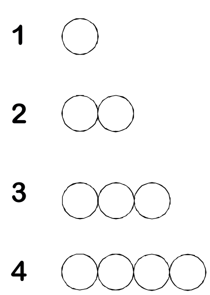
15-251

Great Theoretical Ideas in Computer Science

Ancient Wisdom: Unary and Binary

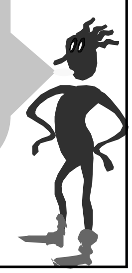
Lecture 5 (September 11, 2007)

Prehistoric Unary



Consider the problem of finding a formula for the sum of the first n numbers

You already used induction to verify that the answer is $\frac{1}{2}n(n+1)$



$1 + 2 + 3 + \ldots + n\text{-}1 + n = S$

$n + n\text{-}1 + n\text{-}2 + \ldots + 2 + 1 = S$

$n{+}1 + n{+}1 + n{+}1 + \ldots + n{+}1 + n{+}1 = 2S$

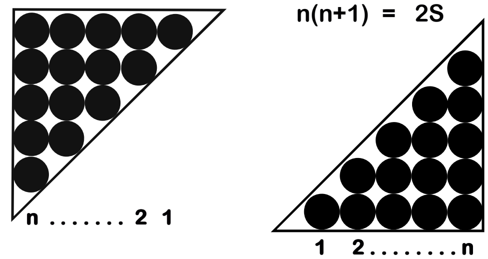
$n(n+1) = 2S$

$$S = \frac{n(n+1)}{2}$$

$1 + 2 + 3 + \ldots + n\text{-}1 + n = S$

$n + n\text{-}1 + n\text{-}2 + \ldots + 2 + 1 = S$

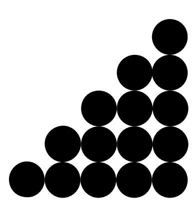
$n(n+1) = 2S$



n^{th} Triangular Number

$\Delta_n = 1 + 2 + 3 + \ldots + n\text{-}1 + n$

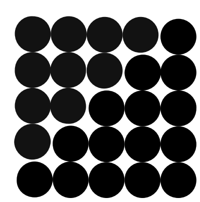
$= n(n+1)/2$

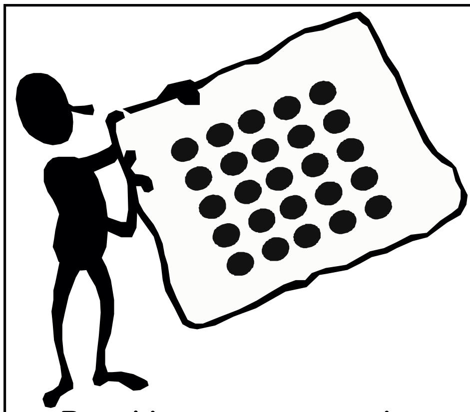


n^{th} Square Number

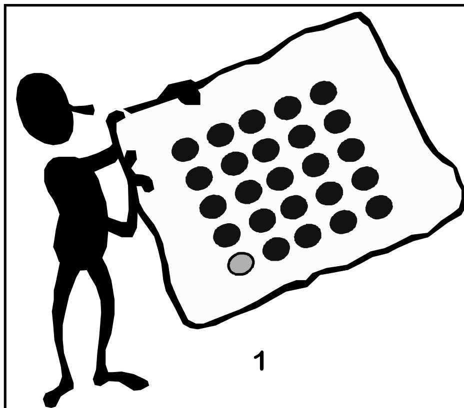
$n = n^2$

$= \Delta_n + \Delta_{n-1}$





Breaking a square up in a new way



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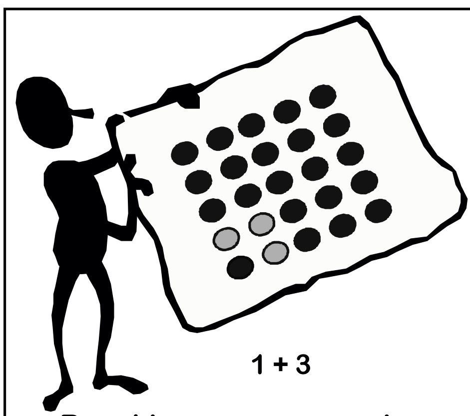
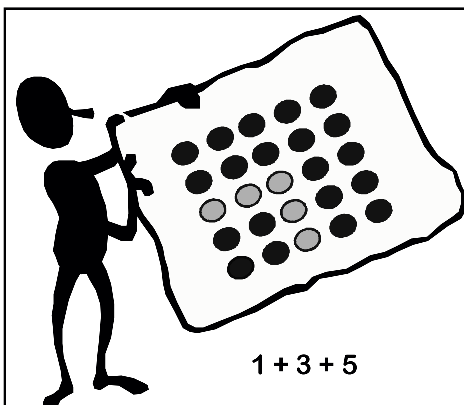
Breaking a square up in a new way



