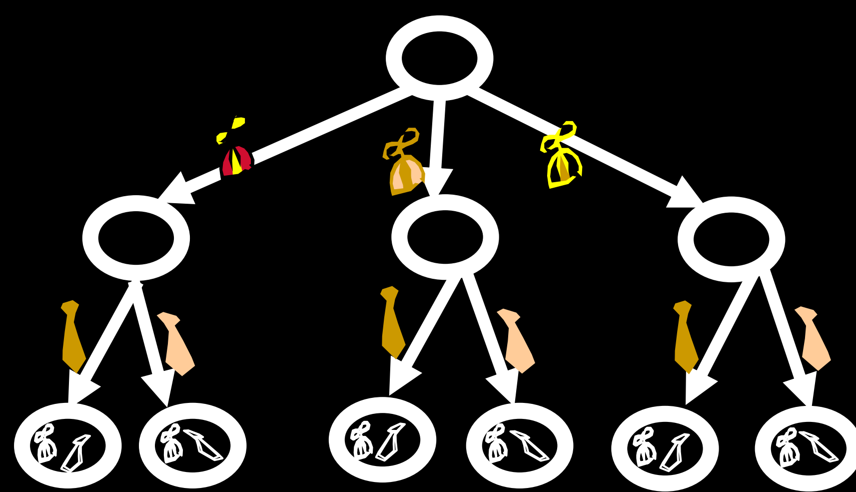
15-251

Great Theoretical Ideas in Computer Science

Counting I: One-To-One Correspondence and Choice Trees

Lecture 6 (September 13, 2007)



If I have 14 teeth on the top and 12 teeth on the bottom, how many teeth do I have in all?

Addition Rule

Let A and B be two disjoint finite sets

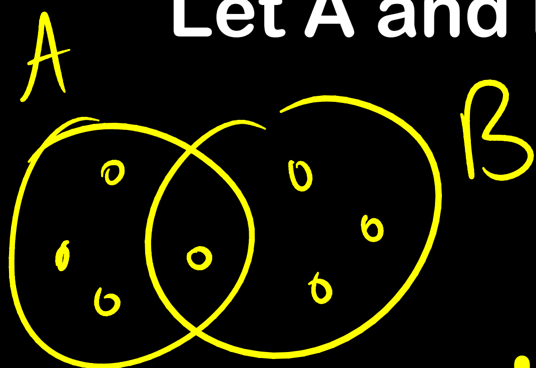
The size of $(A \cup B)$ is the sum of the size of A and the size of B

$$|A \cup B| = |A| + |B|$$

Addition Rule
(2 possibly overlapping sets)

Inclusion Exclusion

Let A and B be two finite sets



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Addition of multiple disjoint sets:

Let A_1, A_2, A_3, …, A_n be disjoint, finite sets.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Partition Method

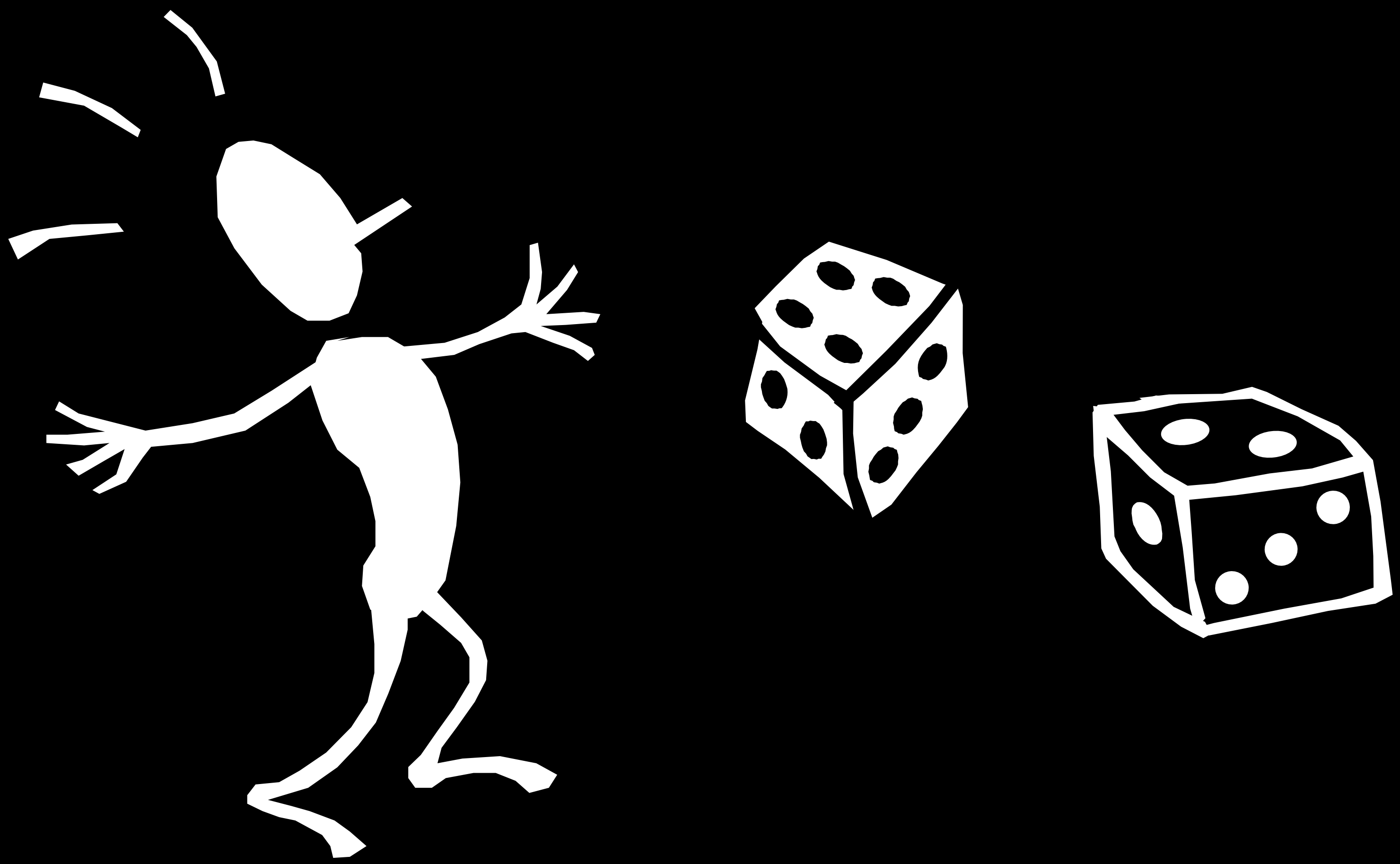
To count the elements of a finite set S, partition the elements into non-overlapping subsets A_1, A_2, A_3, …, A_n .

|S| =

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Partition Method

S = all possible outcomes of one white die and one black die.



Partition Method

S = all possible outcomes of one white die and one black die.

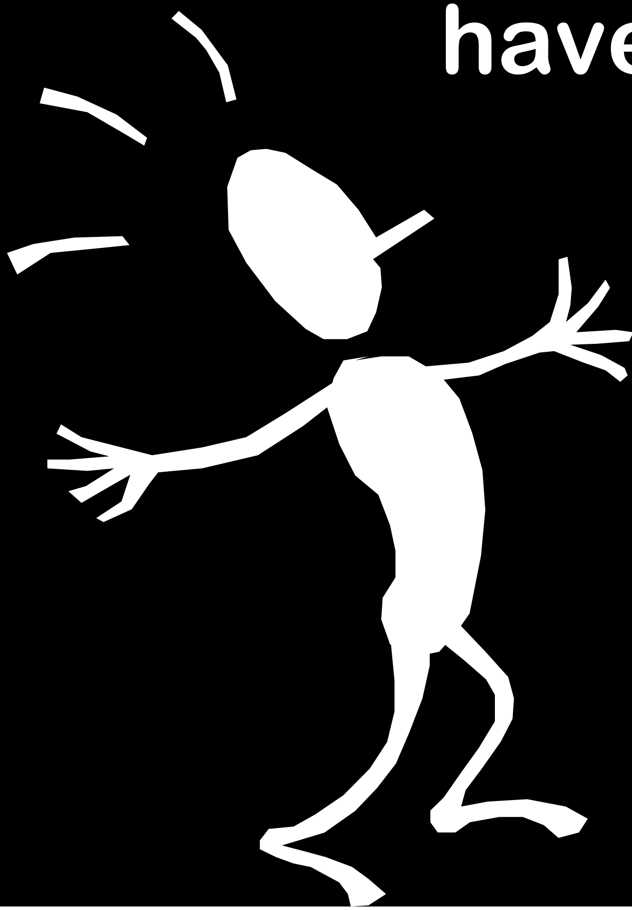
Partition S into 6 sets:

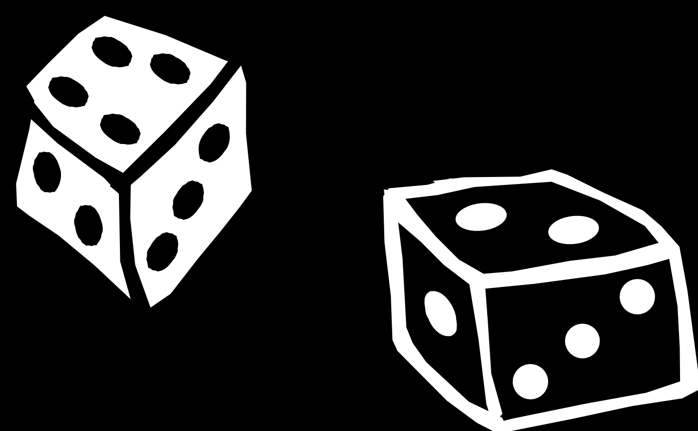
A_1 = the set of outcomes where the white die is 1.
A_2 = the set of outcomes where the white die is 2.
A_3 = the set of outcomes where the white die is 3.
A_4 = the set of outcomes where the white die is 4.
A_5 = the set of outcomes where the white die is 5.
A_6 = the set of outcomes where the white die is 6.

Each of 6 disjoint set have size 6 = 36 outcomes

Partition Method

S = all possible outcomes where the white die and the black die have different values





S $\equiv$ Set of all outcomes where the dice show different values. $|S| = ?$

$A_i \equiv$ set of outcomes where black die says i and the white die says something else.

$$|S| = \left|\bigcup_{i=1}^{6} A_i\right| = \sum_{i=1}^{6} |A_i| = \sum_{i=1}^{6} 5 = 30$$

S ≡ Set of all outcomes where the dice show different values. | S | = ?

