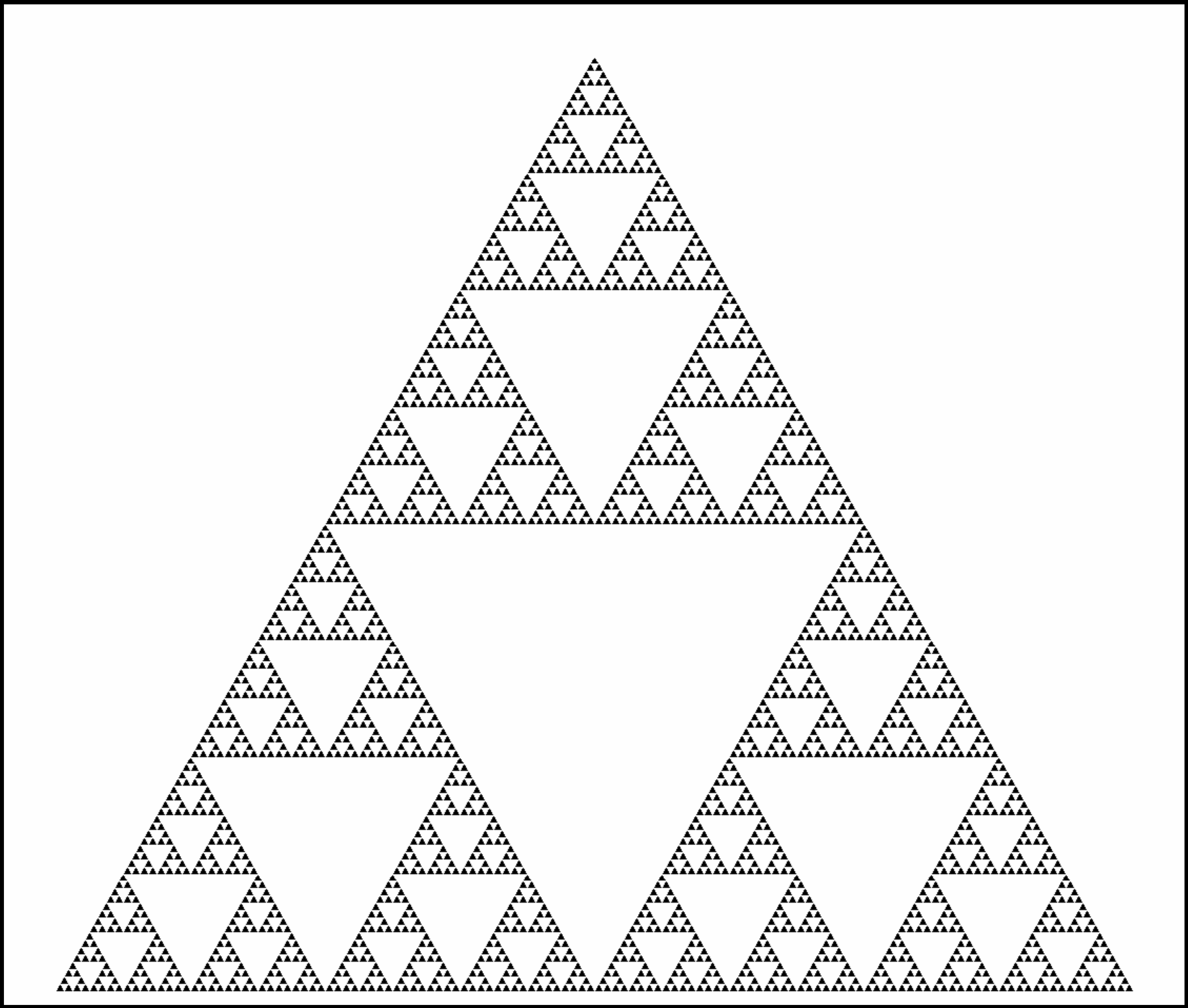
1 3 $+2\cdot$3 1 = 9

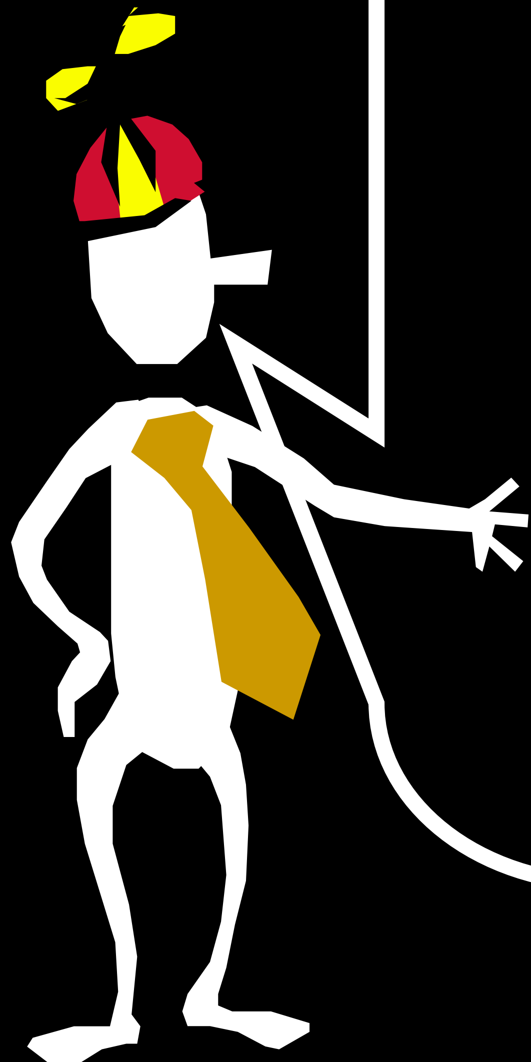
1 4 $+2\cdot$6 4 1 = 16

1 5 $+2\cdot$10 10 5 1 = 25

1 6 $+2\cdot$15 20 15 6 1 = 36

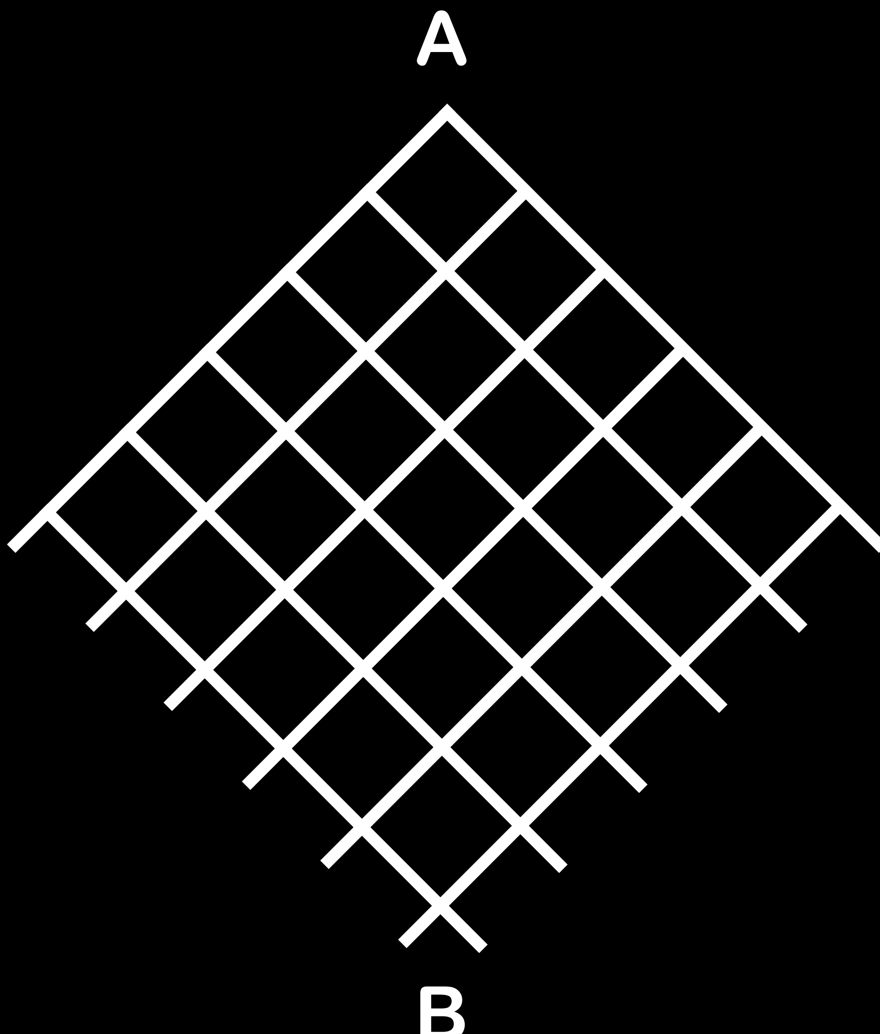
Pascal Mod 2