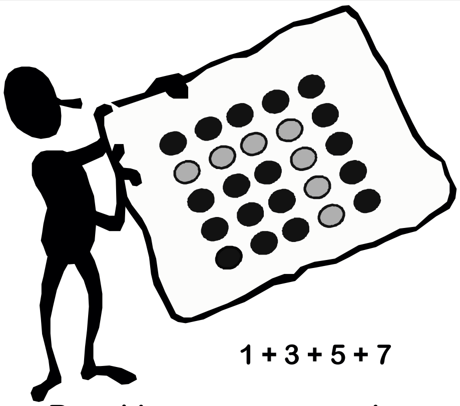
1 + 3

Breaking a square up in a new way



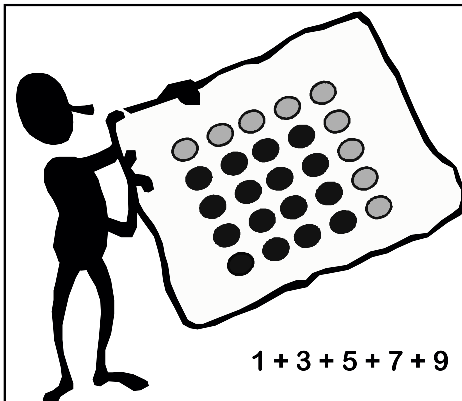
1 + 3 + 5

Breaking a square up in a new way



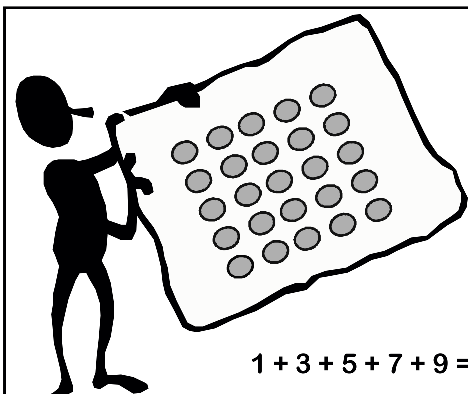
$1 + 3 + 5 + 7$

Breaking a square up in a new way



$1 + 3 + 5 + 7 + 9$

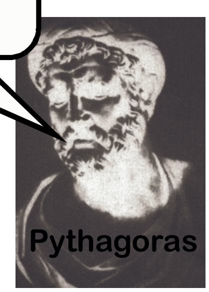
Breaking a square up in a new way

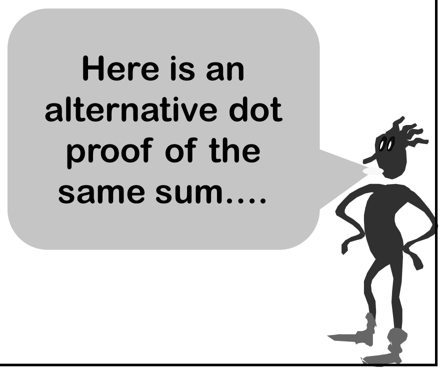


$1 + 3 + 5 + 7 + 9 = 5^2$

Breaking a square up in a new way

The sum of the first n odd numbers is n^2

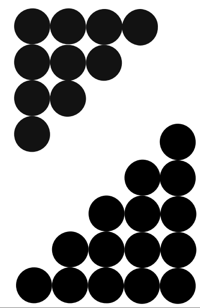




n^{th} Square Number

$n = \Delta_n + \Delta_{n-1}$

$= n^2$



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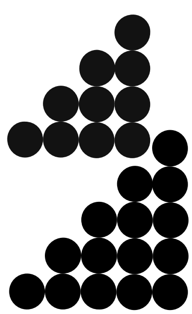

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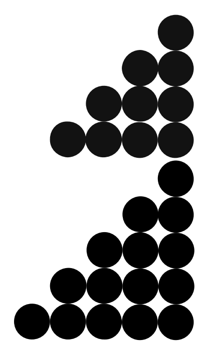
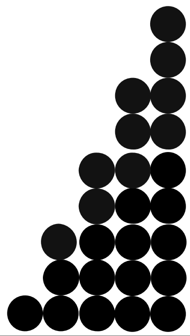


