Counting III: Pascal’s Triangle, Polynomials, and Vector Programs
Last time, we saw that *Polynomials Count!*
Choice tree for terms of $(1+X)^3$

Combine like terms to get $1 + 3X + 3X^2 + X^3$
The Binomial Formula

\[(1 + X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n\]

Binomial Coefficients

binomial expression
The Binomial Formula

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]
One polynomial, two representations

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]

“Product form” or “Generating form”

“Additive form” or “Expanded form”
What is the coefficient of $BA^3N^2$ in the expansion of $(B + A + N)^6$?

\[
\frac{6!}{3!2!1!} = \binom{6}{3,2,1}
\]

The number of ways to rearrange the letters in the word BANANA.
Multinomial Coefficients

\[
\binom{n}{r_1; r_2; \ldots; r_k} \equiv \begin{cases} 
0 & \text{if } r_1 + r_2 + \ldots + r_k \neq n \\
\frac{n!}{r_1! r_2! \ldots r_k!} & \text{if } r_1 + r_2 + \ldots + r_k = n 
\end{cases}
\]

\[
\binom{n}{k; n-k} = \binom{n}{k}
\]
The Multinomial Formula

\[(X_1 + X_2 + \ldots + X_k)^n\]

\[= \sum_{r_1, r_2, \ldots, r_k} \binom{n}{r_1; r_2; \ldots; r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}\]
Power Series Representation

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]

“Closed form” or “Generating form”

\[= \sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k\]

Since \(\binom{n}{k} = 0\) if \(k > n\)

“Power series” (“Taylor series”) expansion
By playing these two representations against each other we obtain a new representation of a previous insight:

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]

Let \(x = 1\).

\[2^n = \sum_{k=0}^{n} \binom{n}{k}\]

The number of subsets of an \(n\)-element set.
By varying $x$, we can discover new identities

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x = -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$
The number of even-sized subsets of an $n$ element set is the same as the number of odd-sized subsets.

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x = -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$
Proofs that work by manipulating algebraic forms are called “algebraic” arguments. Proofs that build a 1-1 onto correspondence are called “combinatorial” arguments.

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]
Let \( O_n \) be the set of binary strings of length \( n \) with an odd number of ones.

Let \( E_n \) be the set of binary strings of length \( n \) with an even number of ones.

We gave an algebraic proof that

\[
\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}
\]
A Combinatorial Proof

Let $O_n$ be the set of binary strings of length $n$ with an odd number of ones.

Let $E_n$ be the set of binary strings of length $n$ with an even number of ones.

A combinatorial proof must construct a one-to-one correspondence between $O_n$ and $E_n$. 
An attempt at a correspondence

Let $f_n$ be the function that takes an $n$-bit string and flips all its bits.

$f_n$ is clearly a one-to-one and onto function for odd $n$. E.g. in $f_7$ we have

$0010011 \rightarrow 1101100$

$1001101 \rightarrow 0110010$

...but do even $n$ work? In $f_6$ we have

$110011 \rightarrow 001100$

$101010 \rightarrow 010101$

Uh oh. Complementing maps evens to evens!
A correspondence that works for all $n$

Let $f_n$ be the function that takes an $n$-bit string and flips only the first bit.

For example,

- $0010011 \rightarrow 1010011$
- $1001101 \rightarrow 0001101$
- $110011 \rightarrow 010011$
- $101010 \rightarrow 001010$
The binomial coefficients have so many representations that many fundamental mathematical identities emerge...
The Binomial Formula

\[(1+X)^0 = 1\]
\[(1+X)^1 = 1 + 1X\]
\[(1+X)^2 = 1 + 2X + 1X^2\]
\[(1+X)^3 = 1 + 3X + 3X^2 + 1X^3\]
\[(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4\]
Pascal’s Triangle:

\[ \text{ } \]

\[ k^{\text{th}} \text{ row are the coefficients of } (1+X)^k \]

\[ (1+X)^0 = 1 \]

\[ (1+X)^1 = 1 + 1X \]

\[ (1+X)^2 = 1 + 2X + 1X^2 \]