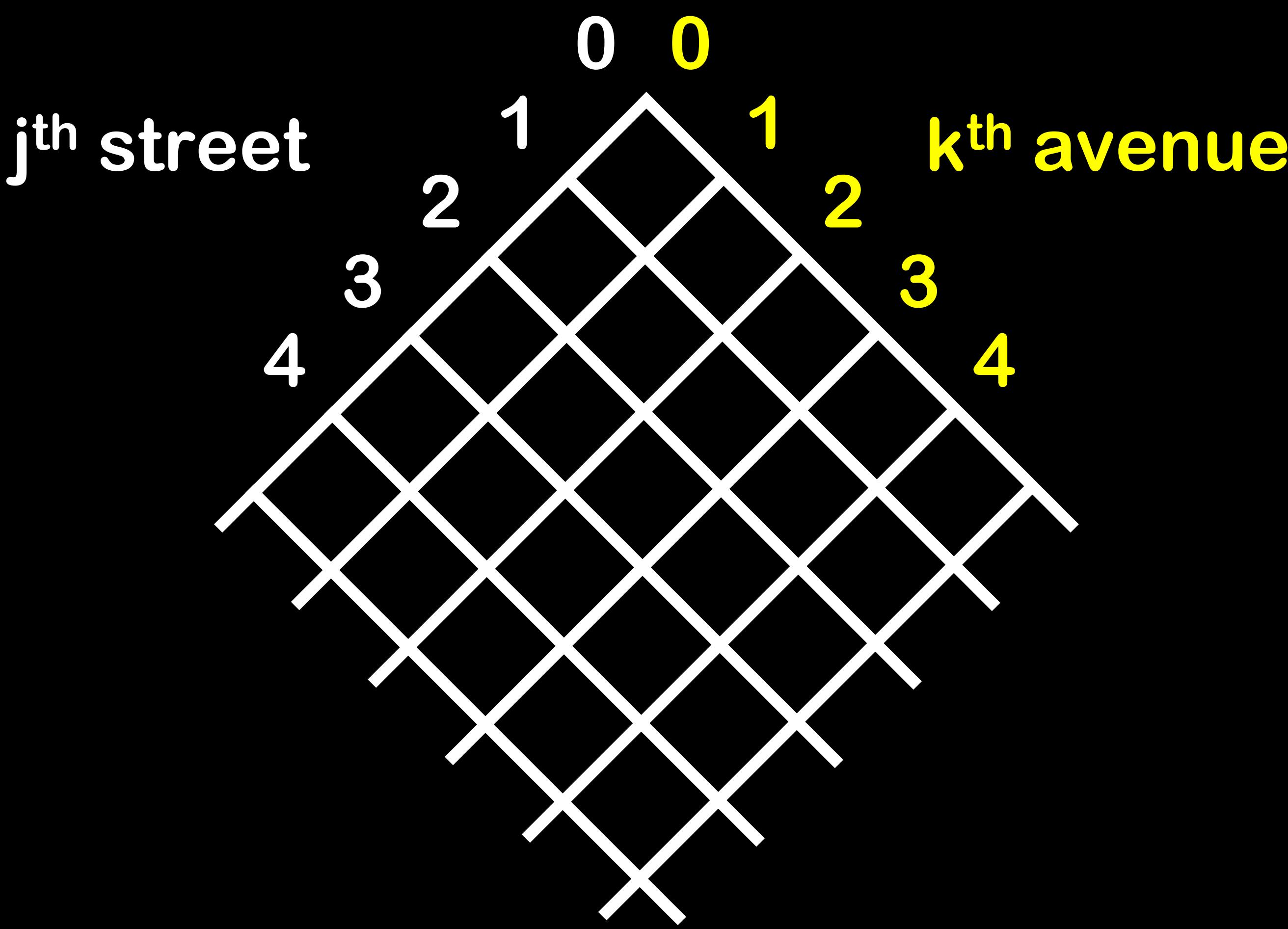
All these properties can be proved inductively and algebraically. We will give *combinatorial* proofs using the **Manhattan block walking representation of binomial coefficients**

How many shortest routes from A to B?