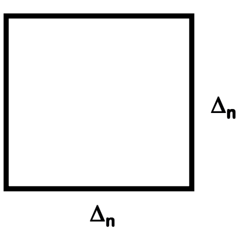
n^{th} Square Number

$n = \Delta_n + \Delta_{n-1}$

= Sum of first n odd numbers



Area of square $= (\Delta_n)^2$



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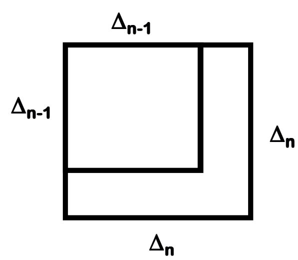

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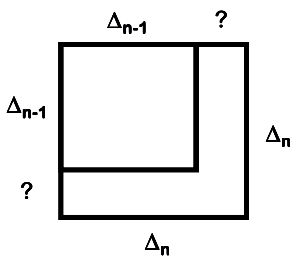

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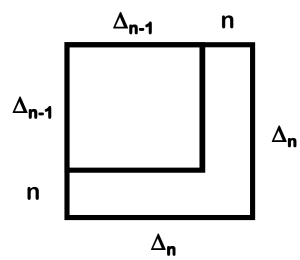
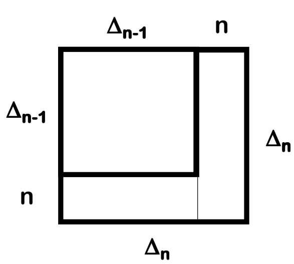


Area of square $= (\Delta_n)^2$



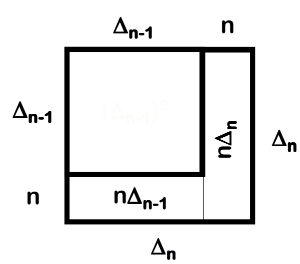
Area of square $= (\Delta_n)^2$

$= (\Delta_{n-1})^2 + n\Delta_{n-1} + n\Delta_n$

$= (\Delta_{n-1})^2 + n(\Delta_{n-1} + \Delta_n)$

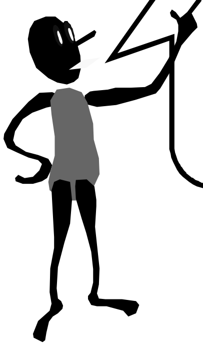
$= (\Delta_{n-1})^2 + n(\quad_n)$

$= (\Delta_{n-1})^2 + n^3$



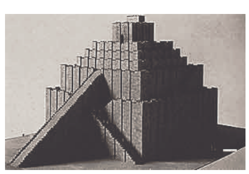
$$(\Delta_n)^2 = n^3 + (\Delta_{n-1})^2$$
$$= n^3 + (n-1)^3 + (\Delta_{n-2})^2$$
$$= n^3 + (n-1)^3 + (n-2)^3 + (\Delta_{n-3})^2$$
$$= n^3 + (n-1)^3 + (n-2)^3 + \ldots + 1^3$$

$$(\Delta_n)^2 = 1^3 + 2^3 + 3^3 + \ldots + n^3$$
$$= [\ n(n+1)/2\ ]^2$$



Can you find a formula for the sum of the first n squares?

Babylonians needed this sum to compute the number of blocks in their pyramids



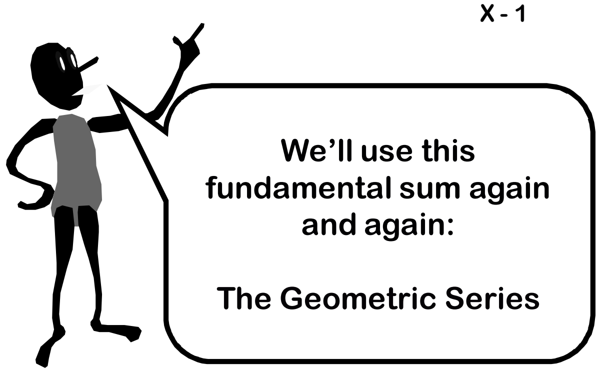
Rhind Papyrus

Scribe Ahmes was Martin Gardener of his day!

A man has 7 houses,
Each house contains 7 cats,
Each cat has killed 7 mice,
Each mouse had eaten 7 ears of spelt,
Each ear had 7 grains on it.
What is the total of all of these?

