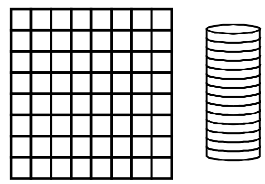
15-251

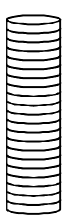
Great Theoretical Ideas in Computer Science

Combinatorial Games

Lecture 2 (August 30, 2007)



A Take-Away Game



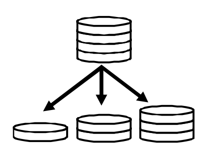
21 chips

- Two Players: I and II
- A move consists of removing one, two, or three chips from the pile
- Players alternate moves, with Player I starting
- The player that removes the last chip wins

Which player would you rather be?

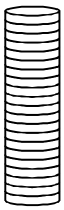
Try Small Examples!

If there were 1, 2, or 3 only, player who moves next wins



If there are 4 chips left, player who moves next must leave 1, 2 or 3 chips left, and his opponent will be able to win

If there are 5, 6 or 7 chips left, the player who moves next can win by moving to the position with four chips left



21 chips

0, 4, 8, 12, 16, … are target positions; if a player moves to that position, they can win the game

Therefore, with 21 chips, Player I can win!

Combinatorial Games

There are two players

There is a finite set of possible positions

The rules of the game specify for both players and each position which moves to other positions are legal moves.

The players alternate moving

The game ends in a finite number of moves (no draws!)

Normal Versus Misère

Normal Play Rule: The last player to move wins

Misère Play Rule: The last player to move loses

A Terminal Position is one where neither player can move anymore

What is Omitted

No random moves

(This rules out games like poker)

No hidden moves

(This rules out games like battleship)

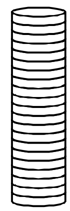
No draws in a finite number of moves

(This rules out tic-tac-toe)

P-Positions and N-Positions

P-Position: Positions that are winning for the Previous player (the player who just moved)

N-Position: Positions that are winning for the Next player (the player who is about to move)



21 chips

0, 4, 8, 12, 16, … are P-positions; if a player moves to that position, they can win the game

21 chips is an N-position

What's a P-Position?

"Positions that are winning for the Previous player (the player who just moved)"

That means:

For any move that N makes
    There exists a move for P such that
        For any move that N makes
            There exists a move for P such that

⋮

                There exists a move for P such that
                    There are no possible moves for N

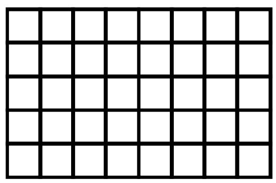
P-positions and N-positions can be defined recursively by the following:

(1) All terminal positions are P-positions

(2) From every N-position, there is at least one move to a P-position

(3) From every P-position, every move is to an N-position

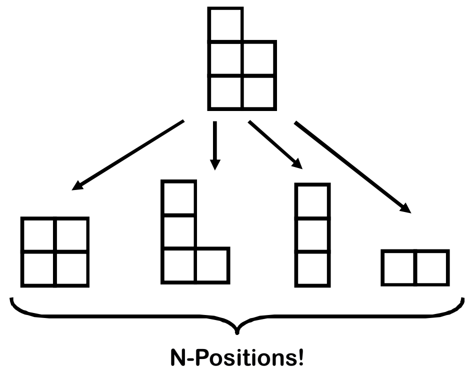
Chomp!



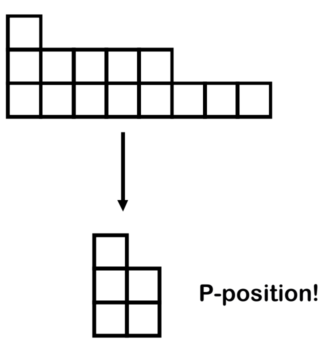
Two-player game, where each move consists of taking a square and removing it and all squares to the right and above.

