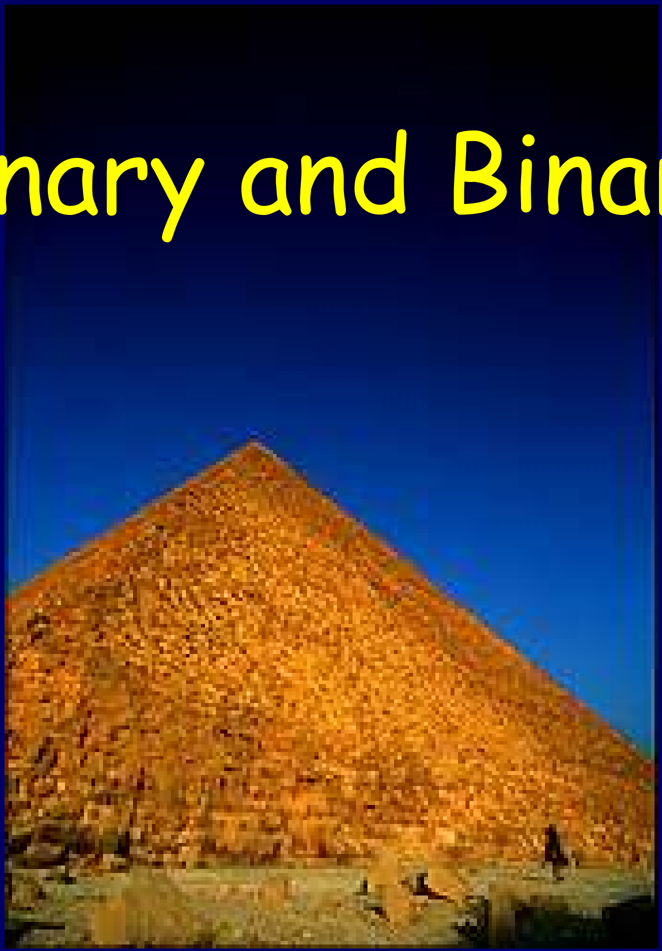
Great Theoretical Ideas In Computer Science

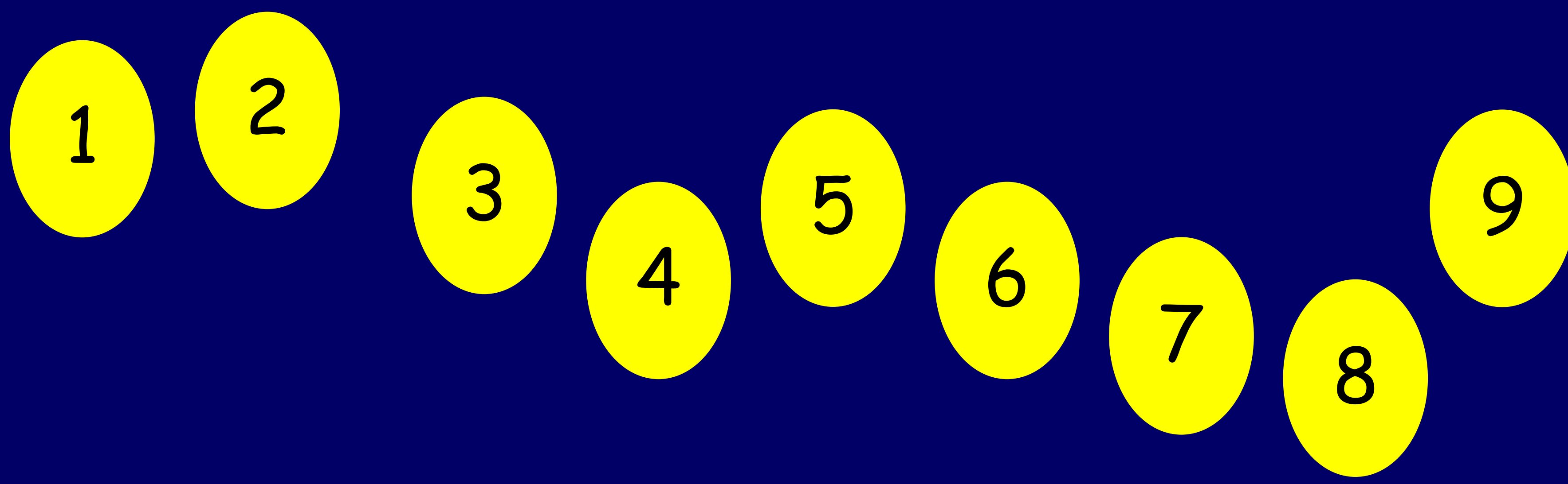
Anupam Gupta CS 15-251 Fall 2006

Lecture 4 Sept 7, 2006 Carnegie Mellon University

Unary and Binary



How to play the 9 stone game?