T ≡ set of outcomes where dice agree.
= { <1,1>, <2,2>, <3,3>,<4,4>,<5,5>,<6,6>}

$|S \cup T|$ = # of outcomes = 36

$|S| + |T| = 36$

$|T| = 6$

$|S| = 36 - 6 = 30$

S ≡ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$A_i \equiv$ set of outcomes where the black die says i and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

$$|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15$$

S ≡ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

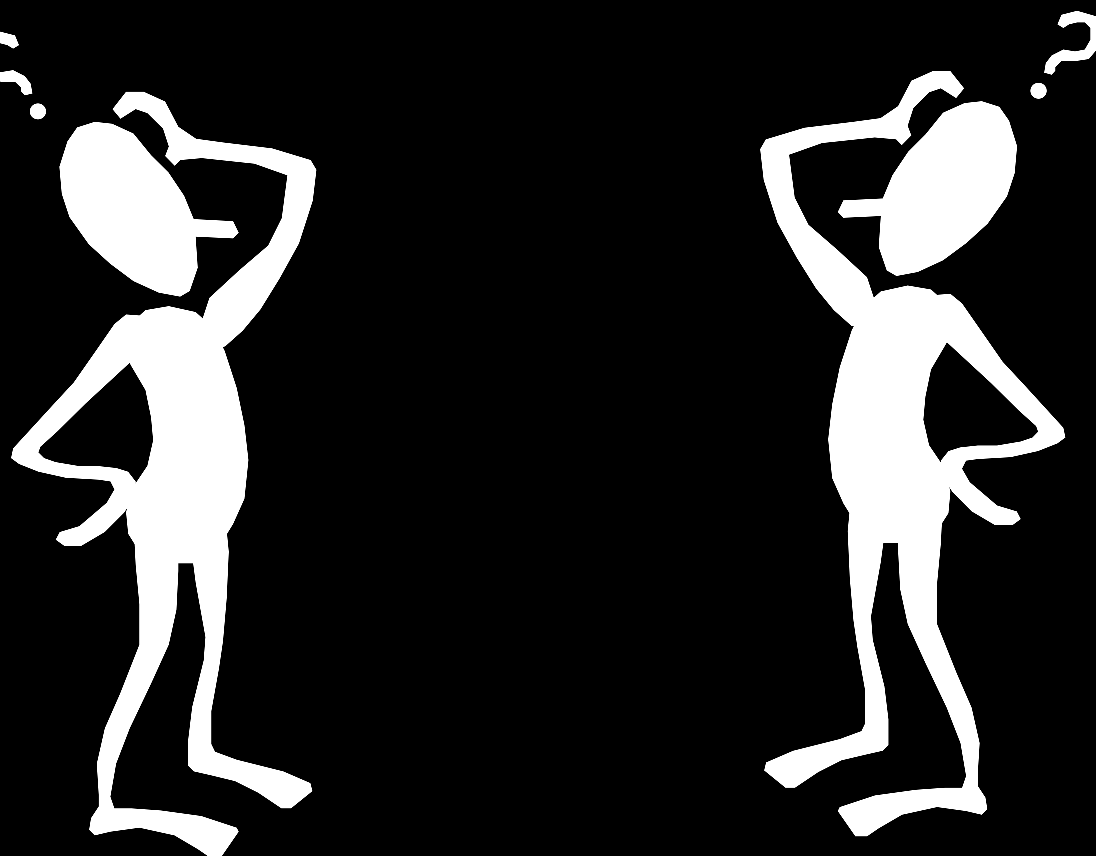
L ≡ set of all outcomes where the black die shows a larger number than the white die.

$$|S| + |L| = 30$$

It is clear by symmetry that | S | = | L |.

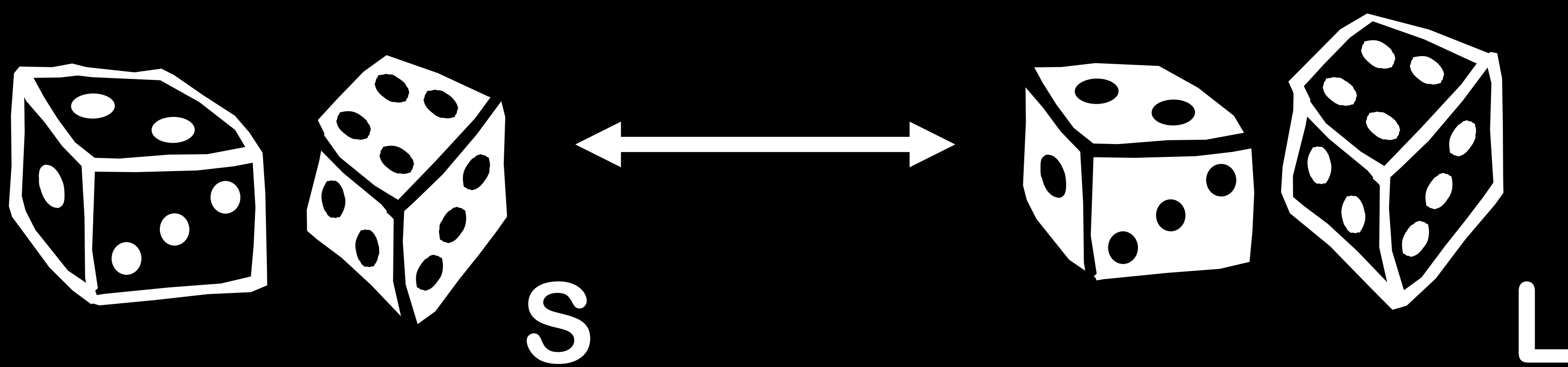
Therefore | S | = 15

“It is **clear** by symmetry that |S| = |L|?”



Pinning Down the Idea of Symmetry by Exhibiting a Correspondence

Put each outcome in S in correspondence with an outcome in L by **swapping** color of the dice.



Each outcome in S gets matched with exactly one outcome in L, with none left over.