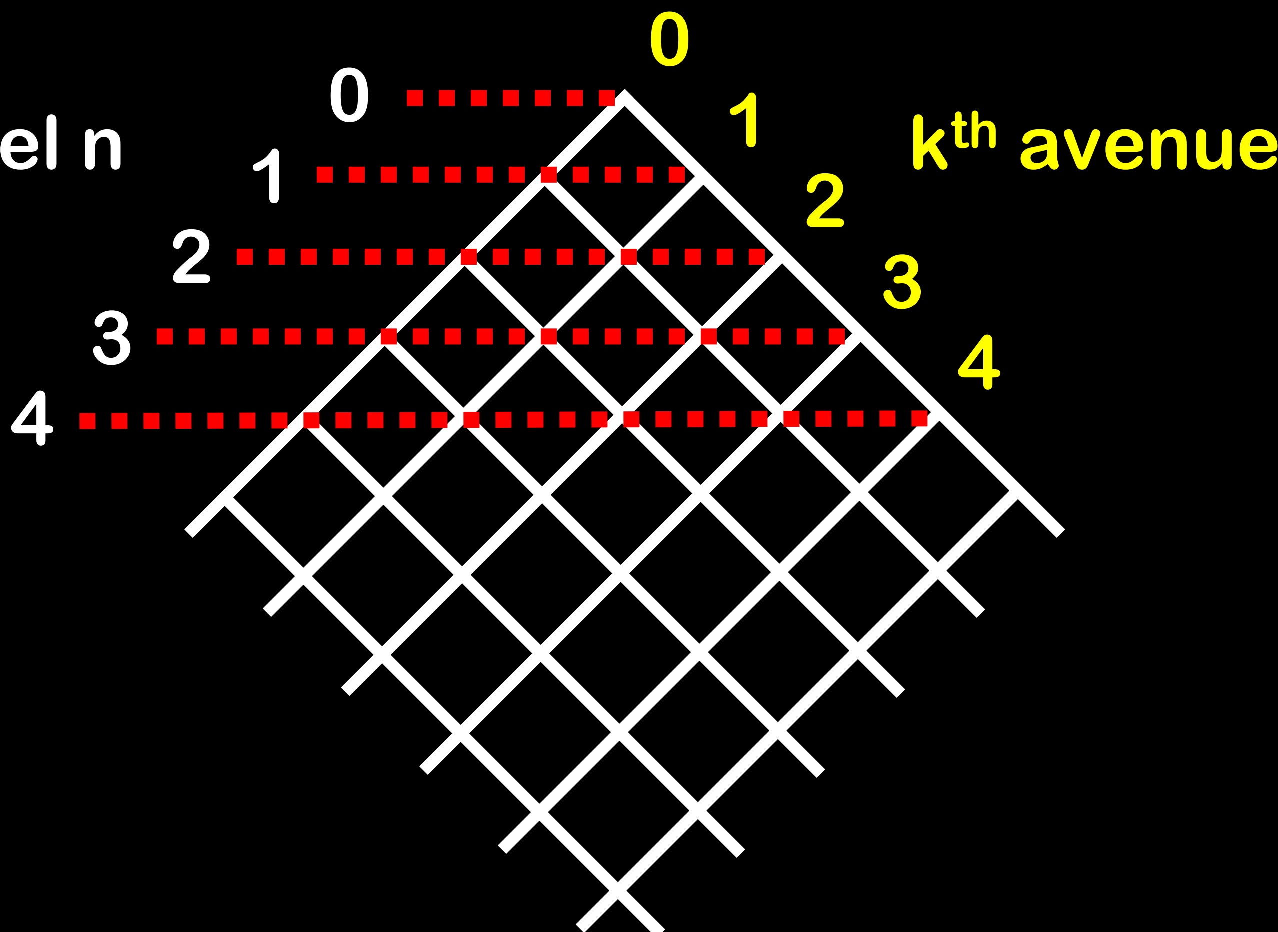
$$\binom{10}{5}$$

Manhattan