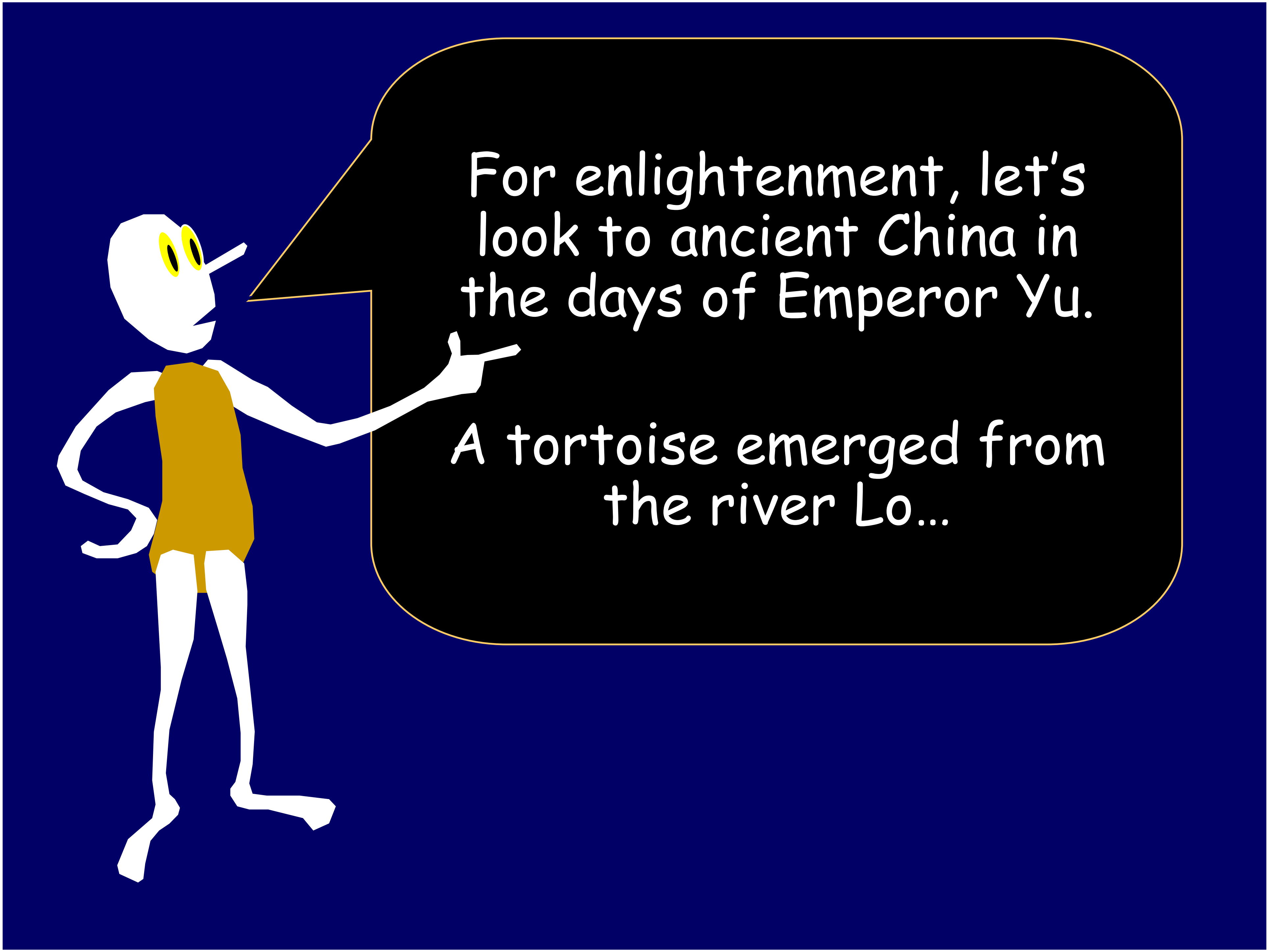


9 stones, numbered 1-9

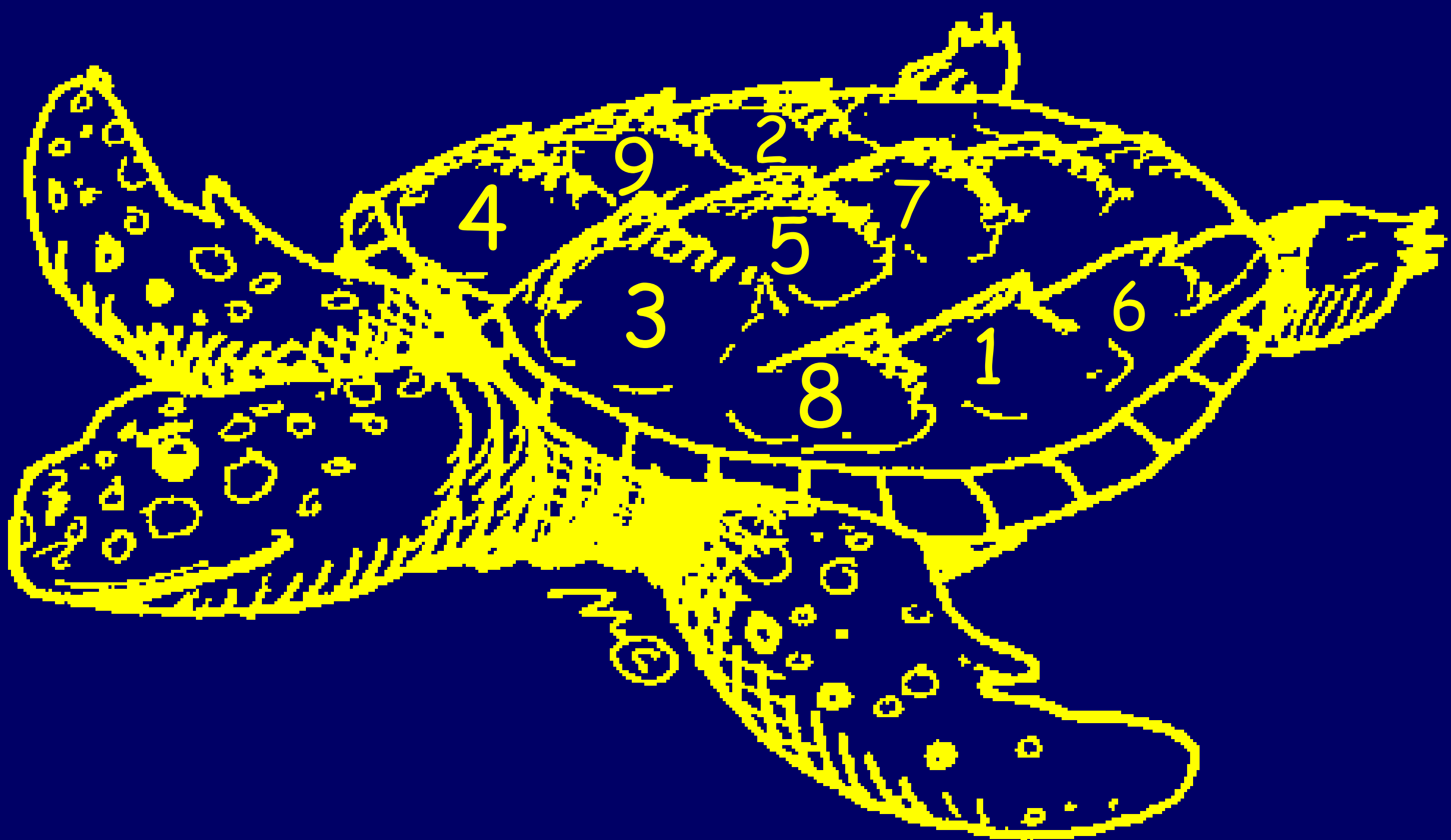
Two players alternate moves.

Each move a player gets to take a new stone

Any subset of 3 stones adding to 15, wins.


For enlightenment, let's look to ancient China in the days of Emperor Yu.
A tortoise emerged from the river Lo...

Magic Square: Brought to humanity on the back of a tortoise from the river Lo in the days of Emperor Yu



Magic Square: Any 3 in a vertical, horizontal, or diagonal line add up to 15.