Sum of powers of 7

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$



A Frequently Arising Calculation

$$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1})$$

$$= \quad X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n$$

$$- \; 1 - X^1 - X^2 - X^3 - \ldots - X^{n-2} - X^{n-1}$$

$$= X^n - 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

(when $x \neq 1$)

Geometric Series for X=2

$$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

(when $x \neq 1$)

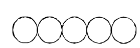
BASE X Representation

$S = a_{n-1}\, a_{n-2} \ldots a_1\, a_0$ represents the number:

$$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$$

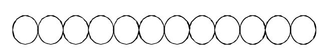
Base 2 [Binary Notation]

101 represents: $1\,(2)^2 + 0\,(2^1) + 1\,(2^0)$

= 

Base 7

015 represents: $0\,(7)^2 + 1\,(7^1) + 5\,(7^0)$

= 

Bases In Different Cultures

Sumerian-Babylonian: 10, 60, 360
Egyptians: 3, 7, 10, 60
Maya: 20
Africans: 5, 10
French: 10, 20
English: 10, 12, 20

BASE X Representation

$S = (a_{n-1}\ a_{n-2} \ldots a_1\ a_0)_X$ represents the number:

$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$

Largest number representable in base-X with n "digits"

$= (X\text{-}1\ X\text{-}1\ X\text{-}1\ X\text{-}1\ X\text{-}1 \ldots X\text{-}1)_X$

$= (X-1)(X^{n-1} + X^{n-2} + \ldots + X^0)$

$= (X^n - 1)$

Fundamental Theorem For Binary

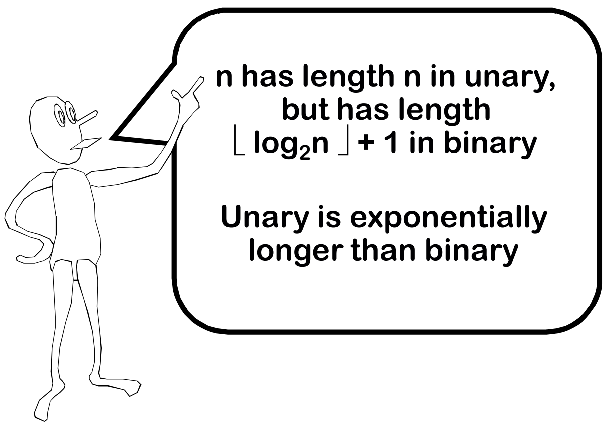
Each of the numbers from 0 to 2^{n-1} is uniquely represented by an n-bit number in binary

k uses $\lfloor \log_2 k \rfloor + 1$ digits in base 2

Fundamental Theorem For Base-X

Each of the numbers from 0 to X^{n-1} is uniquely represented by an n-"digit" number in base X

k uses $\lfloor \log_X k \rfloor + 1$ digits in base X



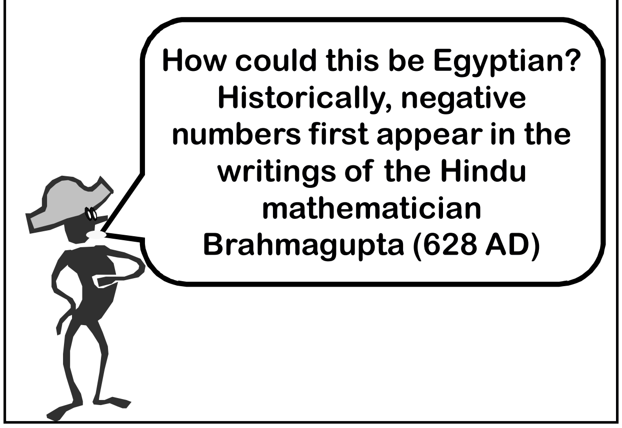
Other Representations: Egyptian Base 3

Conventional Base 3:
Each digit can be 0, 1, or 2

Here is a strange new one:
Egyptian Base 3 uses -1, 0, 1

Example: 1 -1 -1 = 9 - 3 - 1 = 5

We can prove a unique representation theorem



One weight for each power of 3
Left = “negative”. Right = “positive”

Here's What You Need to Know…

Unary and Binary

Triangular Numbers

Dot proofs

$(1+x+x^2 + \ldots + x^{n-1}) = (x^n - 1)/(x-1)$

Base-X representations

k uses $\lfloor \log_2 k \rfloor + 1 = \lceil \log_2 (k+1) \rceil$ digits in base 2