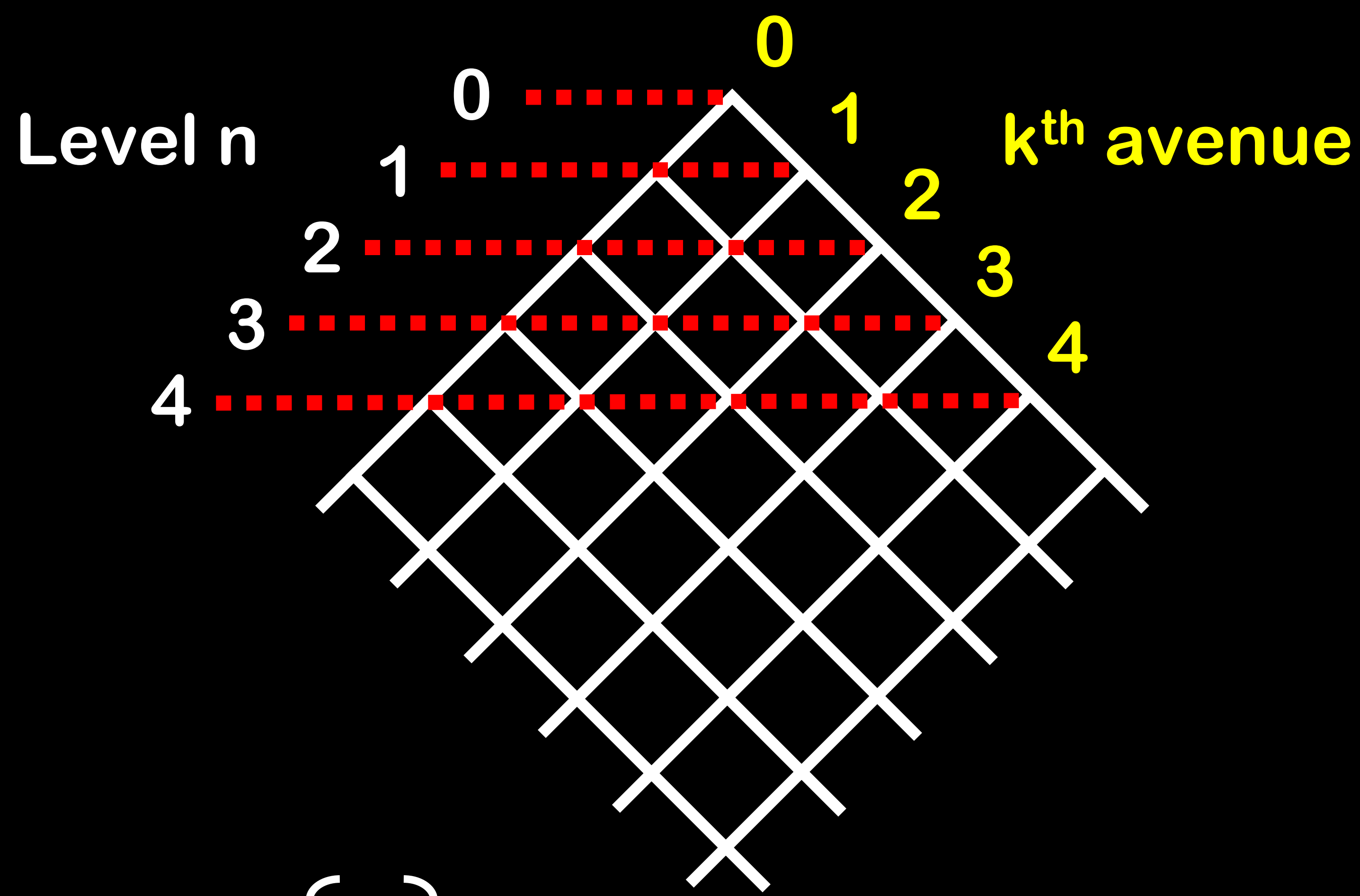
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k)