Thus: $|S| = |L|$

Let f : A → B Be a Function From a Set A to a Set B

f is **1-1** if and only if

$$\forall x,y \in A,\ \ x \neq y \Rightarrow f(x) \neq f(y)$$

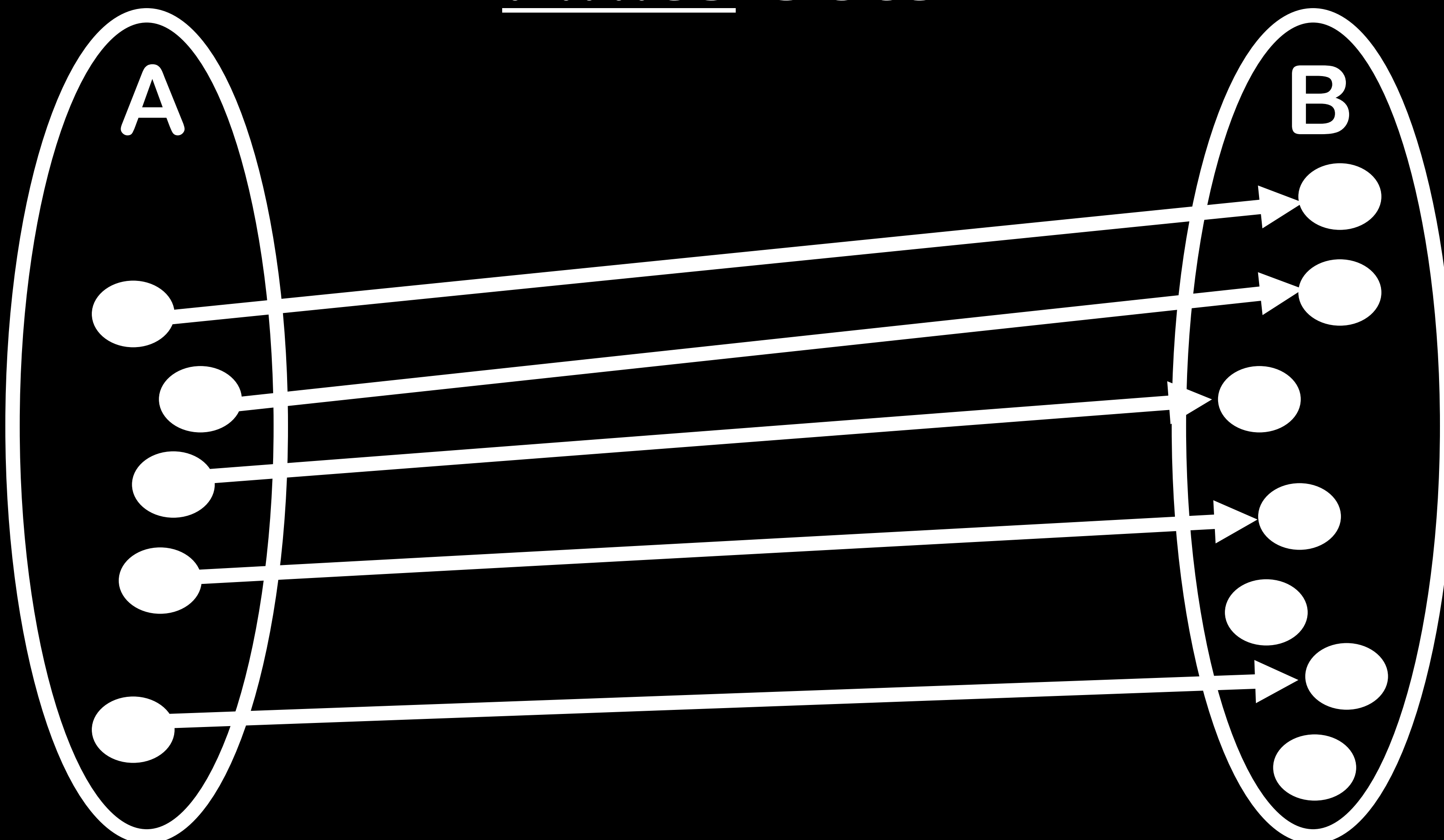
f is **onto** if and only if

$$\forall z \in B\ \ \exists x \in A\ \ f(x) = z$$

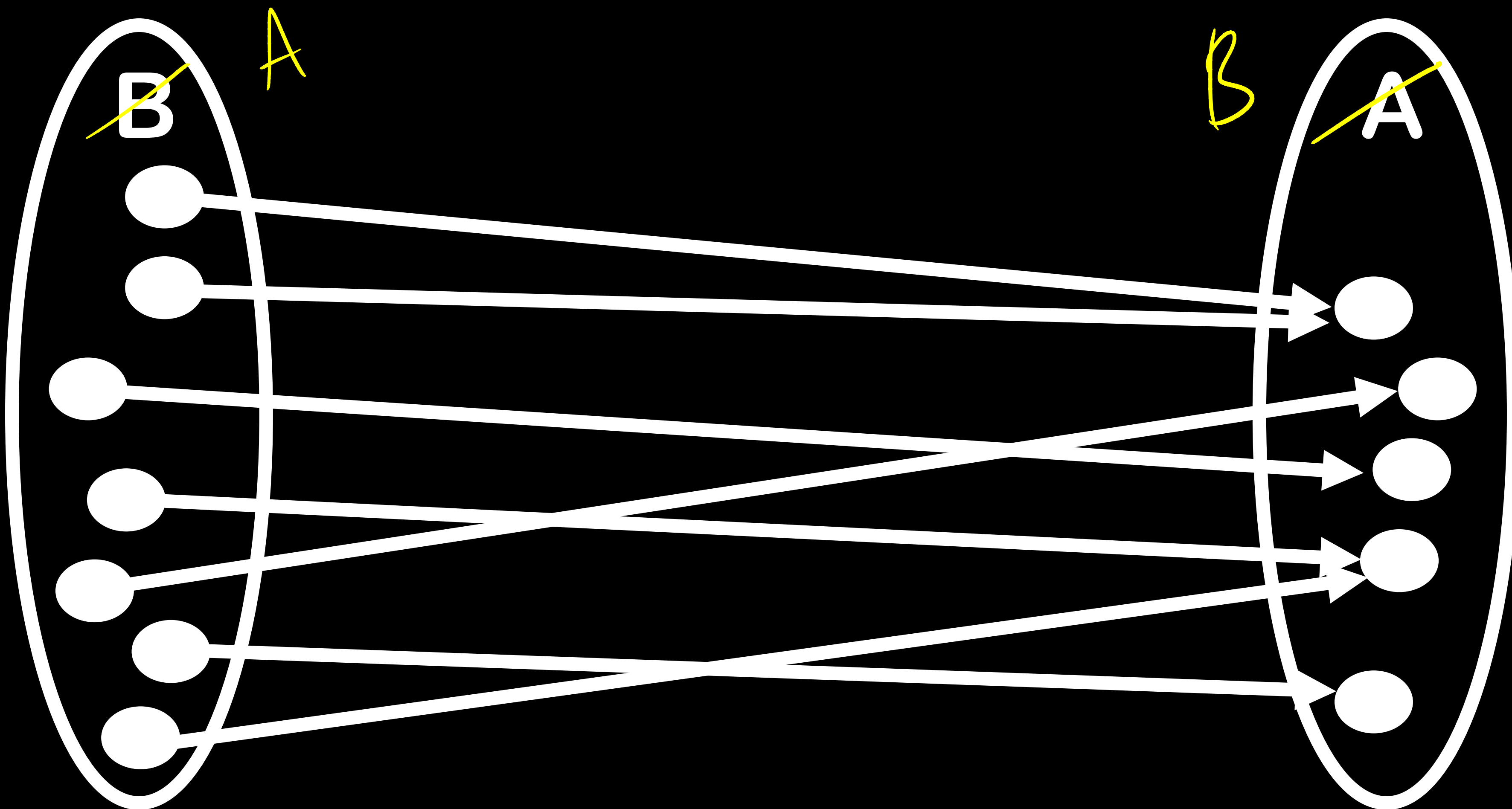
For Every

There Exists

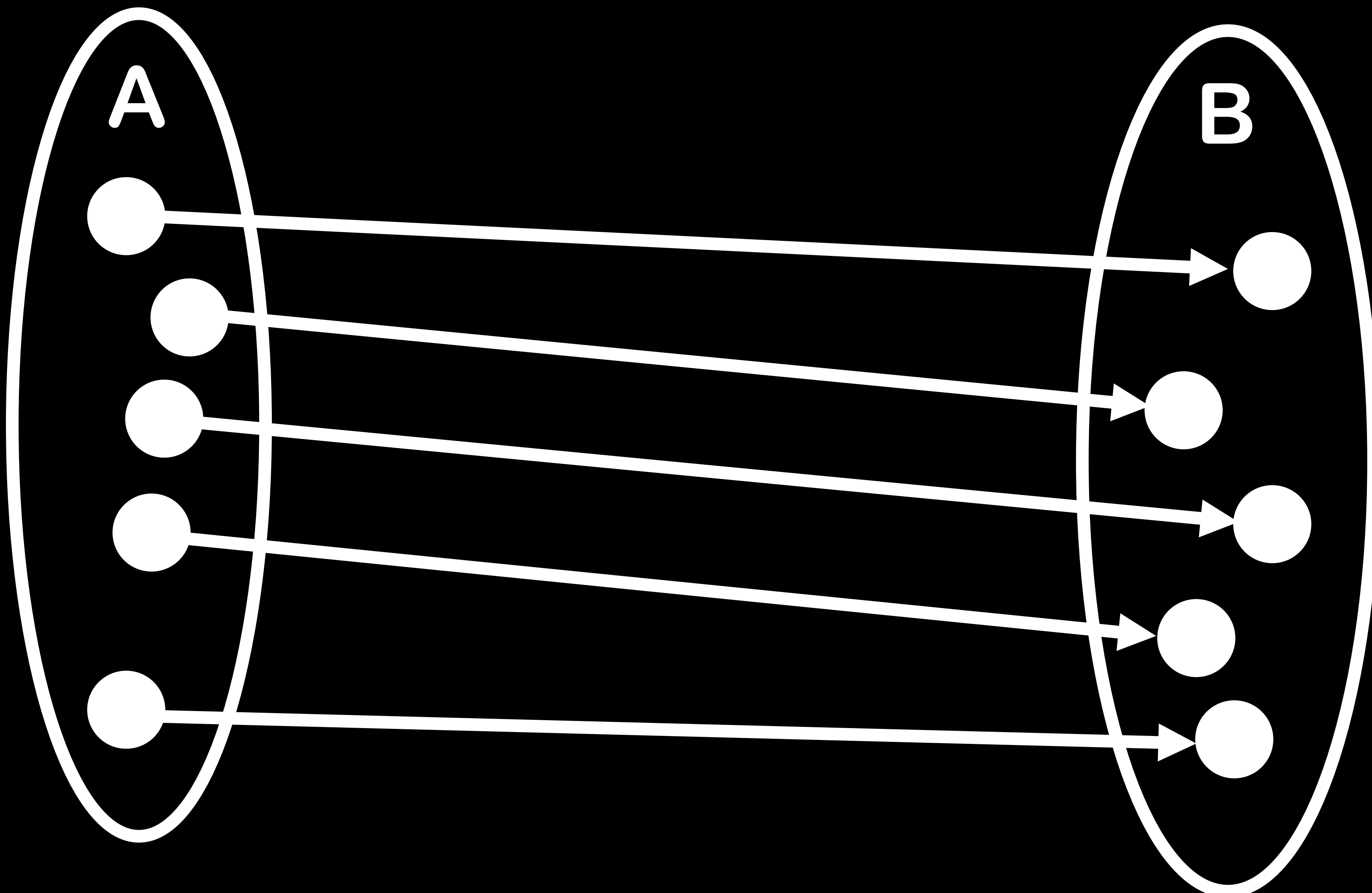
Let's Restrict Our Attention to <u>Finite</u> Sets



$\exists$ 1-1 f : A $\rightarrow$ B $\Rightarrow$ | A | $\leq$ | B |



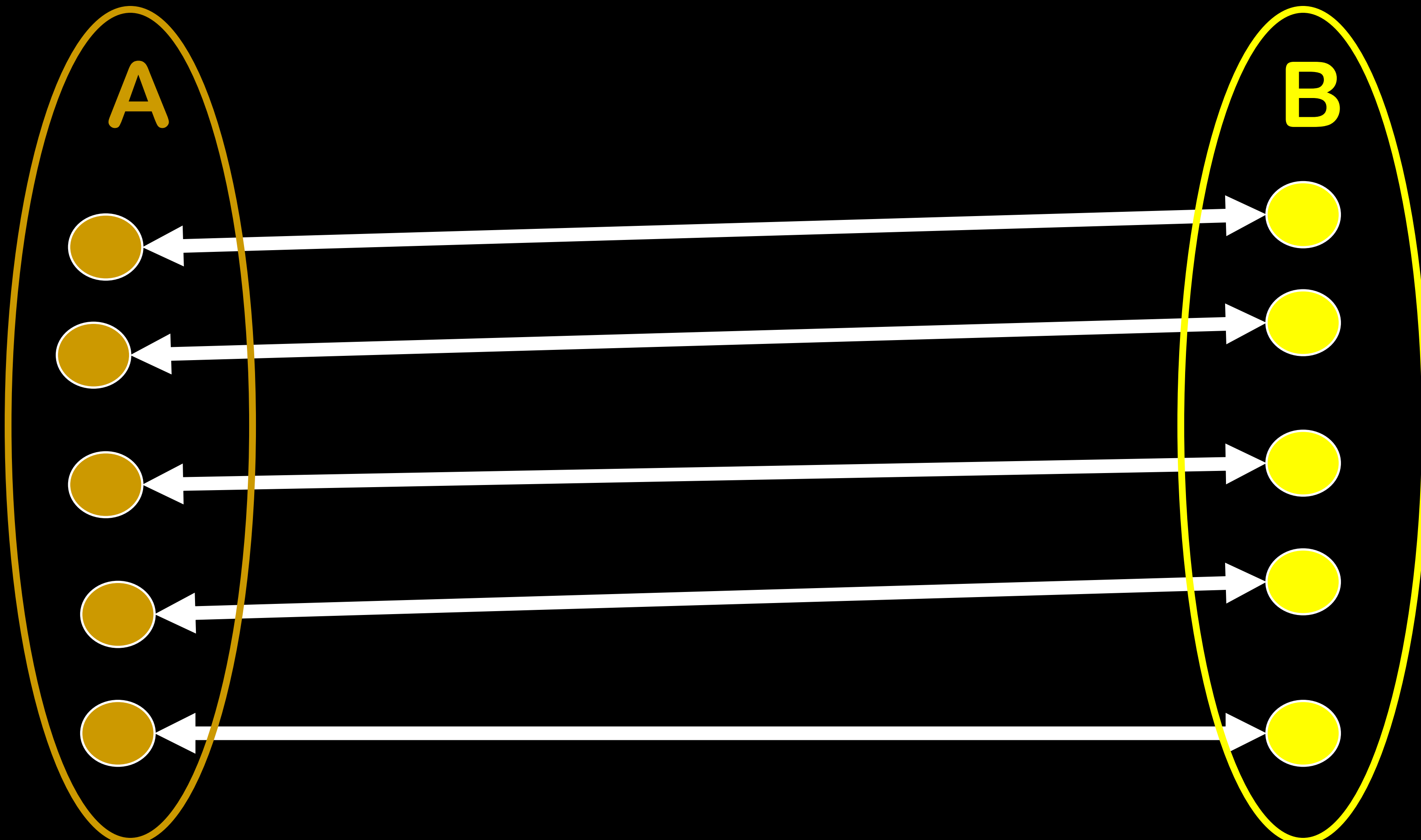
$\exists$ **onto f : A** $\rightarrow$ **B** $\Rightarrow$ **| A |** $\geq$ **| B |**



$\exists$ 1-1 onto $f : A \to B \Rightarrow |A| = |B|$

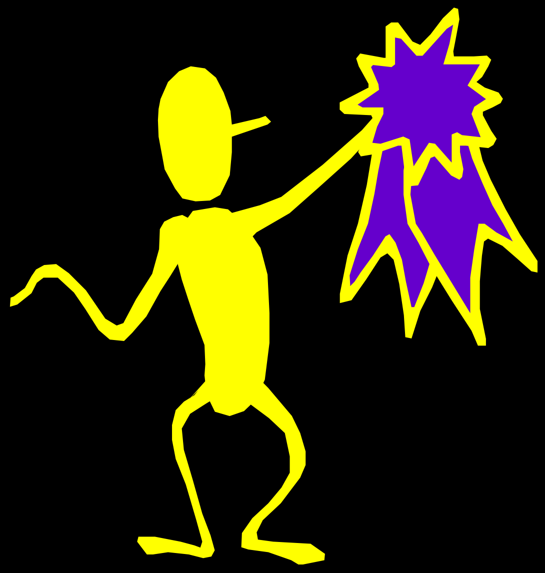
f being 1-1 onto means f^{-1} is well defined and unique

f is a way of pairing up elements



Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size



It's one of the most important mathematical ideas of all time!

Question: How many n-bit sequences are there?

000000	$\leftrightarrow$	0
000001	$\leftrightarrow$	1
000010	$\leftrightarrow$	2
000011	$\leftrightarrow$	3
:	:	:
111111	$\leftrightarrow$	2^n-1

Each sequence corresponds to a unique number from 0 to 2^n-1. Hence 2^n sequences.