\[ (1+X)^3 = 1 + 3X + 3X^2 + 1X^3 \]

\[ (1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4 \]
$k^{th}$ Row Of Pascal’s Triangle:

\[
\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{k}, \ldots, \binom{n}{n}
\]

\[
(1+X)^0 = 1
\]
\[
(1+X)^1 = 1 + 1X
\]
\[
(1+X)^2 = 1 + 2X + 1X^2
\]
\[
(1+X)^3 = 1 + 3X + 3X^2 + 1X^3
\]
\[
(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4
\]
Inductive definition of kth entry of nth row:

\[ \text{Pascal}(n,0) = \text{Pascal} (n,n) = 1; \]
\[ \text{Pascal}(n,k) = \text{Pascal}(n-1,k-1) + \text{Pascal}(n-1,k) \]

\[
\begin{align*}
(1+X)^0 &= 1 \\
(1+X)^1 &= 1 + 1X \\
(1+X)^2 &= 1 + 2X + 1X^2 \\
(1+X)^3 &= 1 + 3X + 3X^2 + 1X^3 \\
(1+X)^4 &= 1 + 4X + 6X^2 + 4X^3 + 1X^4
\end{align*}
\]
"Pascal’s Triangle"

\[
\binom{0}{0} = 1 \\
\binom{1}{0} = 1, \quad \binom{1}{1} = 1 \\
\binom{2}{0} = 1, \quad \binom{2}{1} = 2, \quad \binom{2}{2} = 1 \\
\binom{3}{0} = 1, \quad \binom{3}{1} = 3, \quad \binom{3}{2} = 3, \quad \binom{3}{3} = 1
\]

Al-Karaji, Baghdad 953-1029

Chu Shin-Chieh 1303
The Precious Mirror of the Four Elements
... Known in Europe by 1529

Blaise Pascal 1654
Pascal's Triangle

```
1   1
1   2   1
1   3   3   1
1   4   6   4   1
1   5   10  10  5   1
1   6   15  20  15  6   1
```

“It is extraordinary how fertile in properties the triangle is. Everyone can try his hand.”
\[ 2^n = \sum_{k=0}^{n} \binom{n}{k} \]

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 1 & 1 & & & \\
1 & 1 & 2 & 1 & & \\
1 & 1 & 3 & 3 & 1 & \\
1 & 1 & 4 & 6 & 4 & 1 \\
1 & 1 & 5 & 10 & 10 & 5 & 1 \\
1 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

= 1
= 2
= 4
= 8
= 16
= 32
= 64
\[
\begin{array}{cccccc}
& & & 1 & & \\
& & 1 & 1 & & \\
& 1 & 2 & 1 & & \\
1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

6 + 20 + 6 = 1 + 15 + 15 + 1
Summing on 1\textsuperscript{st} Avenue

\[
\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n \cdot (n+1)}{2}
\]
Summing on $k^{th}$ Avenue

\[ \sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \]
**Al-Karaji Squares**

\[
\begin{align*}
n + \frac{2 \times n \cdot (n-1)}{2} &= n + n^2 - n \\ &= n^2
\end{align*}
\]

\[
\begin{array}{cccc}
1 & 1 & 1 & = 0 \\
1 & 2 & 2 & = 4 \\
1 & 3 & 6 & = 9 \\
1 & 4 & 10 & = 16 \\
1 & 5 & 20 & = 25 \\
1 & 6 & 20 & = 36 \\
\end{array}
\]

The diagram shows a table with rows and columns for the values of the polynomial expression, with additional values for each row.
All these properties can be proved inductively and algebraicallly. We will give **combinatorial** proofs using the **Manhattan block walking representation of binomial coefficients**.
How many shortest routes from A to B?
There are $\binom{j+k}{k}$ shortest routes from $(0,0)$ to $(j,k)$. 
There are \( \binom{n}{k} \) shortest routes from \((0,0)\) to \((n-k,k)\).
There are \( \binom{n}{k} \) shortest routes from (0,0) to Level \( n \) and \( k^{th} \) Avenue.
level $n$

