Graphs
A tree.
methane

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some saturated hydrocarbons
A tree is a connected graph with no cycles.
These are not trees...
The Shy People Party

At the shy people party, people enter one-by-one, and as a person comes in, (s)he shakes hand with only one person already at the party.

Prove that at a shy party with n people (n >= 2), at least two people have shaken hands with only one other person.
The Shy People Party

Case 1: Shake hands with a "1 shaker"; #1 shakers stay same.

Case 2: Shake hands with someone who has shaken > 1 hand.
How many trees on 1-6 vertices?
We’ll pass around a piece of paper. Draw a new 8-node tree, and put your name next to it. (There are 23 of them...)
Theorem: Let $G$ be a graph with $n$ nodes and $e$ edges.

The following are equivalent:

1. $G$ is a tree (connected, acyclic)
2. Every two nodes of $G$ are joined by a unique path
3. $G$ is connected and $n = e + 1$
4. $G$ is acyclic and $n = e + 1$
5. $G$ is acyclic and if any two nonadjacent points are joined by a line, the resulting graph has exactly one cycle.
To prove this, it suffices to show
\[ 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1 \]

If \( G \) is a tree \( \Rightarrow \) every two nodes are joined by a unique path.

\[ 2 \Rightarrow 3 \] \( \Rightarrow \) every pair of nodes connected by a unique path \( \Rightarrow \)

\( 6 \) is connected and \( n = e + 1 \)

remove an edge

\[ n_1 = e_1 + 1 \]

\[ n_2 = e_2 + 1 \]

\[ n = n_1 + n_2 = e_1 + e_2 + 2 = e + 1 \]
Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

\[ n = n_1 + n_2 + n_3 + \ldots \]

\[ n = e + 1 \]

\[ 2e = 2n - 2 \quad \text{total degree} \]
Question:

How many labeled trees are there with three nodes?

3 labelings.
Question:

How many labeled trees are there with four nodes?
Question:

How many labeled trees are there with five nodes?

\[ T_3 = 3 \]
\[ T_4 = 16 \]
\[ T_5 = 125 \]
\[ T_n \text{?} \quad n^{n-2} \]
\[ T_n = n \]
Question:
How many labeled trees on $n$ nodes are there?
Cayley’s formula

The number of labeled trees on \(n\) nodes is

\[ n^{n-2} \]
The proof will use the correspondence principle.

Each labeled tree on $n$ nodes corresponds to

A sequence in $\{1, 2, \ldots, n\}^{n-2}$ that is, $(n-2)$ numbers, each in the range $[1..n]$
How to make a sequence from a tree.

Loop through \( i \) from 1 to \( n-2 \)

Let \( l \) be the degree-1 node with the lowest label.

Define the \( i^{th} \) element of the sequence as the label of the node adjacent to \( l \).

Delete the node \( l \) from the tree.

Example:

```
\begin{array}{c}
  5 \\
  3 \\
  1 \\
  2 \\
  6 \\
  4 \\
  8 \\
  7 \\
\end{array}
```

\( \langle 1, 3, 3, 4, 4, 4 \rangle \)
How to reconstruct the unique tree from a sequence $S$.

Let $I = \{1, 2, 3, \ldots, n\}$

Loop until $S = \epsilon$ empty sequence

Let $l =$ smallest # in $I$ but not in $S$

Let $s =$ first label in sequence $S$

• Add edge $\{l, s\}$ to the tree.
• Delete $l$ from $I$.
• Delete $s$ from $S$.

Add edge $\{l, s\}$ to the tree, where $I = \{l, s\}$
Another example
Another Proof of Cayley's Formula

\[ T_n = n^{n-2} \]

\[ T_{n,k} = k \cdot n^{n-k-1} \]
A graph is planar if it can be drawn in the plane without crossing edges. A plane graph is any such drawing, which breaks up the plane into a number $f$ of faces or regions.

$n - e + f = 5 - 6 + 3 = 2$
Euler’s Formula

If $G$ is a connected plane graph with $n$ vertices, $e$ edges and $f$ faces, then $n - e + f = 2$. 

Leonhard Euler 1707–1783
Rather than using induction, we’ll use the important notion of the *dual graph*.

To construct the dual graph, put a vertex into the interior of every face, and connect two such vertices by an edge if there is a common boundary edge between the faces. (Note that the dual graph may have multiple edges between points.)
We’ll also use the notion of a *spanning tree*.

A *spanning tree* of a graph is a subgraph that is also a tree, with the same vertex set as the original graph.

In other words, it’s a minimal subgraph that connects all of the vertices.
Euler's Formula: If $G$ is a connected plane graph with $n$ vertices, $e$ edges and $f$ faces, then $n - e + f = 2$

Claim: $T^*$ is a spanning tree of $G^*$

$n = e_T + 1$ because $T$ is a spanning tree

$f = e_{T^*} + 1$

$n + f = \frac{e_T + e_{T^*} + 2}{e}$
Euler's Formula: If $G$ is a connected plane graph with $n$ vertices, $e$ edges and $f$ faces, then $n - e + f = 2$

The beauty of Euler's formula is that it yields a *numeric* property from a purely *geometric* or topological property.
Corollary: Let $G$ be a plane graph with $n > 2$ vertices. Then

a) $G$ has a vertex of degree at most 5.
b) $G$ has at most $3n - 6$ edges
Graph Coloring

A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color.
Graph Coloring

Arises surprisingly often in CS.

Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register.
Instructions

b = a+2

c = b*b

b = c+1

return a*b

Live variables

a

a,b

a,c

a,b

a,b
Every plane graph can be 6-colored
Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree $\leq 5$.

4-color theorem remains challenging

http://www.math.gatech.edu/~thomas/FC/fourcolor.html