A = { a,b,c,d,e } Has Many Subsets

{a}, {a,b}, {a,d,e}, {a,b,c,d,e},
{e}, Ø, …

The entire set and the empty set are subsets with all the rights and privileges pertaining thereto

Question: How Many Subsets Can Be Made From The Elements of a 5-Element Set?

a	b	c	d	e
0	1	1	0	1

{ b c e }

1 means "TAKE IT"
0 means "LEAVE IT"

Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

$A = \{a_1, a_2, a_3, \ldots, a_n\}$
B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b = b_1b_2b_3\ldots b_n$, let $f(b) = \{ a_i \mid b_i=1\}$

Claim: f is 1-1

Any two distinct binary sequences b and b′ have a position i at which they differ

Hence, f(b) is not equal to f(b′) because they disagree on element a_i

$A = \{a_1, a_2, a_3, \ldots, a_n\}$

B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

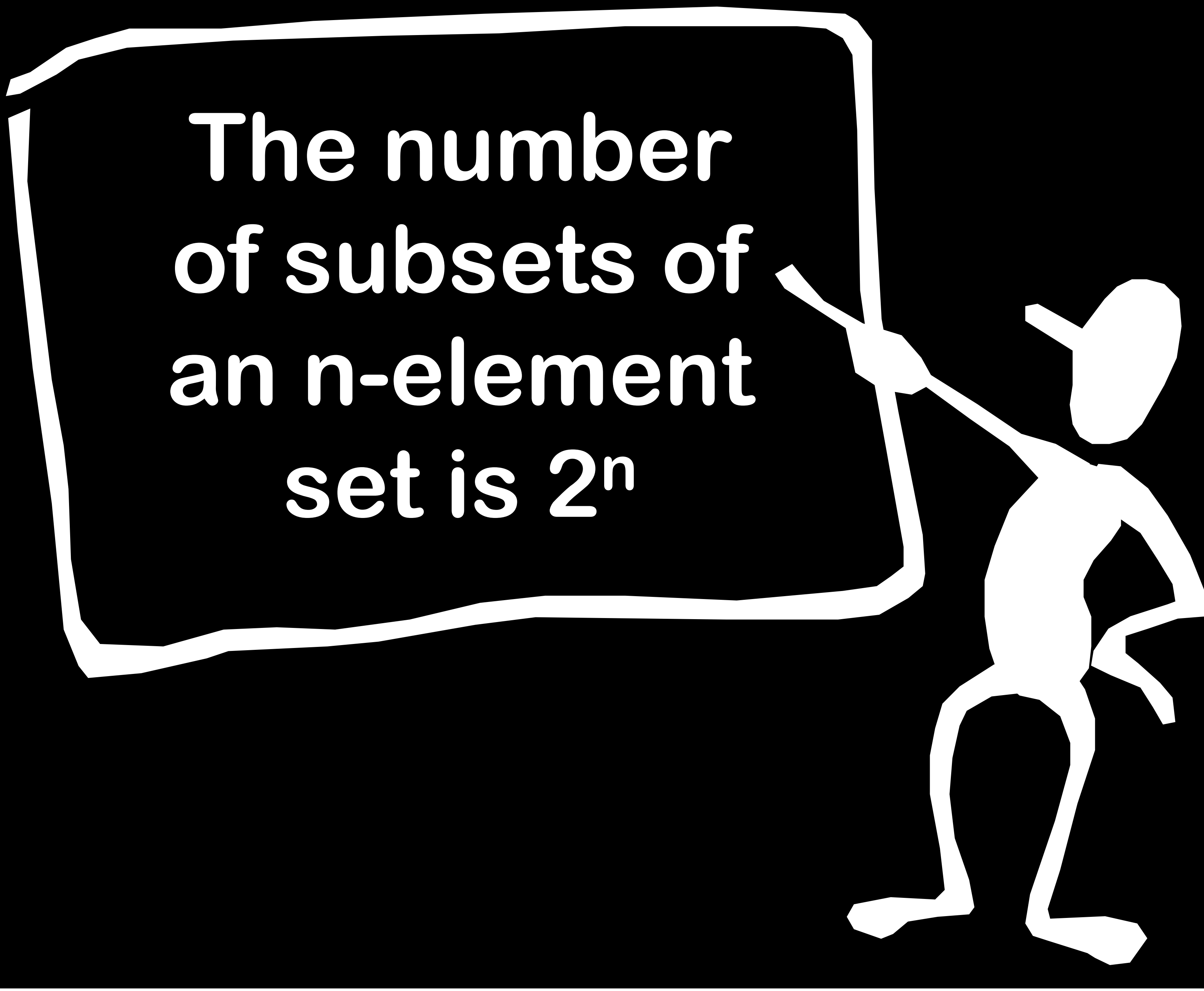
For bit string $b = b_1 b_2 b_3 \ldots b_n$, let $f(b) = \{ a_i \mid b_i = 1\}$

Claim: f is onto

Let S be a subset of $\{a_1, \ldots, a_n\}$.

Define $b_k = 1$ if a_k in S and $b_k = 0$ otherwise.

Note that $f(b_1 b_2 \ldots b_n) = S$.


The number of subsets of an n-element set is 2^n

Let $f : A \to B$ Be a Function From Set A to Set B

f is **1-1** if and only if

$\forall x,y \in A,\ x \neq y \Rightarrow f(x) \neq f(y)$

f is **onto** if and only if

$\forall z \in B\ \ \exists x \in A$ such that $f(x) = z$

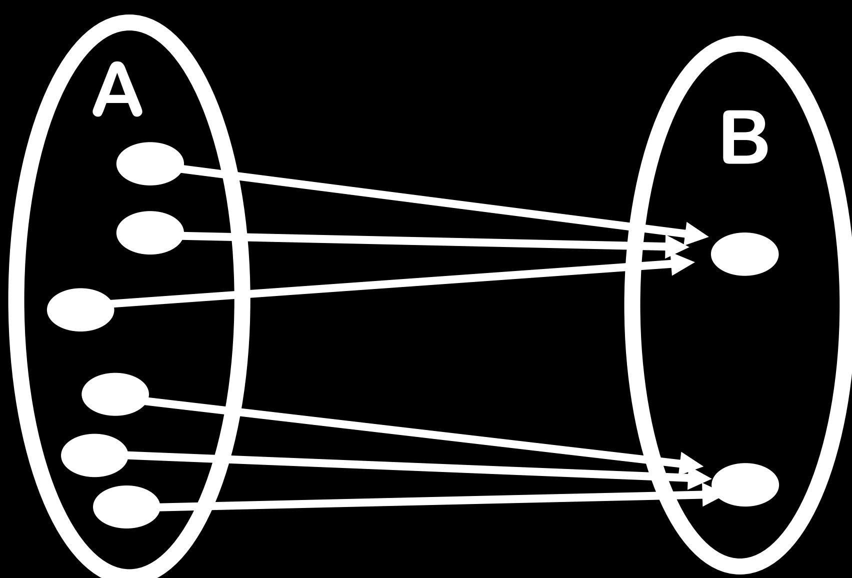
Let f : A → B Be a Function From Set A to Set B

f is a **1-to-1 correspondence** iff

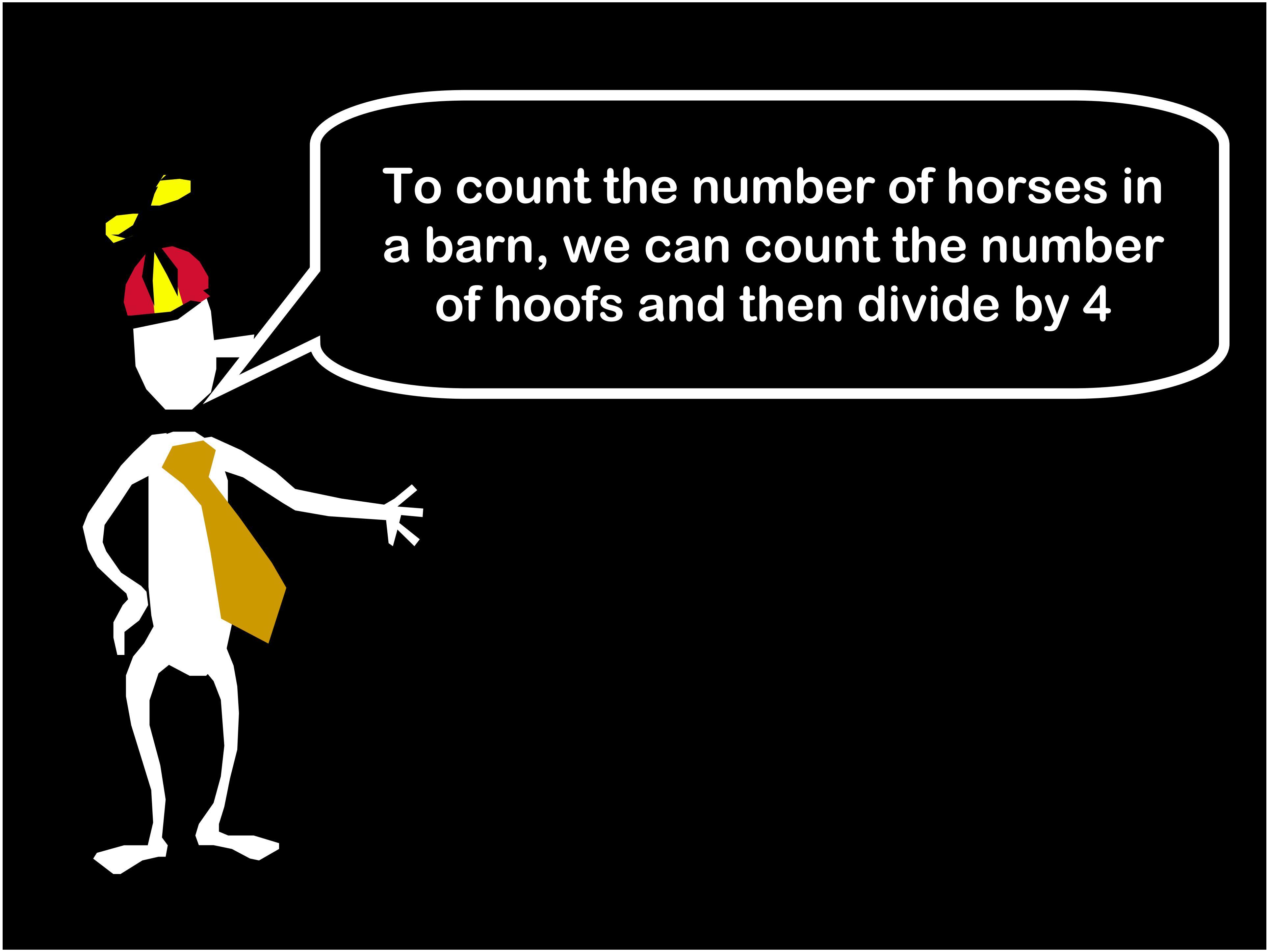
$\forall z \in B$ $\exists$ **exactly one** $x \in A$ such that $f(x) = z$