4	9	2
3	5	7
8	1	6

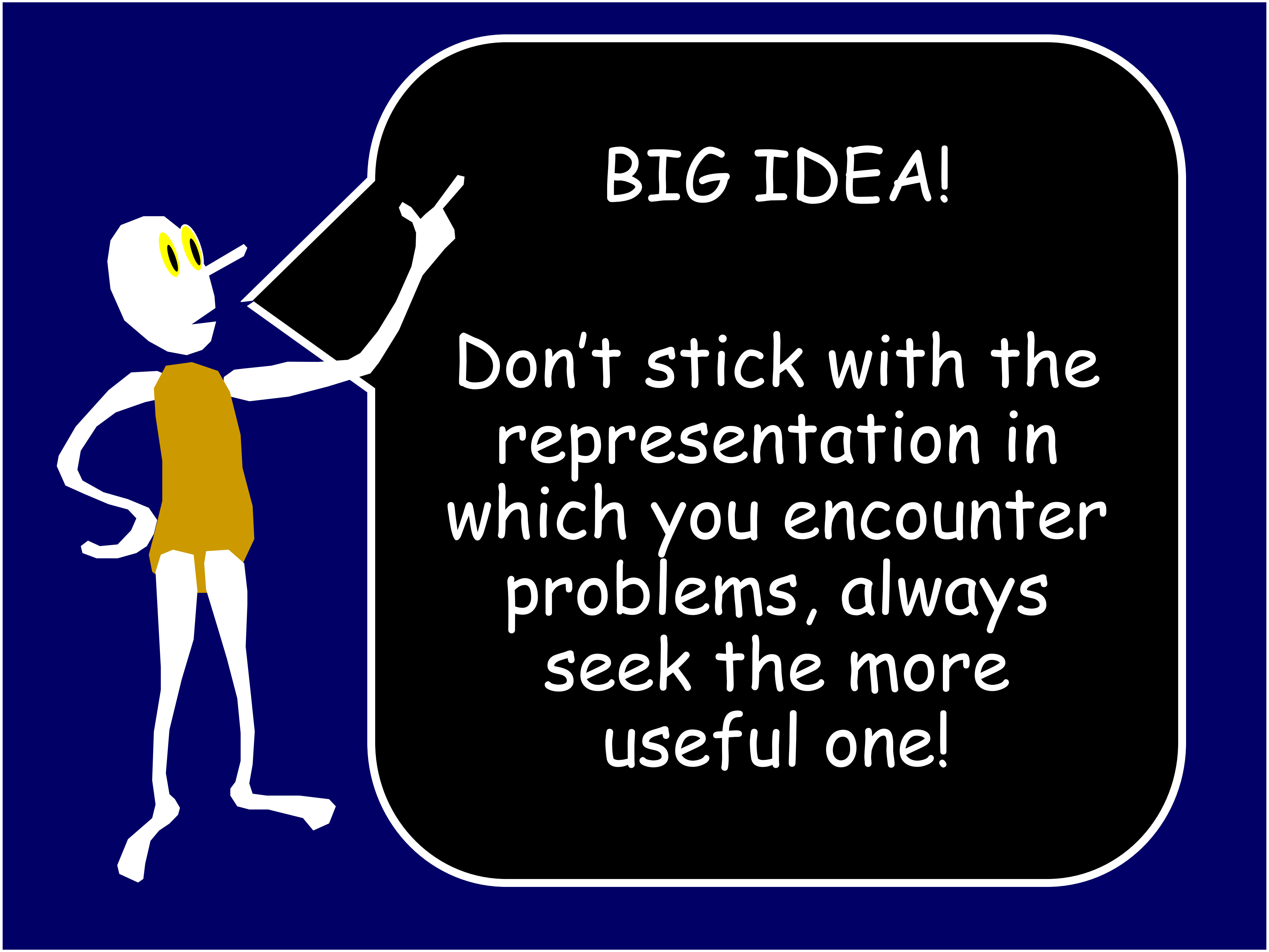
Conversely,
any 3 that add to 15 must be on a line.

4	9	2
3	5	7
8	1	6

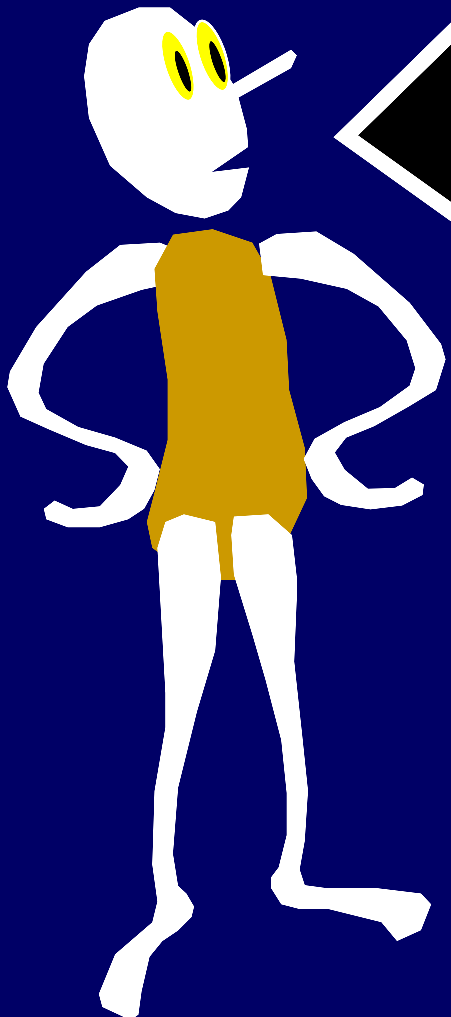
TIC-TAC-TOE on a Magic Square Represents The Nine Stone Game

Alternately choose squares from 1-9.
Get 3 in a row to win.

4	9	2
3	5	7
8	1	6


BIG IDEA!
Don't stick with the representation in which you encounter problems, always seek the more useful one!

This IDEA takes practice, practice, practice to understand and use.