$k$'th Avenue
\begin{align*}
\binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k} \\
\end{align*}
\[
\sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}
\]
\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}
\]
By convention:

\[ 0! = 1 \quad \text{(empty product = 1)} \]

\[ \binom{n}{k} = 1 \quad \text{if } k = 0 \]

\[ \binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n \]
Corollary (1)

\[
\sum_{i=1}^{n} \binom{i}{k} = \binom{n+1}{k+1}
\]

level \( n \)

Corollary \((k = 1)\)

\[
\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}
\]
Application (Al-Karaji):

\[
\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2
\]

\[
= (1 \cdot 0 + 1) + (2 \cdot 1 + 2) + (3 \cdot 2 + 3) + \cdots + (n(n - 1) + n)
\]

\[
= 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + \cdots + n(n - 1) + \sum_{i=1}^{n} i
\]

\[
= 2 \left[ \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2} \right] + \binom{n+1}{2}
\]

\[
= 2 \left( \binom{n+1}{3} + \binom{n+1}{2} \right) = \frac{(2n+1)(n+1)n}{6}
\]
Vector Programs

Let’s define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable $V\rightarrow$ can be thought of as:

\[ \langle \ast, \ast, \ast, \ast, \ast, \ast, \ast, \ast, \ast, \ast, \ldots \rangle \]

\[ 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ \ldots \ldots \]
Vector Programs

Let \( k \) stand for a scalar constant
\(<k>\) will stand for the vector \(<k,0,0,0,\ldots>\)

\(<0> = <0,0,0,0,\ldots>\)
\(<1> = <1,0,0,0,\ldots>\)

\( \vec{V} + \vec{T} \) means to add the vectors position-wise.

\(<4,2,3,\ldots> + <5,1,1,\ldots> = <9,3,4,\ldots>\)
Vector Programs

$\text{RIGHT}(V \rightarrow)$ means to shift every number in $V \rightarrow$ one position to the right and to place a 0 in position 0.

$\text{RIGHT}(\langle 1,2,3, \ldots \rangle) = \langle 0,1,2,3, \ldots \rangle$
Vector Programs

Example:

\[ V \rightarrow := \langle 6 \rangle; \]
\[ V \rightarrow := \text{RIGHT}(V \rightarrow) + \langle 42 \rangle; \]
\[ V \rightarrow := \text{RIGHT}(V \rightarrow) + \langle 2 \rangle; \]
\[ V \rightarrow := \text{RIGHT}(V \rightarrow) + \langle 13 \rangle; \]

\[ V \rightarrow = \langle 13, 2, 42, 6, 0, 0, 0, \ldots \rangle \]
Vector Programs

Example:

\[ \vec{V} := <1>; \]

Loop \( n \) times:

\[ \vec{V} := \vec{V} + \text{RIGHT}(\vec{V}); \]

Store

\[ \vec{V} = <1,0,0,0,..> \]

\[ \vec{V} = <1,1,0,0,..> \]

\[ \vec{V} = <1,2,1,0,..> \]

\[ \vec{V} = <1,3,3,1,..> \]

\[ \vec{V} = n^{th} \text{ row of Pascal’s triangle.} \]
Vector programs can be implemented by polynomials!
The vector $V = <a_0, a_1, a_2, \ldots>$ will be represented by the polynomial:

$$P_V = \sum_{i=0}^{\infty} a_i X^i$$
Formal Power Series

The vector $\mathbf{V} = \langle a_0, a_1, a_2, \ldots \rangle$ will be represented by the formal power series:

$$P_{\mathbf{V}} = \sum_{i=0}^{\infty} a_i X^i$$
V→ = < a₀, a₁, a₂, ... >

\[ P_V = \sum_{i=0}^{\infty} a_i X^i \]

<0> is represented by 0
<k> is represented by k

V→ + T→ is represented by (P_V + P_T)

RIGHT(V→) is represented by (P_V X)
Vector Programs

Example:

\[ V\rightarrow := \langle 1 \rangle; \]

Loop \( n \) times:

\[ V\rightarrow := V\rightarrow + \text{RIGHT}(V\rightarrow); \]

\[ P_V := P_V + P_V \times; \]

\[ V\rightarrow = n^{th} \text{ row of Pascal’s triangle.} \]
Vector Programs

Example:

\[ V \rightarrow := \langle 1 \rangle; \]