f is a **k-to-1 correspondence** iff

$\forall z \in B$ $\exists$ **exactly k** $x \in A$ such that $f(x) = z$



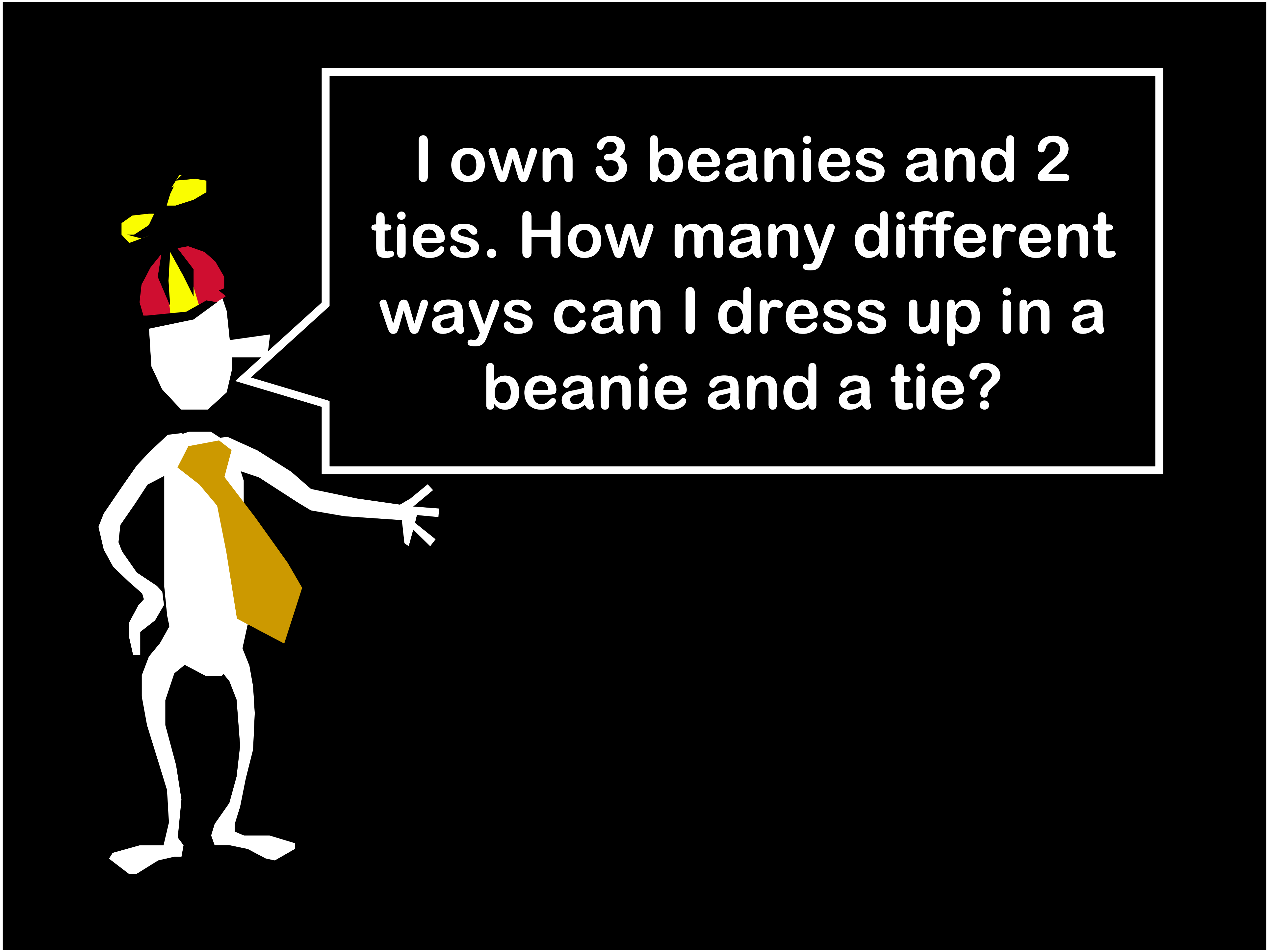
3 to 1 function

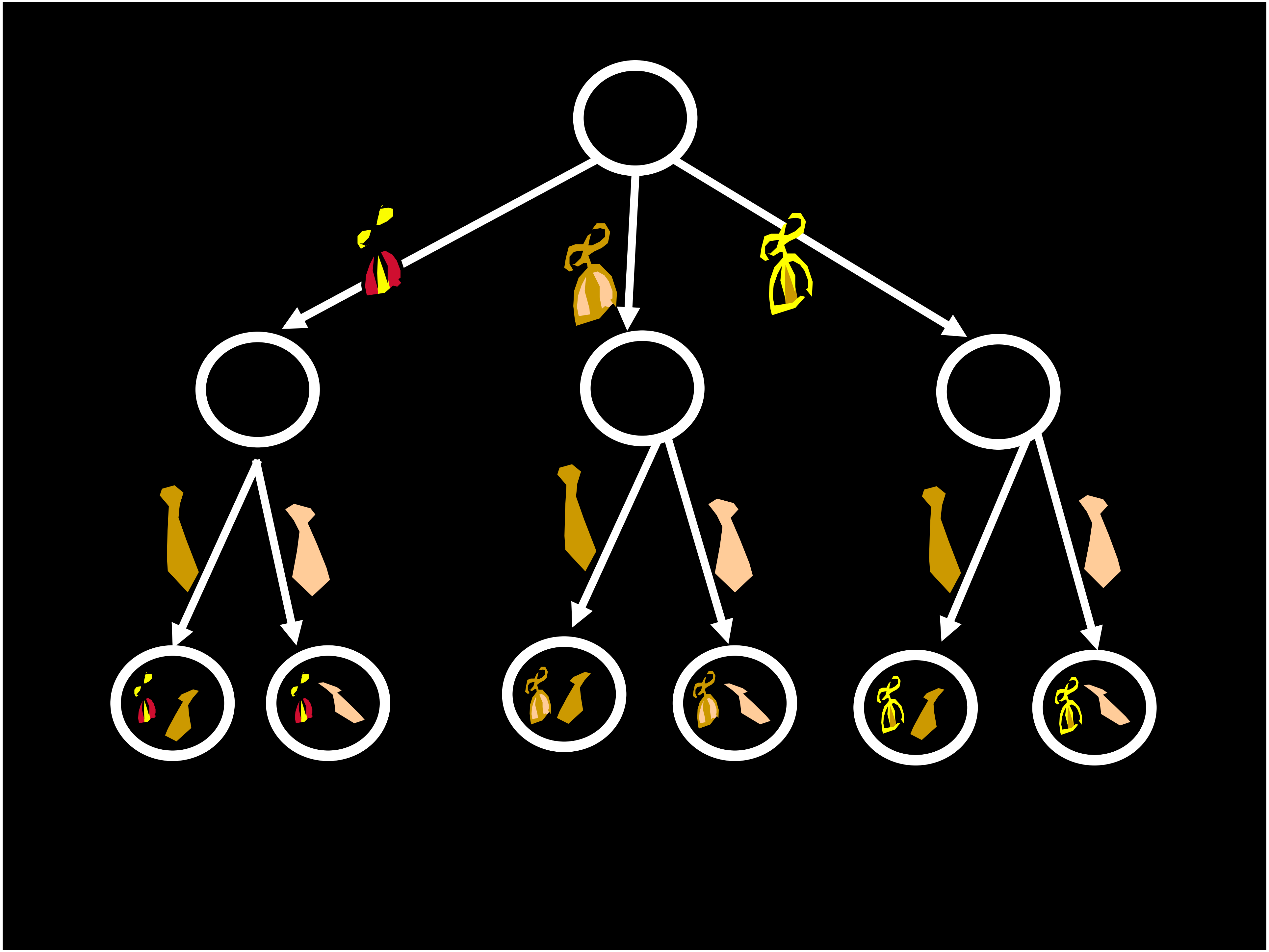

To count the number of horses in a barn, we can count the number of hoofs and then divide by 4

If a finite set A
has a k-to-1
correspondence
to finite set B,
then |B| = |A|/k

Count without Counting:
The auditorium can be partitioned into n rows with k seats each

Thus, we have nk seats in the room


I own 3 beanies and 2 ties. How many different ways can I dress up in a beanie and a tie?



A Restaurant Has a Menu With 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts

How many items on the menu?

$$5 + 6 + 3 + 7 = 21$$

How many ways to choose a complete meal?

$$5 \times 6 \times 3 \times 7 = 630$$

How many ways to order a meal if I am allowed to skip some (or all) of the courses?