Your Ancient Heritage

Let's take a historical view on abstract representations

Mathematical Prehistory
30,000 BC

Paleolithic peoples in Europe record unary numbers on bones

1 represented by 1 mark

2 represented by 2 marks

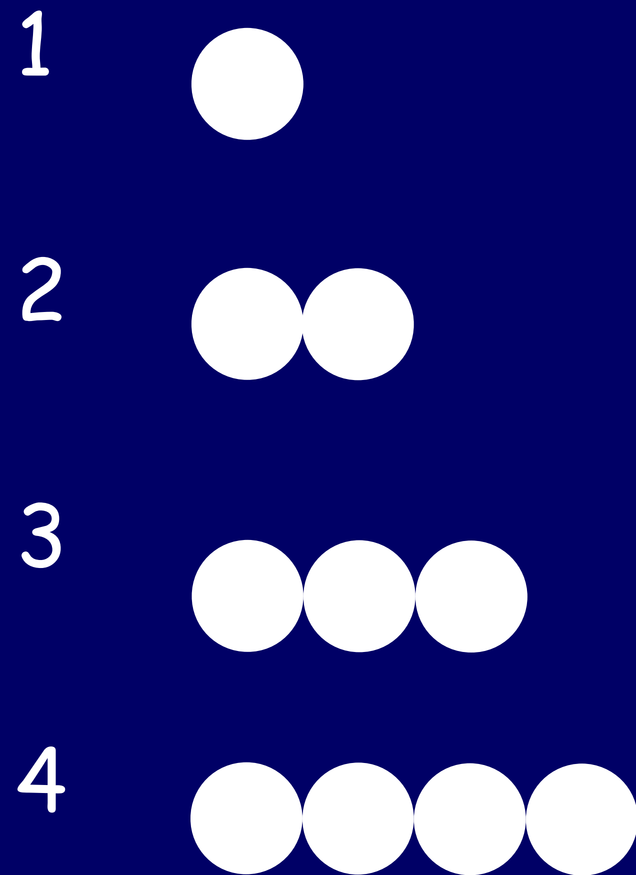
3 represented by 3 marks

.

.

.

Prehistoric Unary



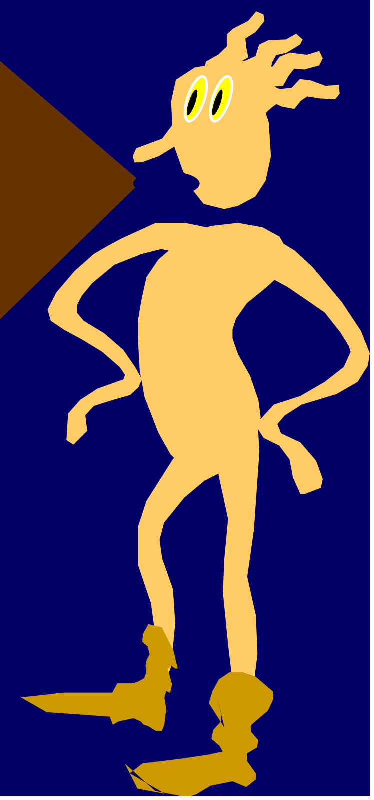
Hang on a minute!

Isn't unary too literal as a representation?

Does it deserve to be an "abstract" representation?

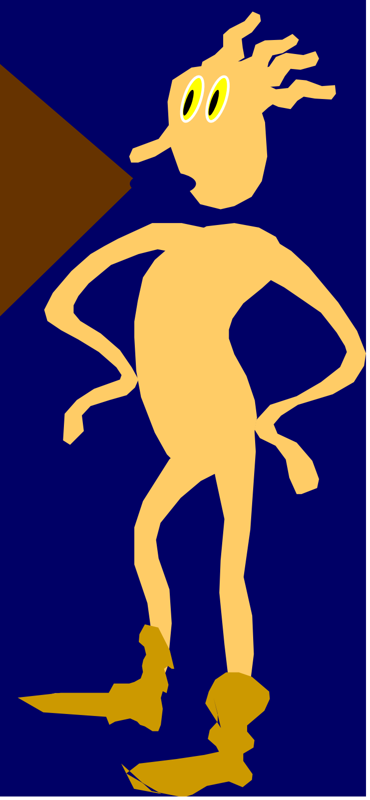
It's important to respect each representation, no matter how primitive

Unary is a perfect example



Consider the problem of finding a formula for the sum of the first n numbers

You already used induction to verify that the answer is $\frac{1}{2}n(n+1)$



$$1 + 2 + 3 + \ldots + n-1 + n = S$$

$$n + n-1 + n-2 + \ldots + 2 + 1 = S$$

$$n+1 + n+1 + n+1 + \ldots + n+1 + n+1 = 2S$$

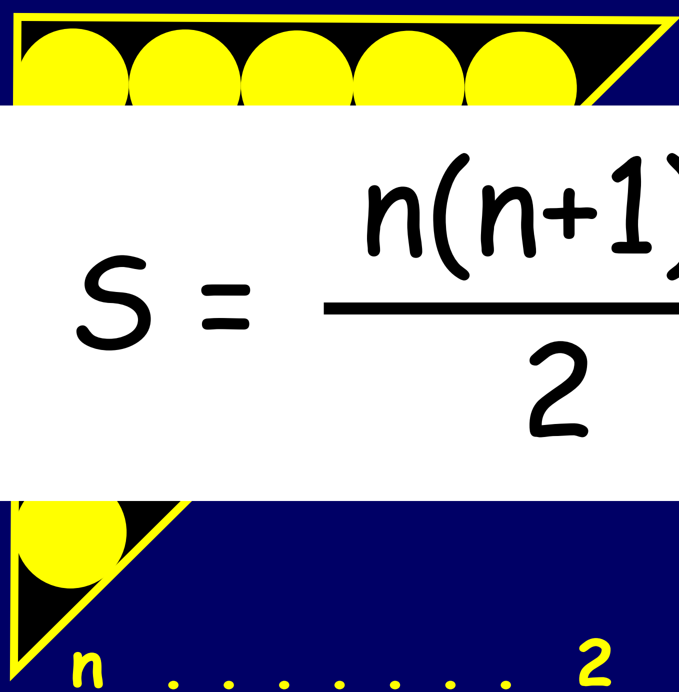
$$n(n+1) = 2S$$

$$S = \frac{n(n+1)}{2}$$

Gauss

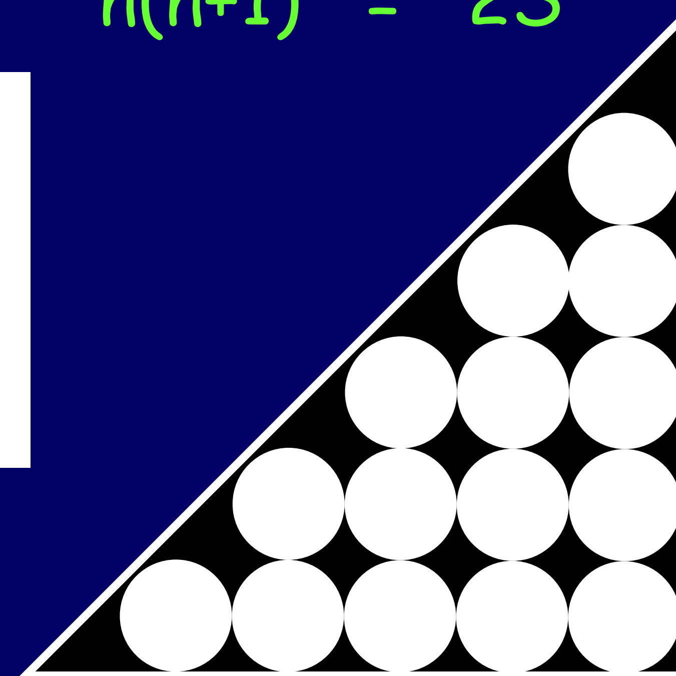
$$1 + 2 + 3 + \dots + n-1 + n = S$$

$$n + n-1 + n-2 + \dots + 2 + 1 = S$$



$$S = \frac{n(n+1)}{2}$$

$$n(n+1) = 2S$$



Very convincing!
Unary brings out the geometry of the problem and makes each step look natural