Loop n times:

\[ V \rightarrow := V \rightarrow + \text{RIGHT}(V \rightarrow); \]

\[ P_V := P_V (1+ X); \]

\[ V \rightarrow = n^{th} \text{ row of Pascal's triangle.} \]
Vector Programs

Example:

$V \rightarrow := \langle 1 \rangle$;

Loop n times:

$V \rightarrow := V \rightarrow + \text{RIGHT}(V \rightarrow)$;

$V \rightarrow = n^{\text{th}}$ row of Pascal’s triangle.

$P_V = (1 + X)^n$
The Geometric Series

\[ 1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1} \]
The Infinite Geometric Series

\[ 1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots = \frac{1}{1 - X} \]
\[
1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots = \frac{1}{1 - X}
\]

\[
(X-1) \left( 1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots \right)
= X^1 + X^2 + X^3 + \ldots + X^n + X^{n+1} + \ldots
- 1 - X^1 - X^2 - X^3 - \ldots - X^{n-1} - X^n - X^{n+1} - \ldots
= 1
\]
1 + aX^1 + a^2X^2 + a^3X^3 + \ldots + a^nX^n + \ldots = \frac{1}{1 - aX}

Geometric Series (Linear Form)
\[(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots)(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) = \]
\[
\frac{1}{(1 - aX)(1-bX)}
\]

Geometric Series (Quadratic Form)
\[(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots) \cdot (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) = 1 + c_1X^1 + \ldots + c_kX^k + \ldots\]

Suppose we multiply this out to get a single, infinite polynomial.

What is an expression for \(C_n\)?
\[(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots) \ (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) =
\]
\[1 + c_1X^1 + \ldots + c_kX^k + \ldots\]

\[c_n =
\]
\[a^0b^n + a^1b^{n-1} + \ldots + a^ib^{n-i} + \ldots + a^{n-1}b^1 + a^n b^0\]
\[(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots) \times (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots) = 1 + c_1X^1 + \ldots + c_kX^k + \ldots\ldots\]

If \(a = b\) then

\[c_n = (n+1)(a^n)\]

\[a^0b^n + a^1b^{n-1} + \ldots + a^ib^{n-i} + \ldots + a^{n-1}b^1 + a^n b^0\]
\[ a^0b^n + a^1b^{n-1} + \ldots + a^ib^{n-i} + \ldots + a^{n-1}b^1 + a^nb^0 = \frac{a^{n+1} - b^{n+1}}{a - b} \]

\[
(a-b) (a^0b^n + a^1b^{n-1} + \ldots + a^ib^{n-i} + \ldots + a^{n-1}b^1 + a^nb^0) = a^1b^n + \ldots + a^{i+1}b^{n-i} + \ldots + a^n b^1 + a^{n+1} b^0
\]

\[
- a^0b^{n+1} - a^1b^n - a^{i+1}b^{n-i} - \ldots - a^{n-1}b^2 - a^n b^1
\]

\[
= - b^{n+1} + a^{n+1}
\]

\[
= a^{n+1} - b^{n+1}
\]
\[(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots) \times (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) = 1 + c_1X^1 + \ldots + c_kX^k + \ldots\]

if \(a \neq b\) then

\[c_n = \frac{a^{n+1} - b^{n+1}}{a - b}\]

\[a^0b^n + a^1b^{n-1} + \ldots + a^ib^{n-i} + \ldots + a^{n-1}b^1 + a^n b^0\]
Geometric Series (Quadratic Form)

\[ (1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots) \cdot (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) = \]

\[ \frac{1}{(1 - aX)(1 - bX)} \]

\[ \sum_{n=0}^{\infty} a^n X^n = \sum_{n=0}^{\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n \]

or

\[ \sum_{n=0}^{\infty} (n+1)a^n X^n \]

when \( a = b \)
• Polynomials count
• Binomial formula
• Multinominal coefficients
• Combinatorial proofs of binomial identities
• Vector programs
• Geometric series