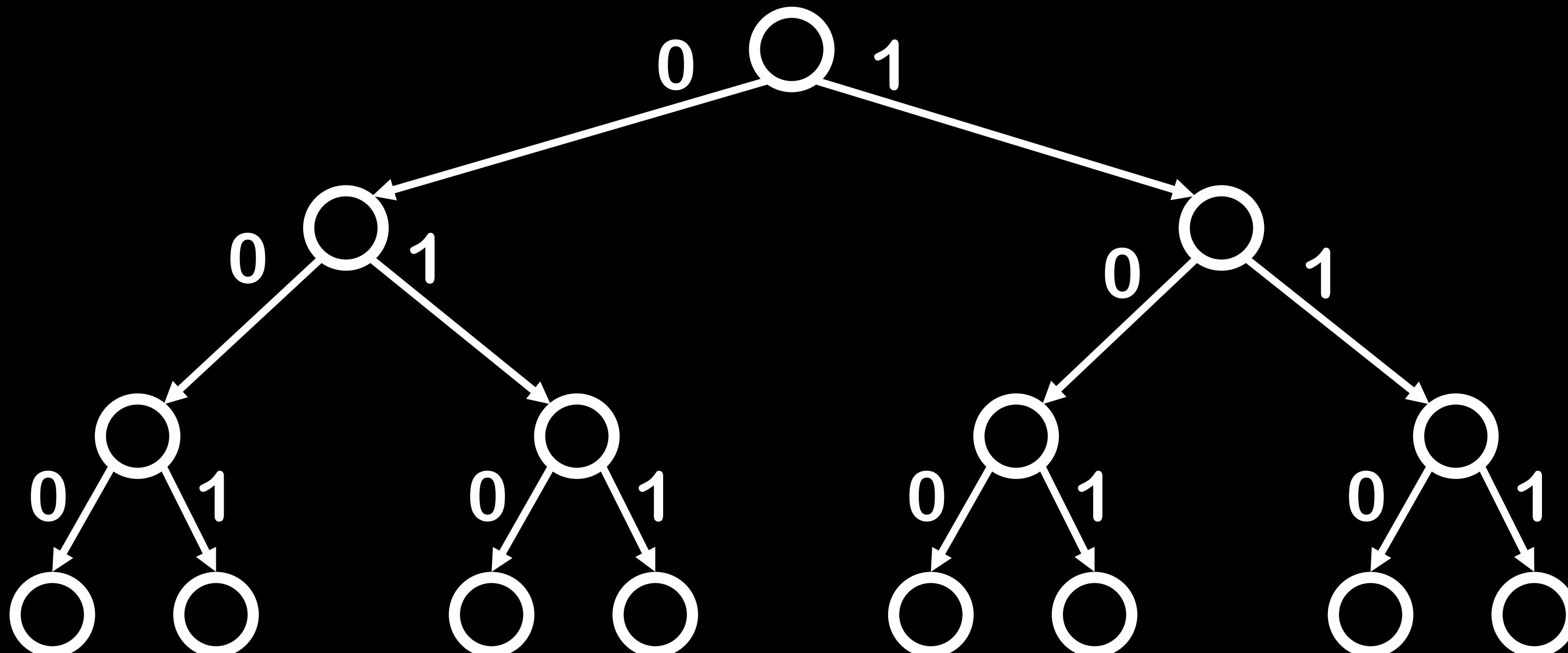
$$6 \times 7 \times 4 \times 8 = 1344$$

Hobson's Restaurant Has Only 1 Appetizer, 1 Entree, 1 Salad, and 1 Dessert

2^4 ways to order a meal if I might not have some of the courses

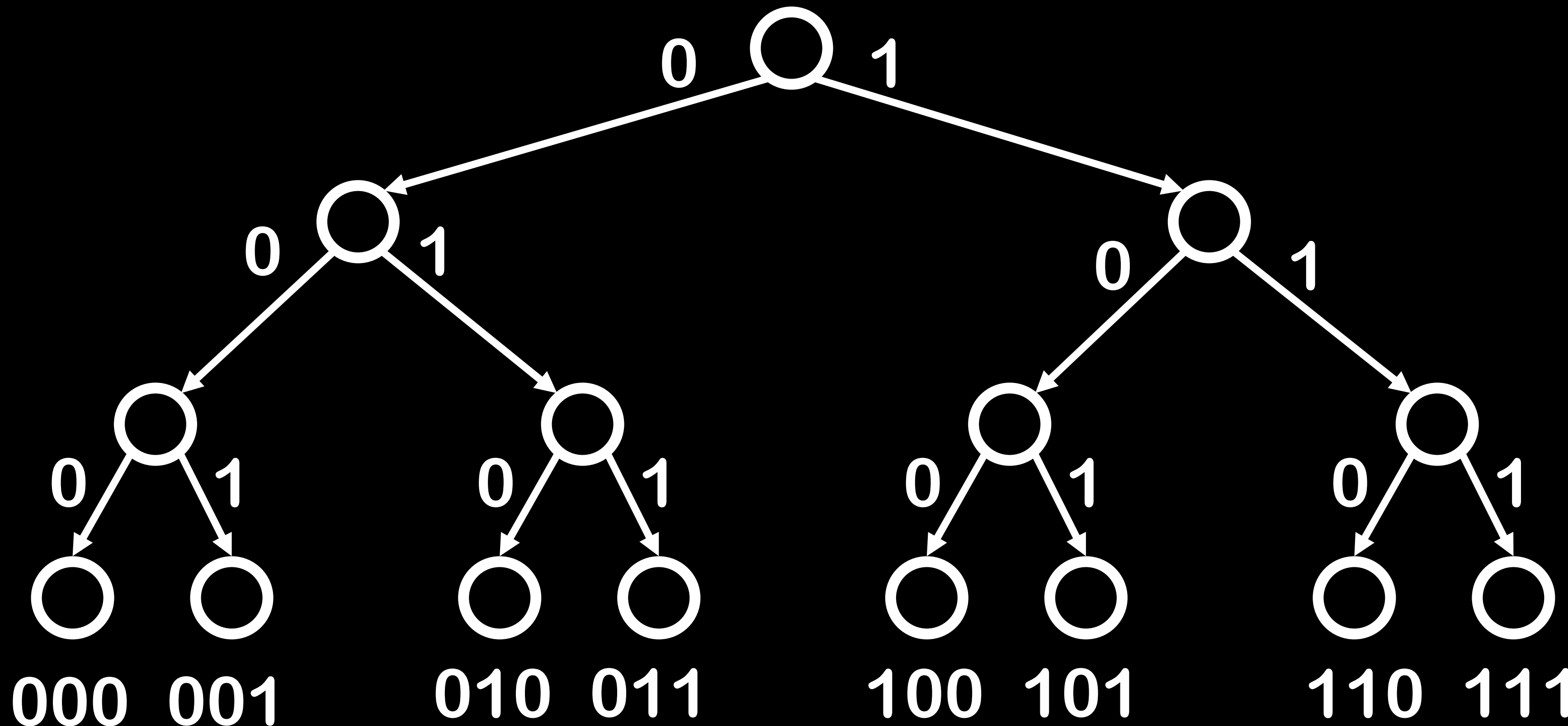
Same as number of subsets of the set
{Appetizer, Entrée, Salad, Dessert}

Choice Tree For 2^n n-bit Sequences

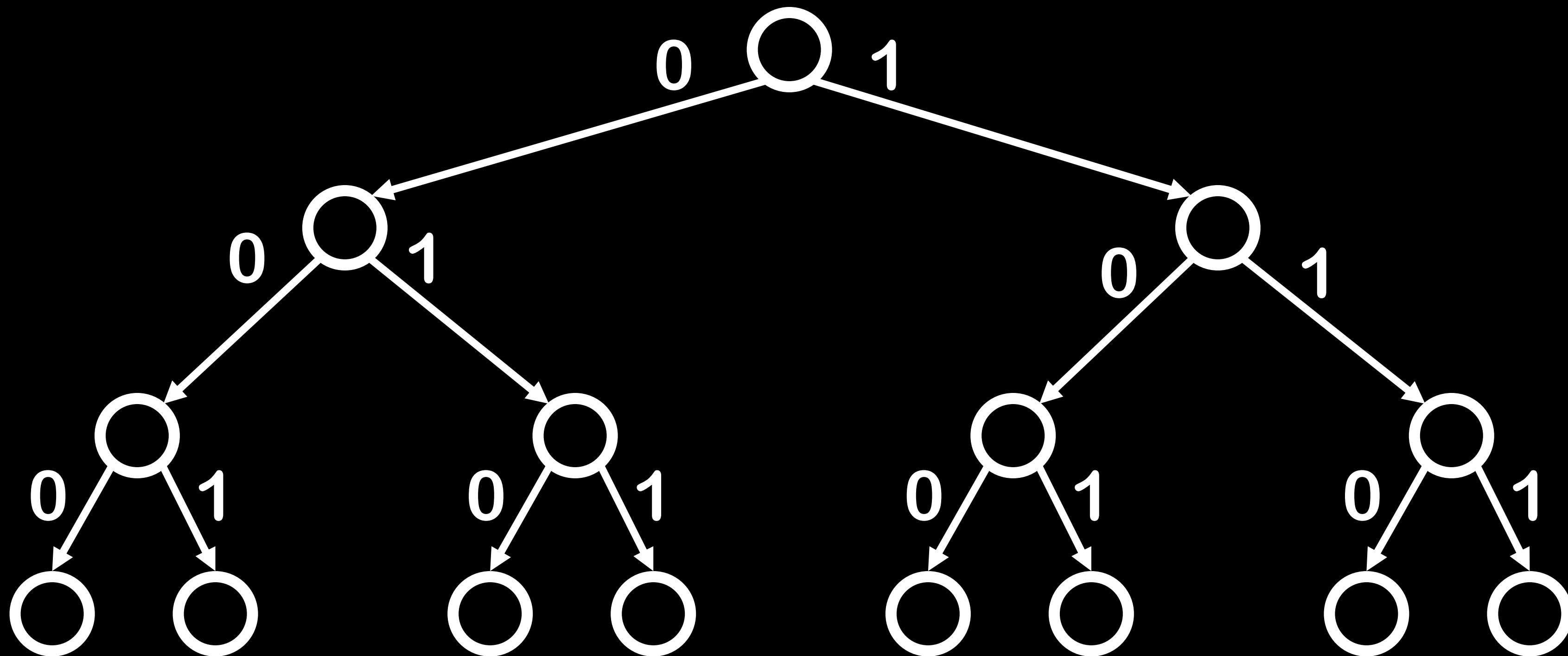


We can use a “choice tree” to represent the construction of objects of the desired type

Choice Tree For 2^n n-bit Sequences



Label each leaf with the object constructed by the choices along the path to the leaf



2 choices for first bit
× 2 choices for second bit
× 2 choices for third bit
:   :
× 2 choices for the n^{th}

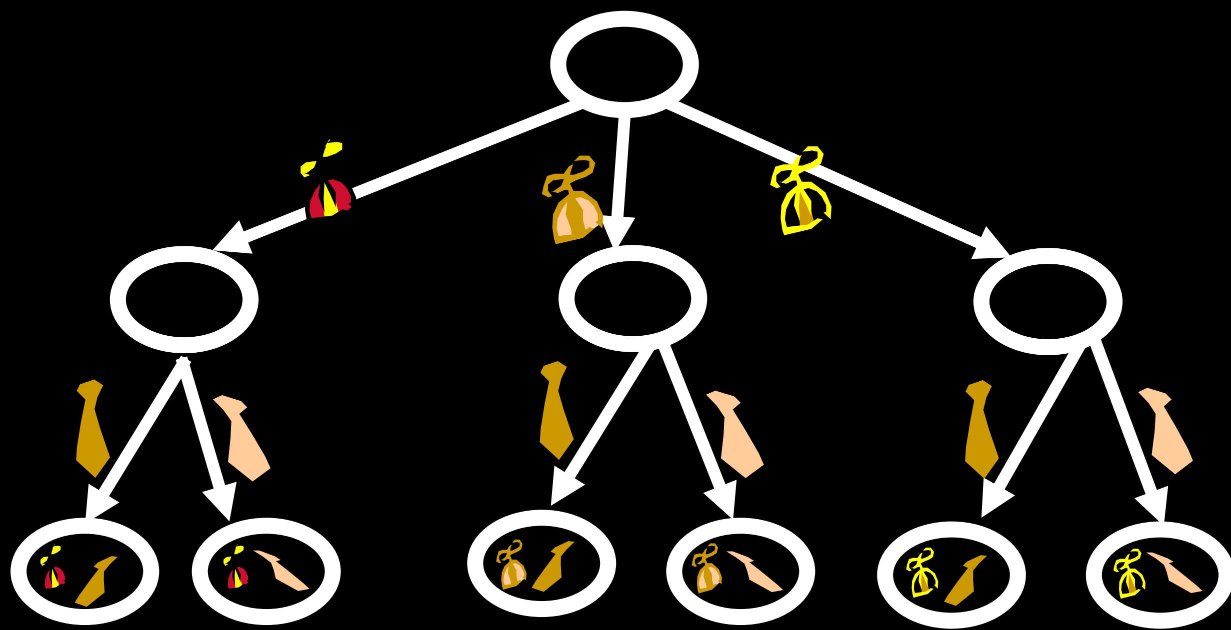
Leaf Counting Lemma

Let T be a depth-n tree when each node at depth $0 \leq i \leq n-1$ has P_{i+1} children

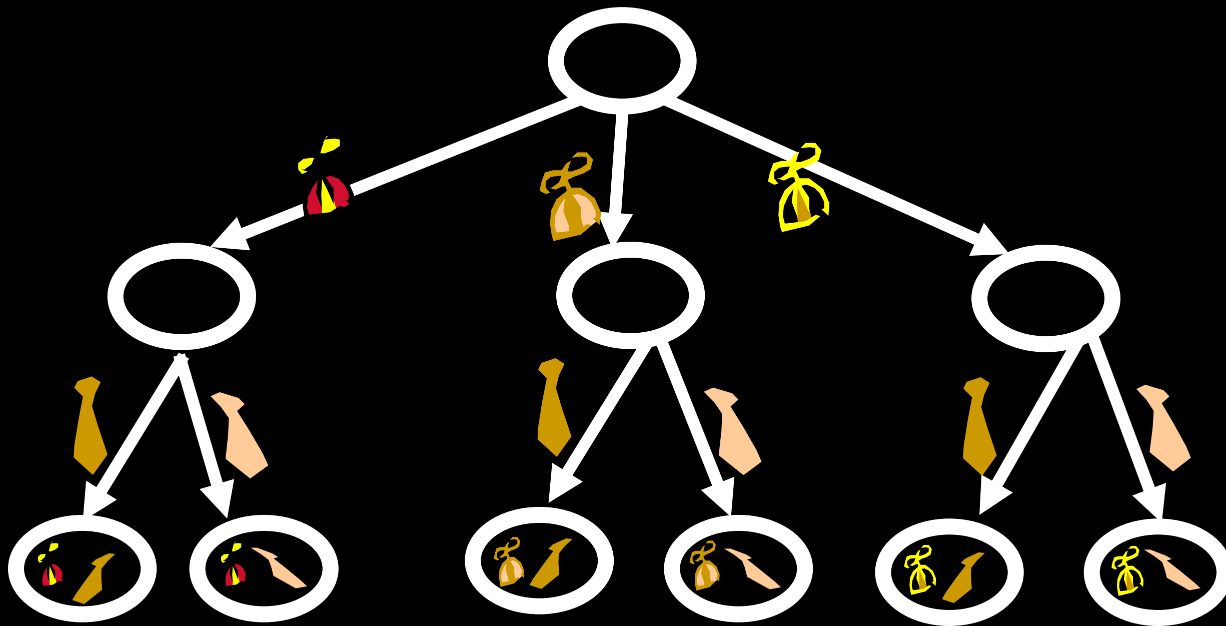
The number of leaves of T is given by:

$P_1P_2 \ldots P_n$

Choice Tree



A **choice tree** is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf



A choice tree provides a “choice tree representation” of a set S, if

1. Each leaf label is in S, and each element of S is some leaf label

2. No two leaf labels are the same


We will now combine the correspondence principle with the leaf counting lemma to make a powerful counting rule for choice tree representation.