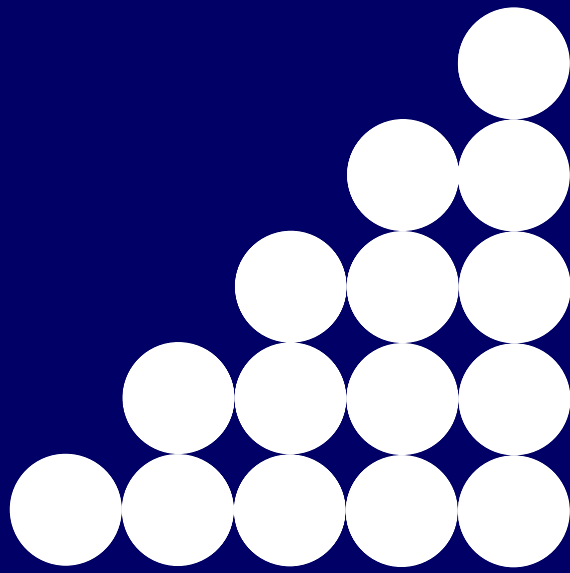
By the way, my name is Bonzo. And you are?


Odette
Yes, Bonzo. Let's take it even further...

n^{th} Triangular Number

$$\Delta_n = 1 + 2 + 3 + \ldots + n\text{-}1 + n$$