Manhattan

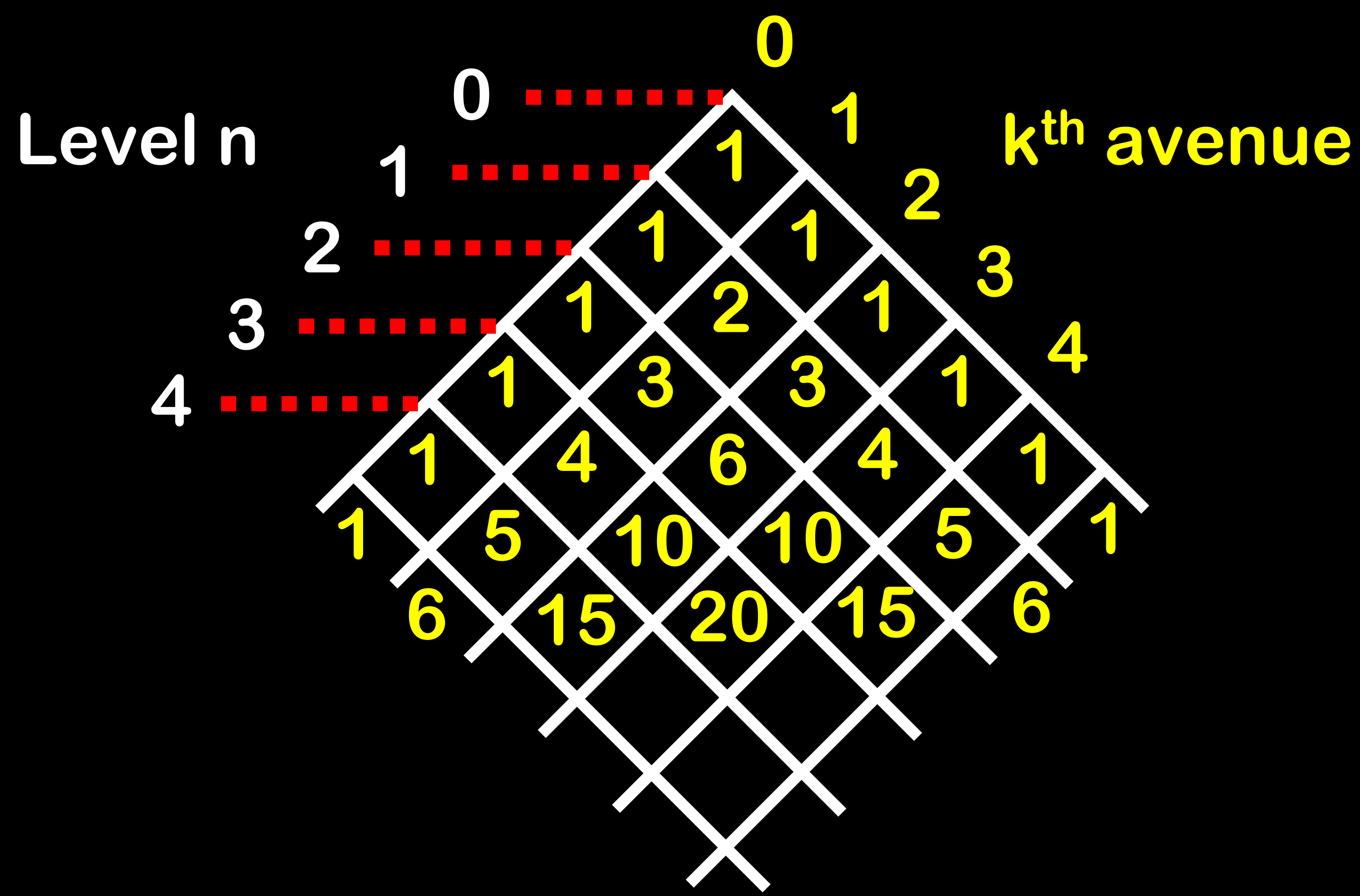


There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k)

Manhattan



There are $\binom{n}{k}$ shortest routes from (0,0) to level n and k^{th} avenue



Level n
0
1
2
3
4
k^th avenue
0
1
2
3
4
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6