Product Rule

IF set S has a choice tree representation with
P_1 possibilities for the first choice,
P_2 for the second, P_3 for the third,
and so on,

THEN

there are $P_1P_2P_3 \ldots P_n$ objects in S

Proof:

There are $P_1P_2P_3 \ldots P_n$ leaves of the choice tree which are in 1-1 onto correspondence with the elements of S.

Product Rule (Rephrased)

Suppose **every** object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

and

every object in S must be created

THEN

There are $P_1P_2P_3 \ldots P_n$ objects of type S

How Many Different Orderings of Deck With 52 Cards?

What object are we making? Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

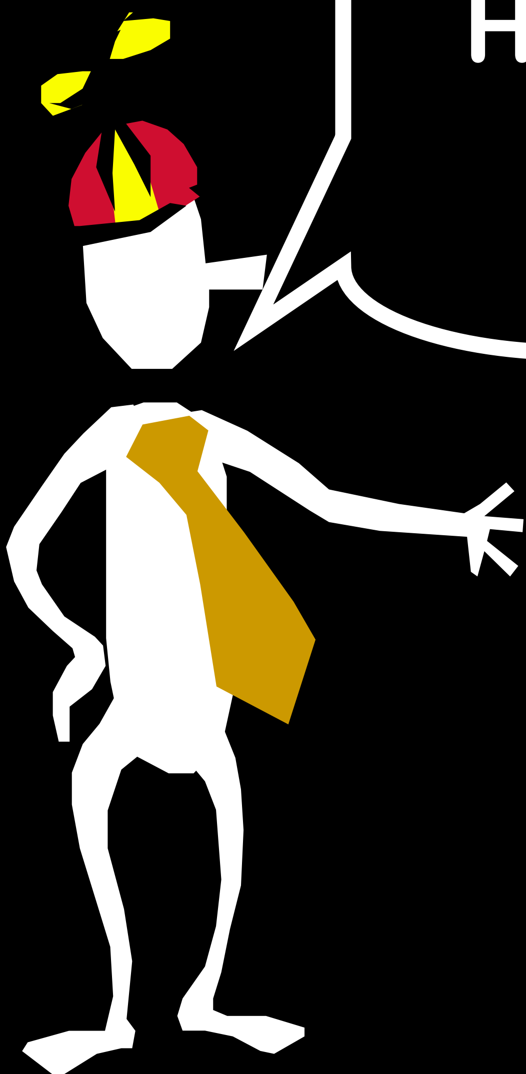
: :

1 possible choice for the 52^{nd} card.

By product rule: $52 \times 51 \times 50 \times \ldots \times 2 \times 1 = 52!$

A **permutation** or **arrangement** of n objects is an ordering of the objects

The number of permutations of n distinct objects is n!



How many sequences of 7 letters are there?

26^7

(26 choices for each of the 7 positions)

How many sequences of 7 letters contain **at least** two of the same letter?

$$\underbrace{26^7 - 26\times25\times24\times23\times22\times21\times20}$$

number of sequences containing all different letters

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.

Helpful Advice:

In logic, it can be useful to represent a statement in the contra positive.

In counting, it can be useful to represent a set in terms of its complement.

If 10 horses race, how many orderings of the top three finishers are there?

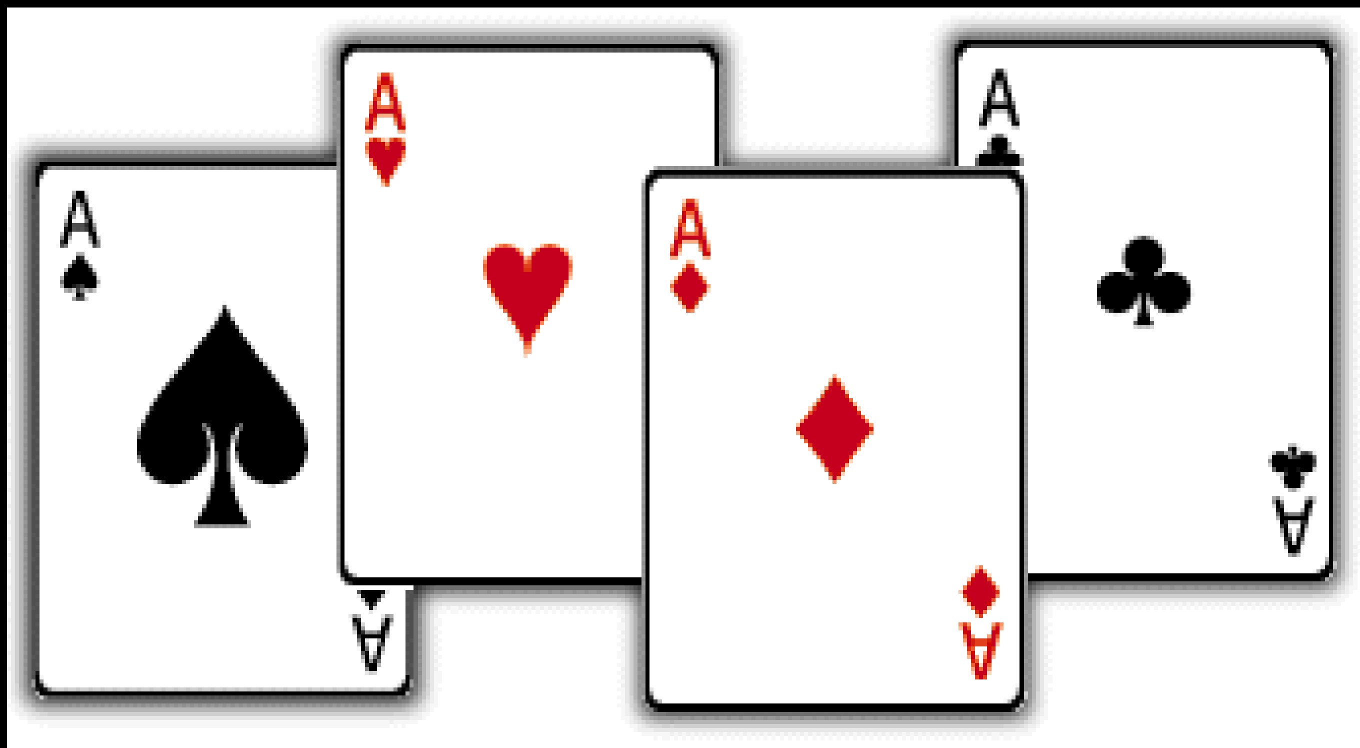
$$10 \times 9 \times 8 = 720$$

Number of ways of ordering, permuting, or arranging r out of n objects

n choices for first place, n-1 choices for second place, . . .

$$n \times (n-1) \times (n-2) \times \ldots \times (n-(r-1))$$

$$= \frac{n!}{(n-r)!}$$



Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52×51

How many unordered pairs?

$52 \times 51 / 2$ ← divide by overcount

Each unordered pair is listed twice on a list of the ordered pairs

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52×51

How many unordered pairs?

$52 \times 51 / 2$ ← divide by overcount

We have a 2-1 map from ordered pairs to unordered pairs.

Hence #unordered pairs = (#ordered pairs)/2

Ordered Versus Unordered

How many ordered 5 card sequences can be formed from a 52-card deck?

$52 \times 51 \times 50 \times 49 \times 48$

How many orderings of 5 cards?

$5!$

How many unordered 5 card hands?

$(52\times51\times50\times49\times48)/5! = 2{,}598{,}960$

A **combination** or **choice** of r out of n objects is an (unordered) set of r of the n objects

The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

n "choose" r

The number of subsets of size r that can be formed from an n-element set is:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Product Rule (Rephrased)

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

THEN

There are $P_1 P_2 P_3 \ldots P_n$ objects of type S

How Many 8-Bit Sequences Have 2 0's and 6 1's?

Tempting, but incorrect:

8 ways to place first 0, times
7 ways to place second 0

Violates condition 2 of product rule!

Choosing position i for the first 0 and then position j for the second 0 gives same sequence as choosing position j for the first 0 and position i for the second 0 } 2 ways of generating same object!

How Many 8-Bit Sequences Have 2 0's and 6 1's?

1. Choose the set of 2 positions to put the 0's. The 1's are forced.

$$\binom{8}{2}$$

2. Choose the set of 6 positions to put the 1's. The 0's are forced.

$$\binom{8}{6}$$

Symmetry In The Formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

"# of ways to pick r out of n elements"

=

"# of ways to choose the (n-r) elements to omit"

How Many Hands Have at Least 3 As?

4560

~~$\binom{4}{3}\binom{48}{2}$~~

4704

$\binom{4}{3}\binom{49}{2}$

$4 \cdot \frac{49 \cdot 48}{2} = 4704$

2496

How Many Hands Have at Least 3 As?

$\binom{4}{3}$ = 4 ways of picking 3 out of 4 aces

$\binom{49}{2}$ = 1176 ways of picking 2 cards out of the remaining 49 cards

$$4 \times 1176 = 4704$$

How Many Hands Have at Least 3 As?

How many hands have **exactly** 3 aces?

$\binom{4}{3}$ = 4 ways of picking 3 out of 4 aces

$\binom{48}{2}$ = 1128 ways of picking 2 cards out of the 48 non-ace cards

$$\begin{array}{r} 4 \\ \times\ 1128 \\ \hline 4512 \end{array}$$

How many hands have **exactly** 4 aces?