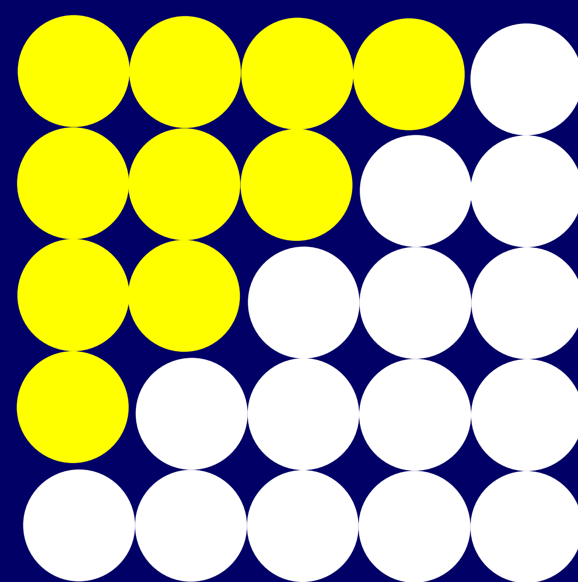
$$= n(n+1)/2$$

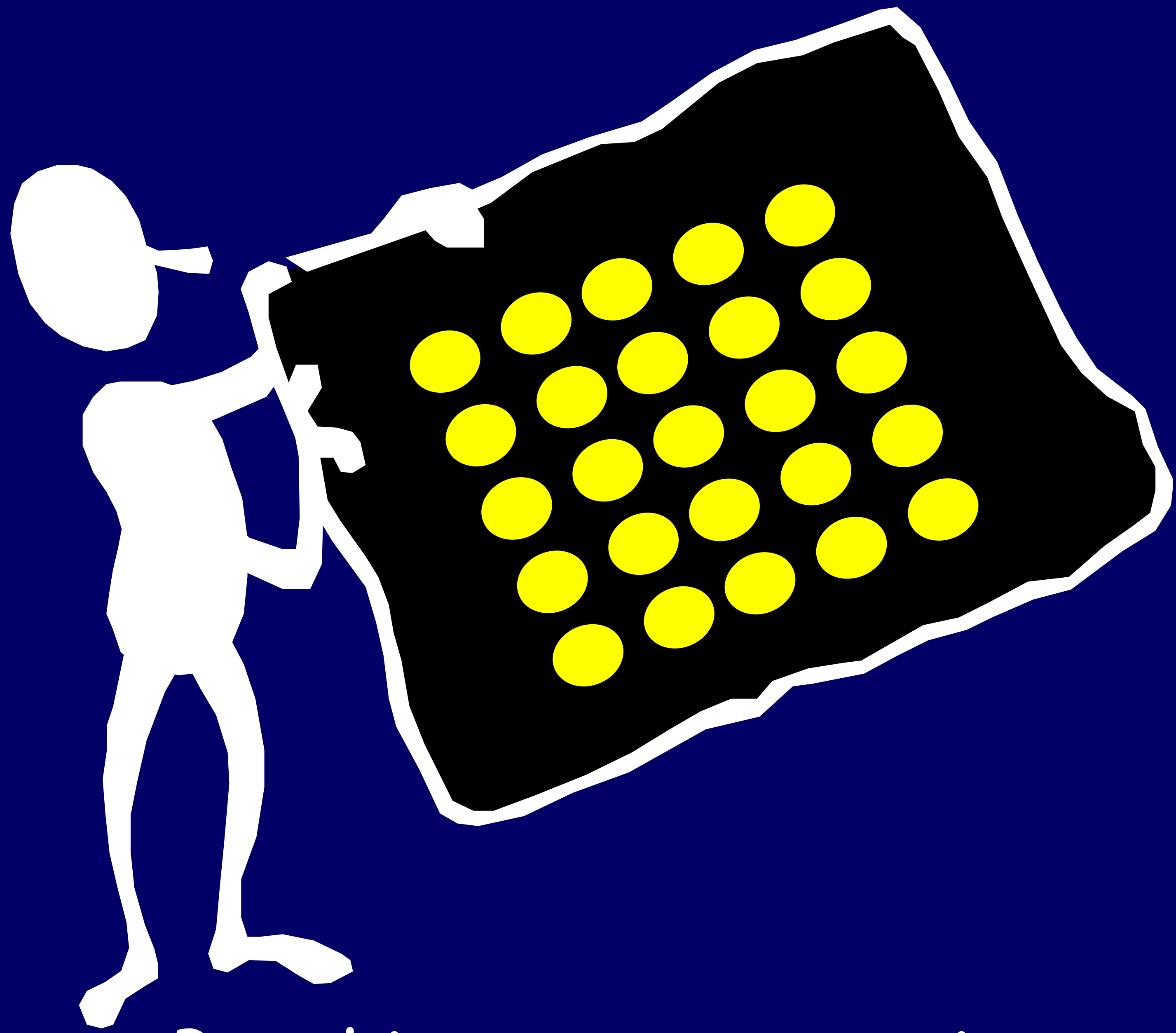


n^{th} Square Number

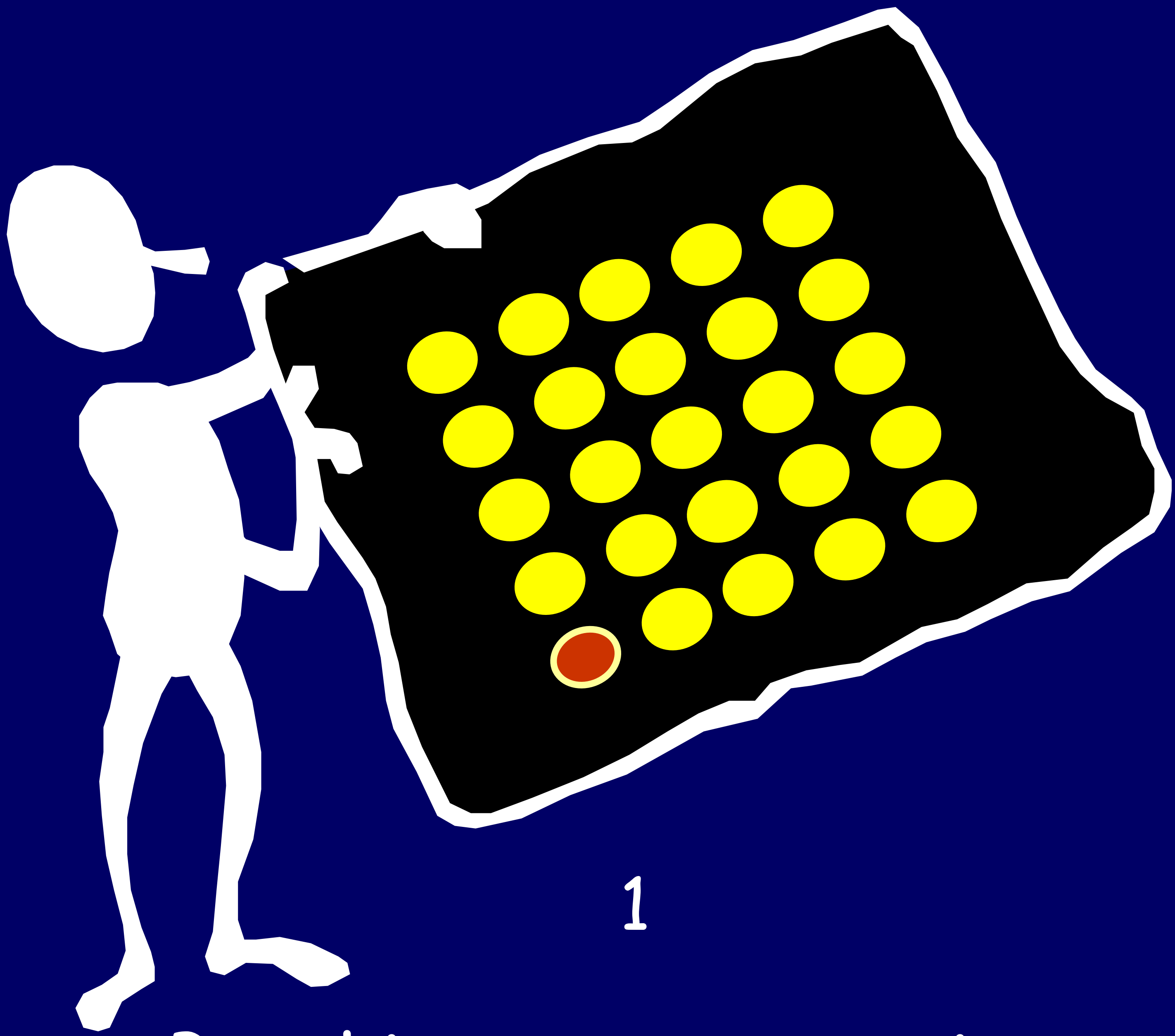
$$\square_n = n^2$$

$$= \Delta_n + \Delta_{n-1}$$