Level n — k^{th} avenue

0: 1

1: 1 1

2: 1 2 1

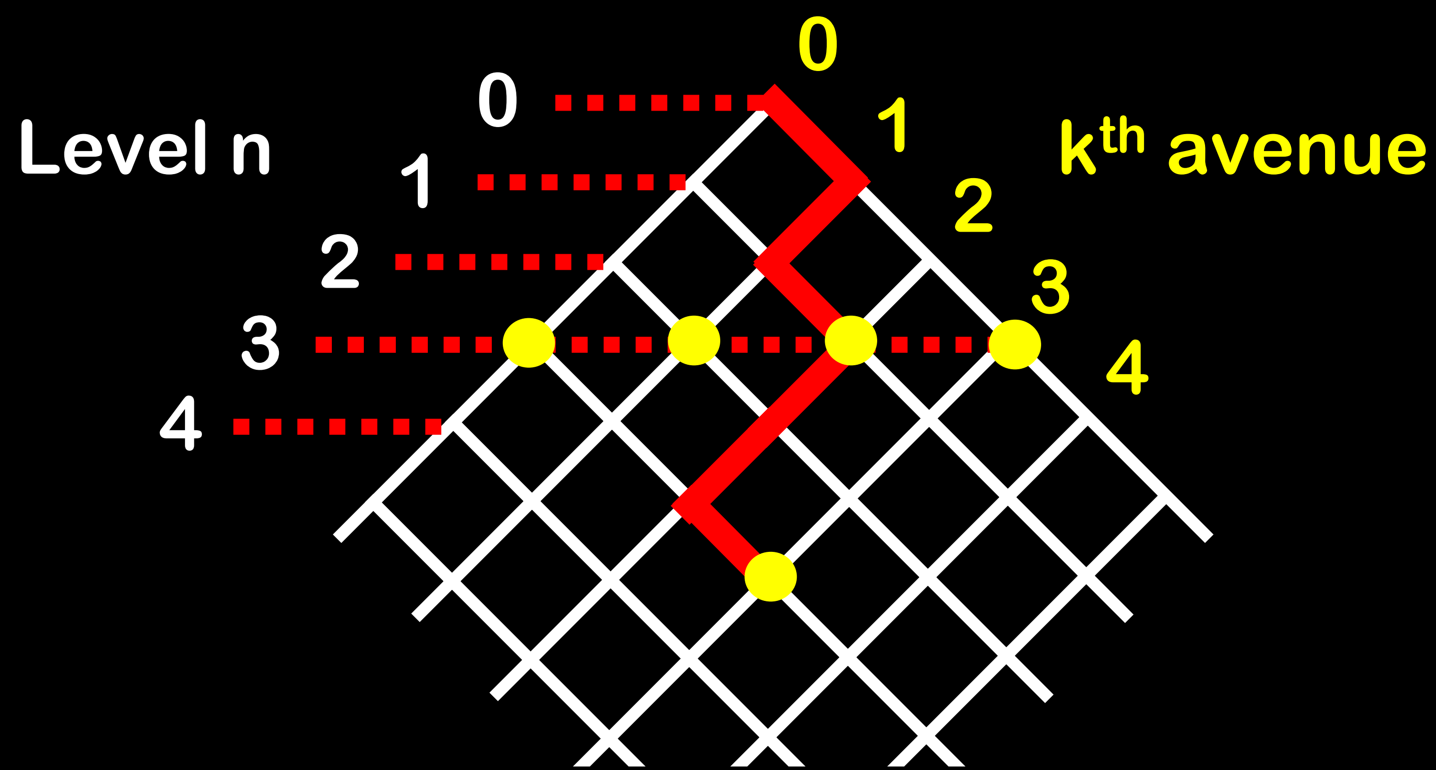
3: 1 3 3 1

4: 1 4 6 4 1

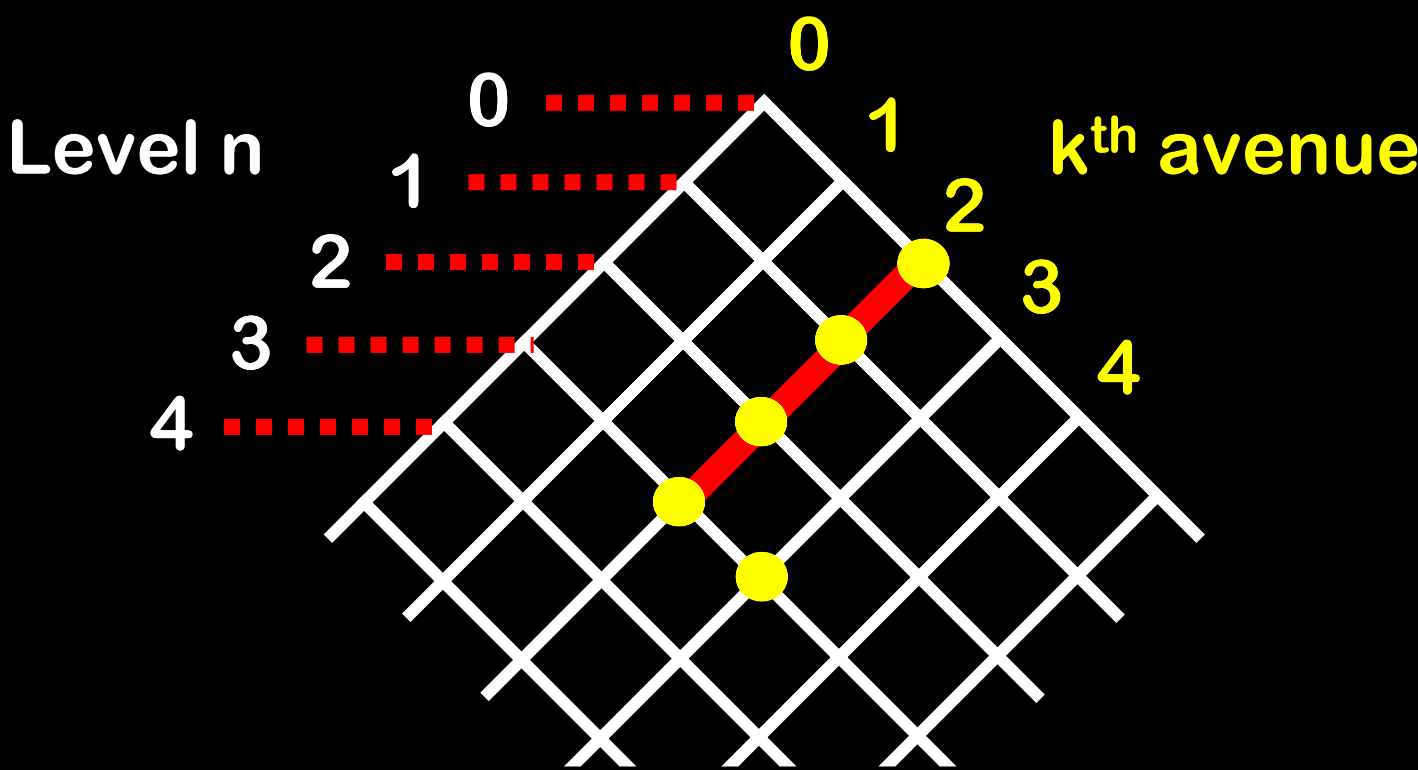
1 5 10 + 10 5 1

6 15 20 15 6

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$



$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable $V^{\rightarrow}$ can be thought of as:

< * , * , * , * , *, *, … >

Vector Programs

Let k stand for a scalar constant

<k> will stand for the vector <k,0,0,0,…>

<0> = <0,0,0,0,…>

<1> = <1,0,0,0,…>

$V^{\rightarrow} + T^{\rightarrow}$ means to add the vectors position-wise

<4,2,3,…> + <5,1,1,….> = <9,3,4,…>

Vector Programs

RIGHT($V^{\rightarrow}$) means to shift every number in $V^{\rightarrow}$ one position to the right and to place a 0 in position 0

RIGHT(<1,2,3, …>) = <0,1,2,3,…>

Vector Programs

Example:

$V^{\rightarrow} := \langle 6 \rangle;$

$V^{\rightarrow} := \text{RIGHT}(V^{\rightarrow}) + \langle 42 \rangle;$

$V^{\rightarrow} := \text{RIGHT}(V^{\rightarrow}) + \langle 2 \rangle;$

$V^{\rightarrow} := \text{RIGHT}(V^{\rightarrow}) + \langle 13 \rangle;$

Store:

$V^{\rightarrow} = \langle 6,0,0,0,\ldots \rangle$

$V^{\rightarrow} = \langle 42,6,0,0,\ldots \rangle$

$V^{\rightarrow} = \langle 2,42,6,0,\ldots \rangle$

$V^{\rightarrow} = \langle 13,2,42,6,\ldots \rangle$

$$V^{\rightarrow} = \langle 13, 2, 42, 6, 0, 0, 0, \ldots \rangle$$

Vector Programs

Example:

$V^{\to}$:= <1>;

Loop n times

$V^{\to}$:= $V^{\to}$ + RIGHT($V^{\to}$);

Store:

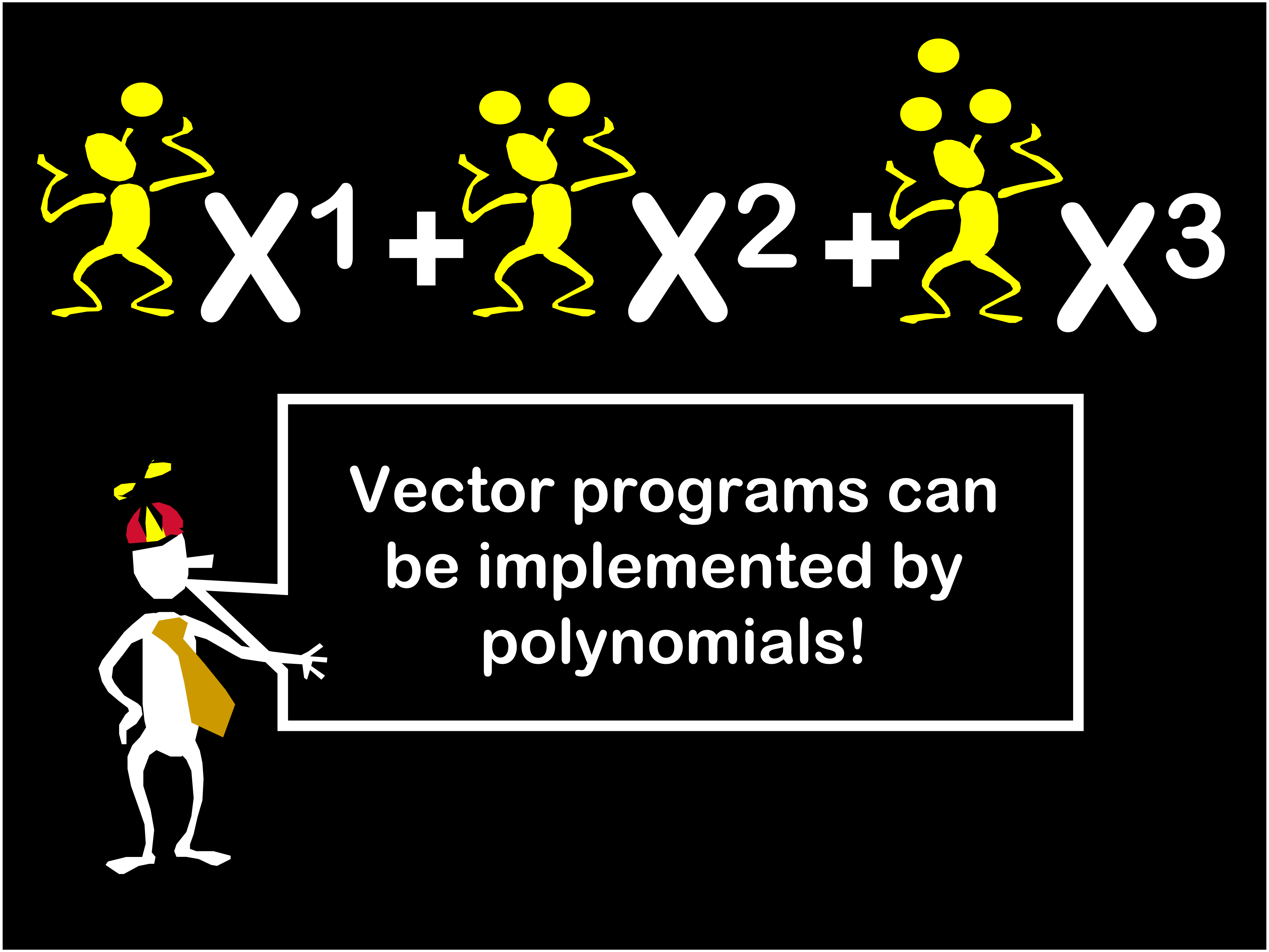
$V^{\to}$ = <1,0,0,0,…>

$V^{\to}$ = <1,1,0,0,…>

$V^{\to}$ = <1,2,1,0,…>

$V^{\to}$= <1,3,3,1,…>

$V^{\to}$ = n^{th} row of Pascal's triangle


X1 + X2 + X3
Vector programs can be implemented by polynomials!

Programs $\rightarrow$ Polynomials

The vector $V^{\rightarrow} = < a_0, a_1, a_2, \ldots >$ will be represented by the polynomial:

$$P_V = \sum_{i=0}^{\infty} a_i X^i$$

Formal Power Series

The vector $\vec{V} = < a_0, a_1, a_2, \ldots >$ will be represented by the **formal power series**:

$$P_V = \sum_{i=0}^{\infty} a_i X^i$$

$$\vec{V} = < a_0, a_1, a_2, \ldots >$$

$$P_V = \sum_{i=0}^{\infty} a_i X^i$$

$<0>$ is represented by	0
$<k>$ is represented by	k
$\vec{V} + \vec{T}$ is represented by	$(P_V + P_T)$
RIGHT$(\vec{V})$ is represented by	$(P_V X)$

Vector Programs

Example:

$V^{\rightarrow} := \langle 1 \rangle;$

$P_V := 1;$

Loop n times

$V^{\rightarrow} := V^{\rightarrow} + \mathrm{RIGHT}(V^{!});$

$P_V := P_V + P_V X;$

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle

Vector Programs

Example:

$\vec{V}$:= <1>; $P_V := 1;$

Loop n times

$\vec{V} := \vec{V} + \text{RIGHT}(\vec{V});$ $P_V := P_V(1+X);$

$\vec{V}$ = n^th^ row of Pascal's triangle

Vector Programs

Example:

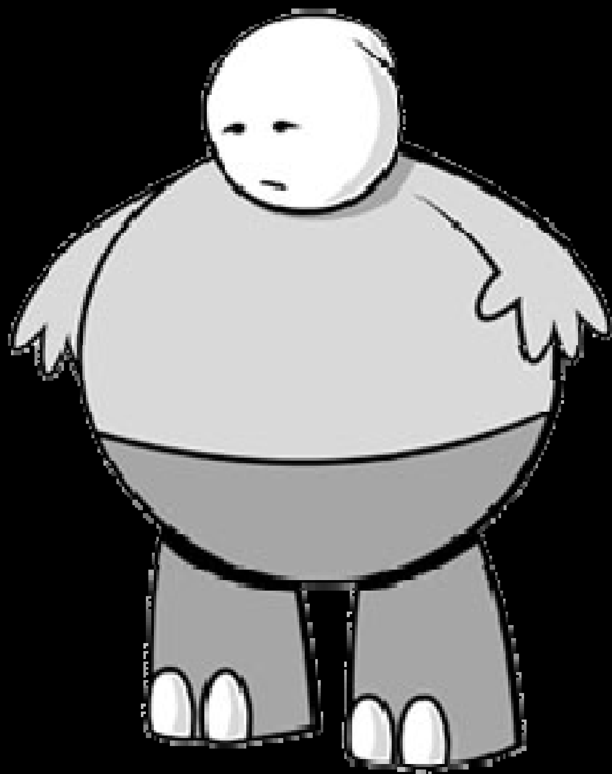
$\vec{V} := \langle 1 \rangle;$

Loop n times

$\vec{V} := \vec{V} + \text{RIGHT}(V!);$

$\Big\}\quad P_V = (1+X)^n$

$\vec{V}$ = n^{th} row of Pascal’s triangle



Here's What You Need to Know…

- Polynomials count
- Binomial formula
- Combinatorial proofs of binomial identities
- Vector programs