$\binom{4}{4}$ = 1 way of picking 4 out of 4 aces

$\binom{48}{1}$ = 48 ways of picking 1 cards out of the 48 non-ace cards

$$\begin{array}{r} 4512 \\ +\ 48 \\ \hline 4560 \end{array}$$

$4704 \neq 4560$

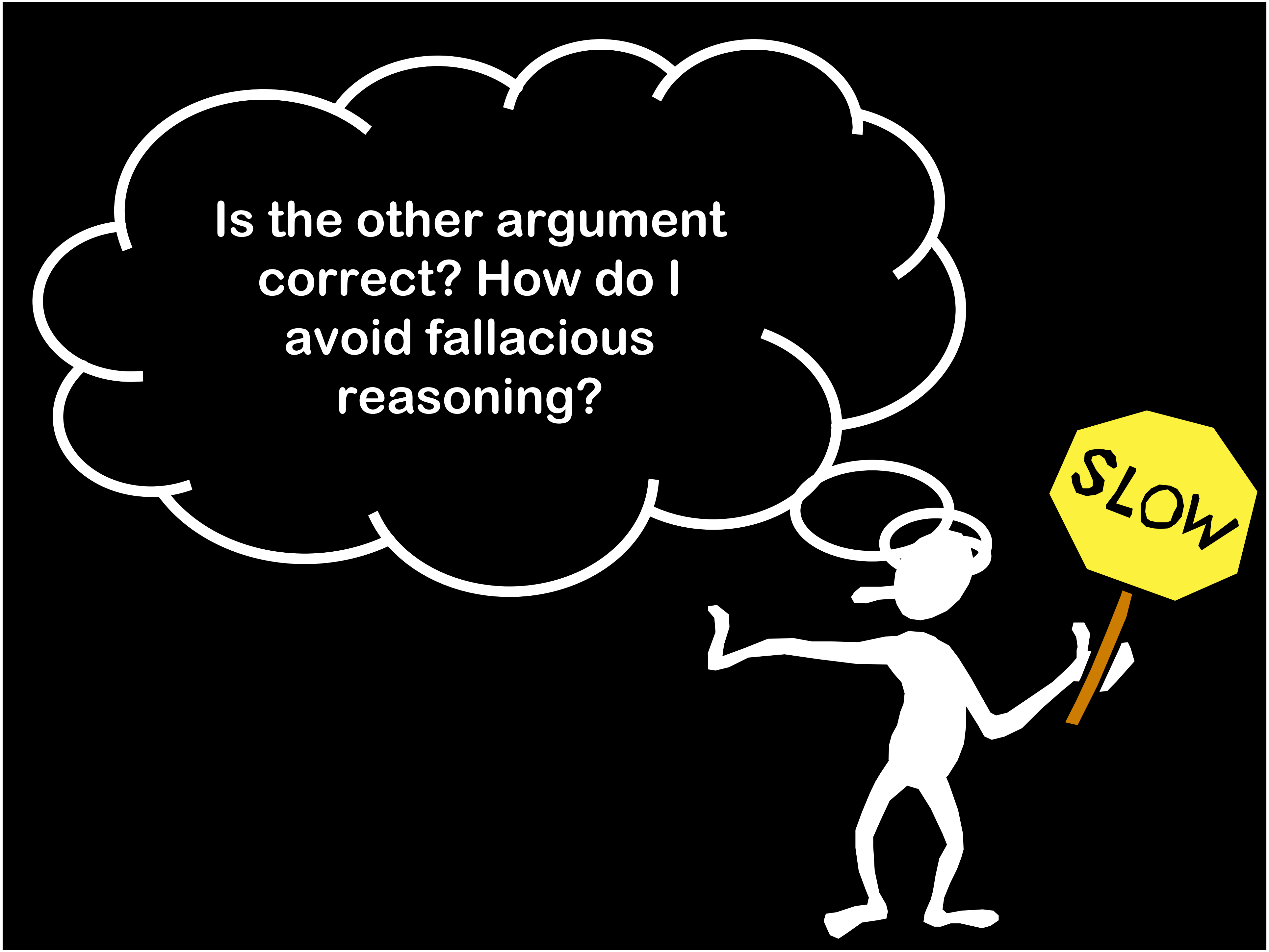
At least one of the two counting arguments is not correct!

Four Different Sequences of Choices Produce the Same Hand

$\binom{4}{3}$ = 4 ways of picking 3 out of 4 aces

$\binom{49}{2}$ = 1176 ways of picking 2 cards out of the remaining 49 cards

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦


Is the other argument correct? How do I avoid fallacious reasoning?
SLOW

REVERSIBILTY CHECK:

For each object can I reverse engineer the unique sequence of choices that constructed it?

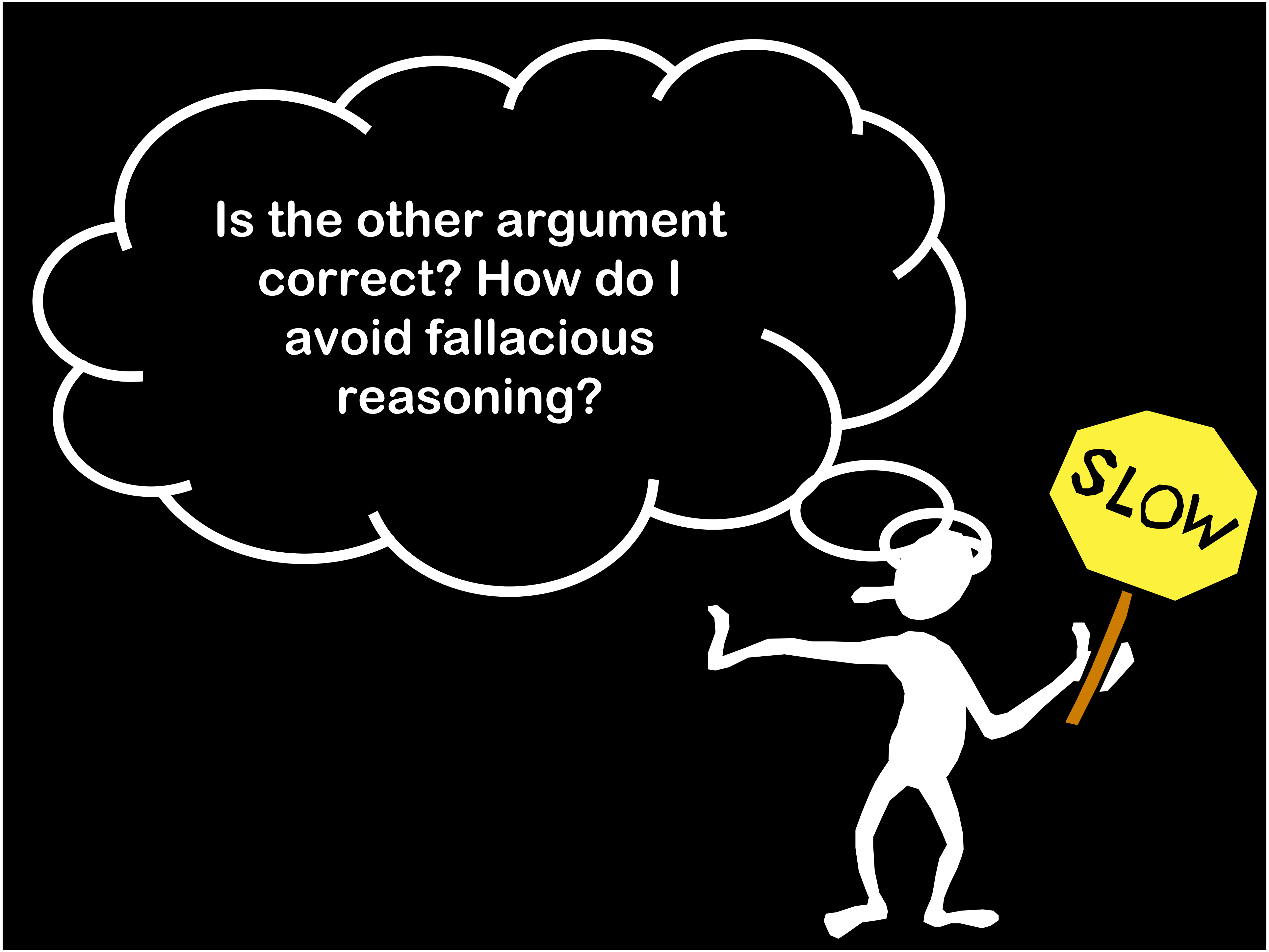
Scheme I

1. Choose 3 of 4 aces
2. Choose 2 of the remaining cards

A♣ A♦ A♥ A♠ K♦

For this hard – you can't reverse to a unique choice sequence.

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A ♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦


Is the other argument correct? How do I avoid fallacious reasoning?
SLOW

Scheme II

1. Choose 3 out of 4 aces
2. Choose 2 out of 48 non-ace cards

A♣ A♦ Q♦ A♠ K♦

REVERSE TEST: Aces came from choices in (1) and others came from choices in (2)

Scheme II

1. Choose 4 out of 4 aces
2. Choose 1 out of 48 non-ace cards

A♣ A♦ A♥ A♠ K♦

REVERSE TEST: Aces came from choices in (1) and others came from choices in (2)

Product Rule (Rephrased)

Suppose **every** object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

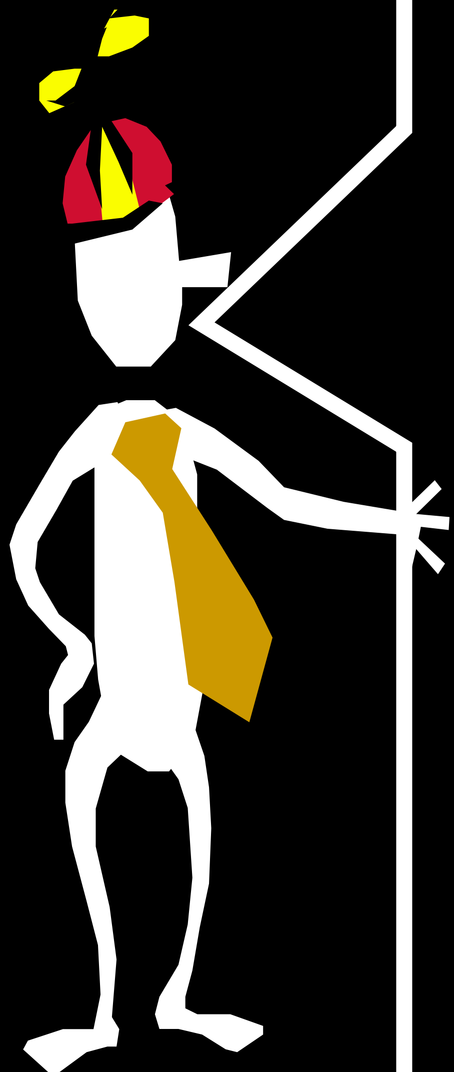
IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

THEN

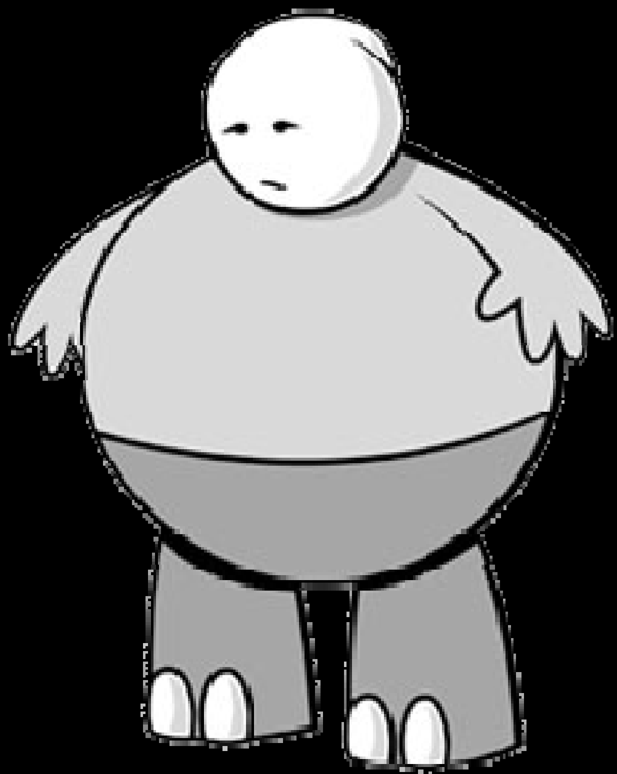
There are $P_1 P_2 P_3 \ldots P_n$ objects of type S



DEFENSIVE THINKING
ask yourself:

Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?



Here's What You Need to Know…

Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size

Choice Tree

Product Rule

two conditions

Reverse Test

Counting by complementing

Binomial coefficient