Breaking a square up in a new way



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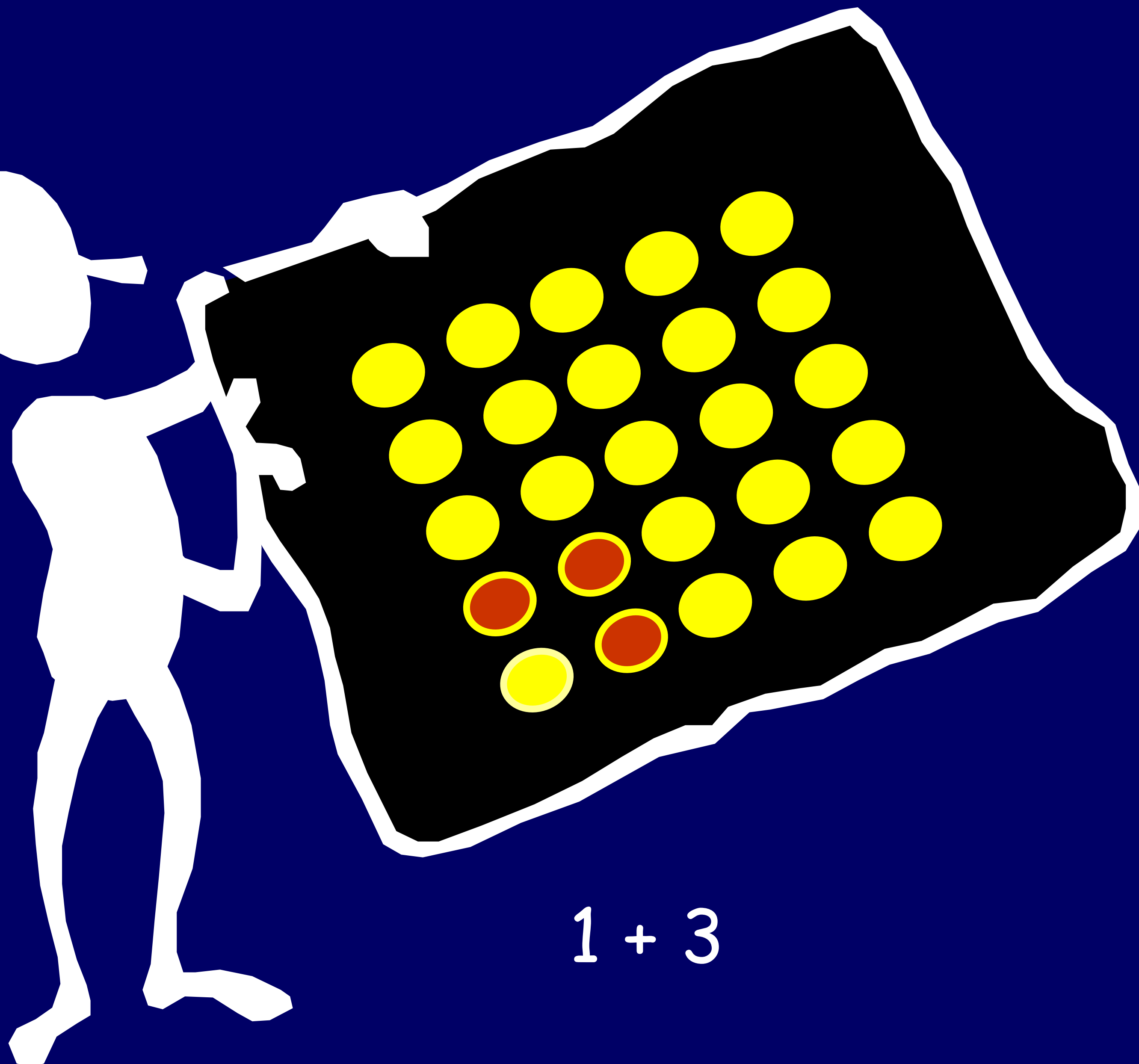
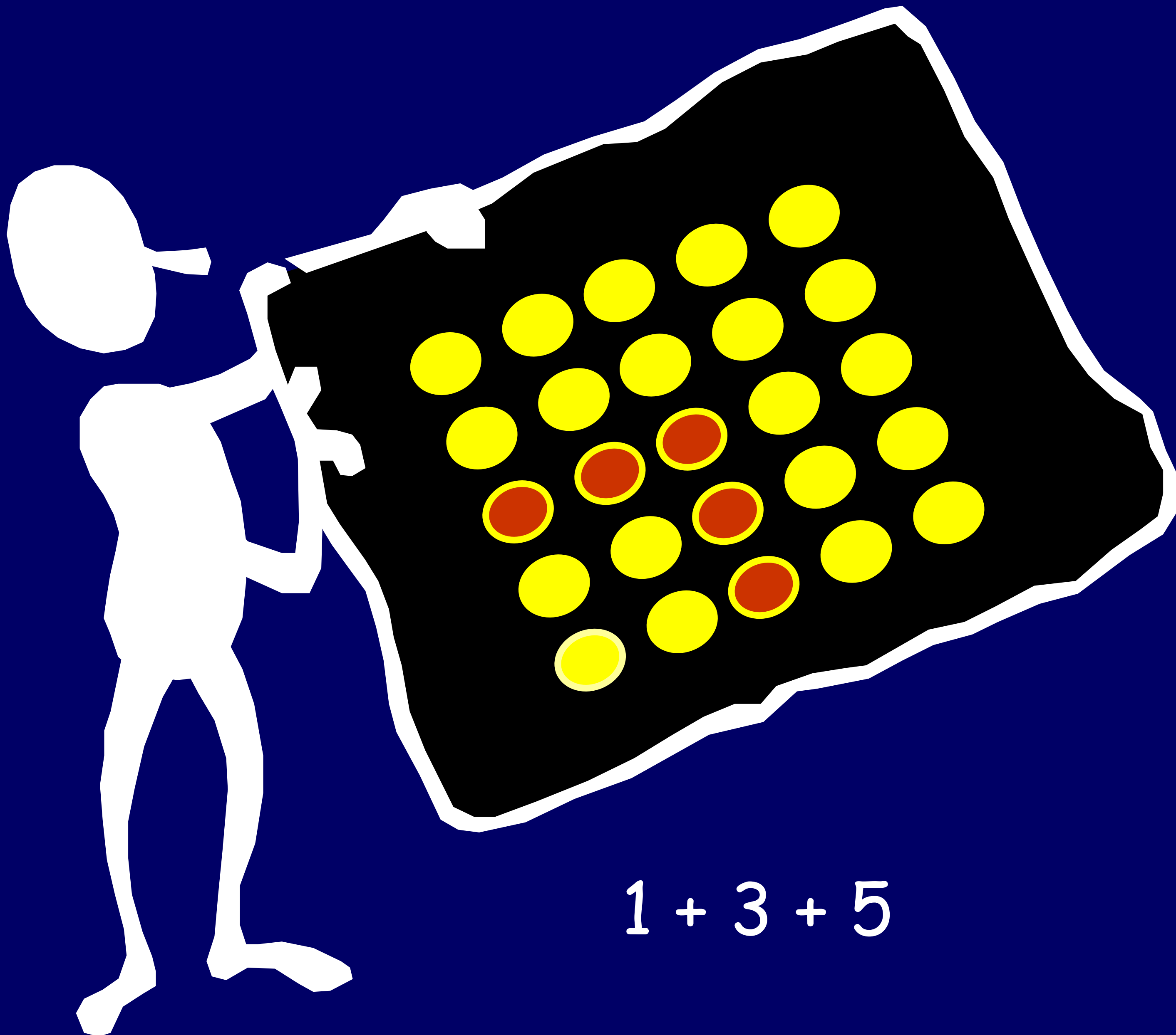
Breaking a square up in a new way