Player who takes position (1,1) loses

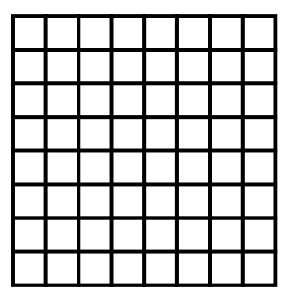
Show That This is a P-position

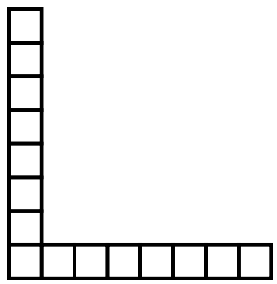


Show That This is an N-position



Let’s Play! I’m player I





No matter what you do, I can mirror it!

Mirroring is an extremely important strategy in combinatorial games!

Theorem: Player I can win in any square starting position of Chomp

Proof:

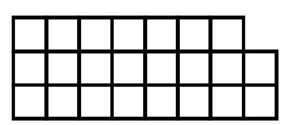
The winning strategy for player I is to chomp on (2,2), leaving only an “L” shaped position

Then, for any move that Player II takes, Player I can simply mirror it on the flip side of the “L”

Theorem: Player I can win in any rectangular starting position

Proof:

Look at this first move:

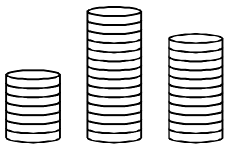


If this is a P-position, then player 1 wins

Otherwise, there exists a P-position that can be obtained from this position

And player 1 could have just taken that move originally

The Game of Nim



x y z

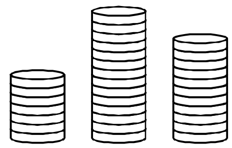
Two players take turns moving

Winner is the last player to remove chips

Each move consists of selecting one of the piles and removing chips from it (you can take as many as you want, but you have to at least take one)

In one move, you cannot remove chips from more than one pile

Analyzing Simple Positions



x y z

We use (x,y,z) to denote this position

(0,0,0) is a: P-position

One-Pile Nim

What happens in positions of the form (x,0,0)?

The first player can just take the entire pile, so (x,0,0) is an N-position

Two-Pile Nim

P-positions are those for which the two piles have an equal number of chips

If it is the opponent's turn to move from such a position, he must change to a position in which the two piles have different number of chips

From a position with an unequal number of chips, you can easily go to one with an equal number of chips (perhaps the terminal position)

Nim-Sum

The nim-sum of two non-negative integers is their addition without carry in base 2

We will use $\oplus$ to denote the nim-sum

$2 \oplus 3 = (10)_2 \oplus (11)_2 = (01)_2 = 1$

$5 \oplus 3 = (101)_2 \oplus (011)_2 = (110)_2 = 6$

$7 \oplus 4 = (111)_2 \oplus (100)_2 = (011)_2 = 3$

$\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

$\oplus$ is commutative: $a \oplus b = b \oplus a$

For any non-negative integer x,

$$x \oplus x = 0$$

Cancellation Property Holds

If $x \oplus y = x \oplus z$

Then $x \oplus x \oplus y = x \oplus x \oplus z$

So $y = z$

Bouton's Theorem: A position (x,y,z) in Nim is a P-position if and only if $x \oplus y \oplus z = 0$

Proof:

Let Z denote the set of Nim positions with nim-sum zero

Let NZ denote the set of Nim positions with non-zero nim-sum

We prove the theorem by proving that Z and NZ satisfy the three conditions of P-positions and N-positions

(1) All terminal positions are in Z

The only terminal position is (0,0,0)

(2) From each position in NZ, there is a move to a position in Z

$$
\begin{array}{rr}
 & 001010001 \\
 & 100010111 \\
\oplus & 111010000 \\
\hline
 & 010010110
\end{array}
\quad \Rightarrow \quad
\begin{array}{rr}
 & 001010001 \\
 & 100010111 \\
\oplus & 101000110 \\
\hline
 & 000000000
\end{array}
$$

Look at leftmost column with an odd # of 1s

Rig any of the numbers with a 1 in that column so that everything adds up to zero

(3) Every move from a position in Z is to a position in NZ

If (x,y,z) is in Z, and x is changed to $x' < x$, then we cannot have

$$x \oplus y \oplus z = 0 = x' \oplus y \oplus z$$

Because then $x = x'$

Here's What You Need to Know…

Combinatorial Games

- P-positions versus N-positions
- When there are no draws, every position is either P or N

Nim

- Definitions of the game
- Nim-sum
- Bouton's Theorem