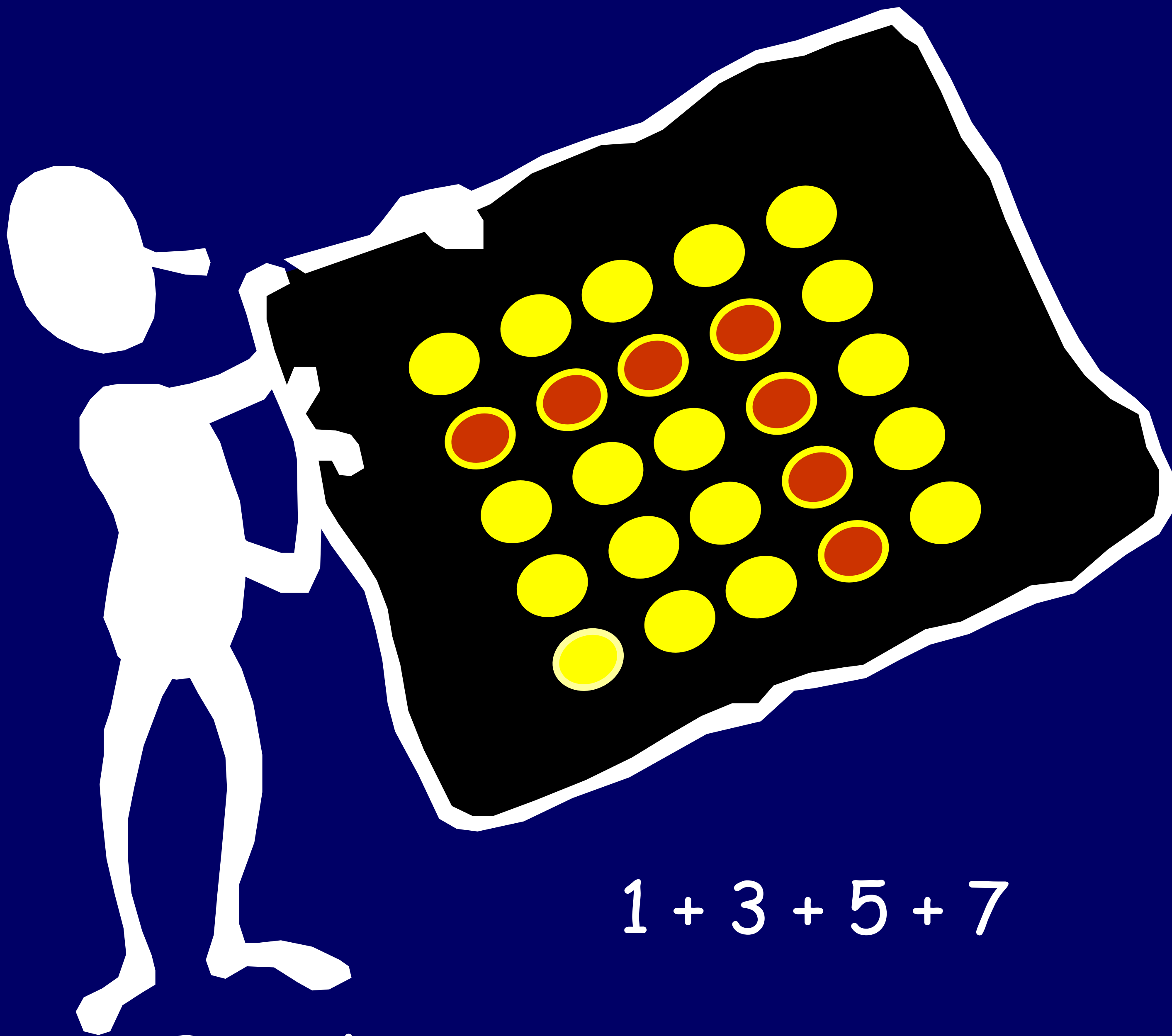
1 + 3

Breaking a square up in a new way



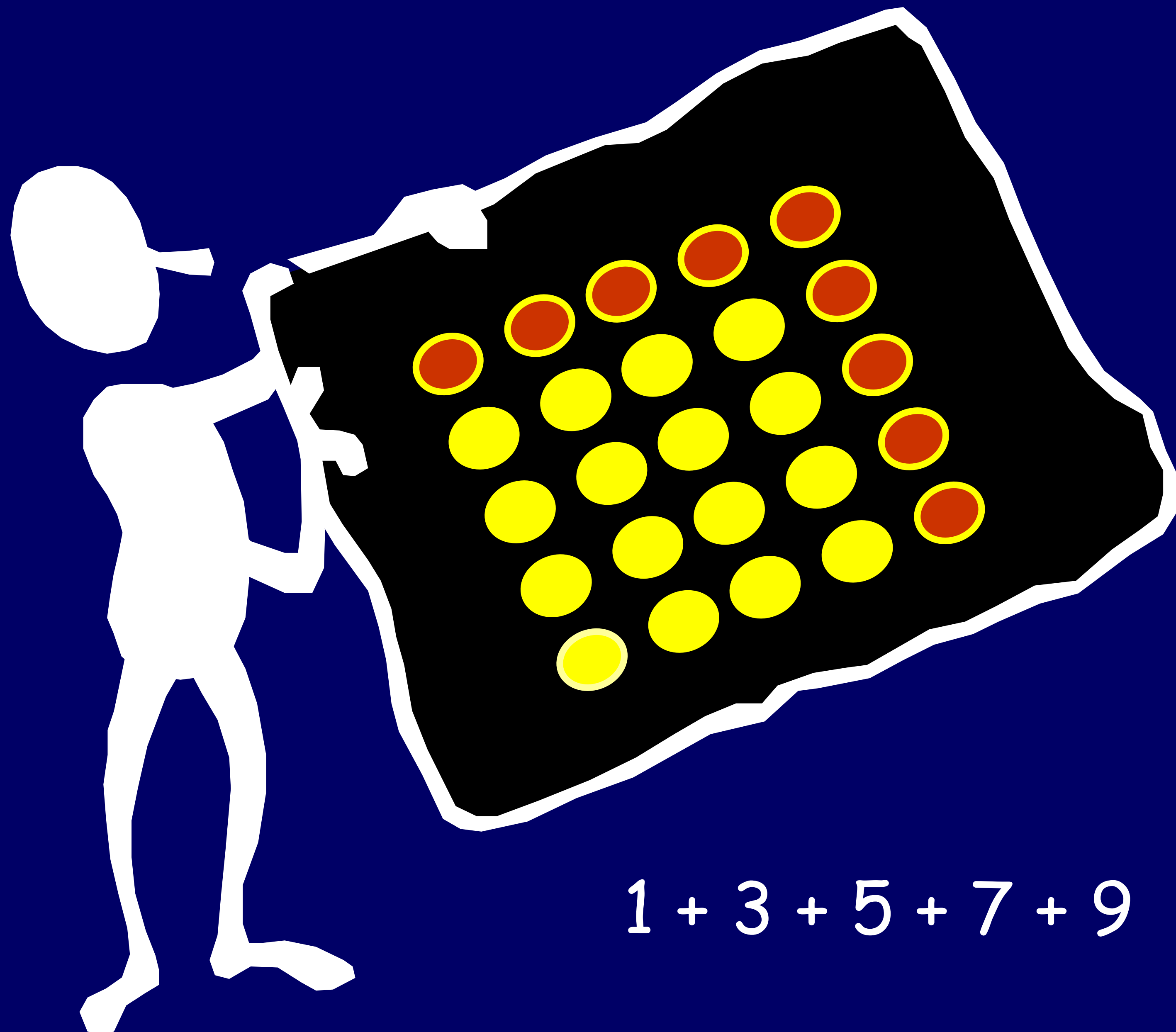
1 + 3 + 5

Breaking a square up in a new way



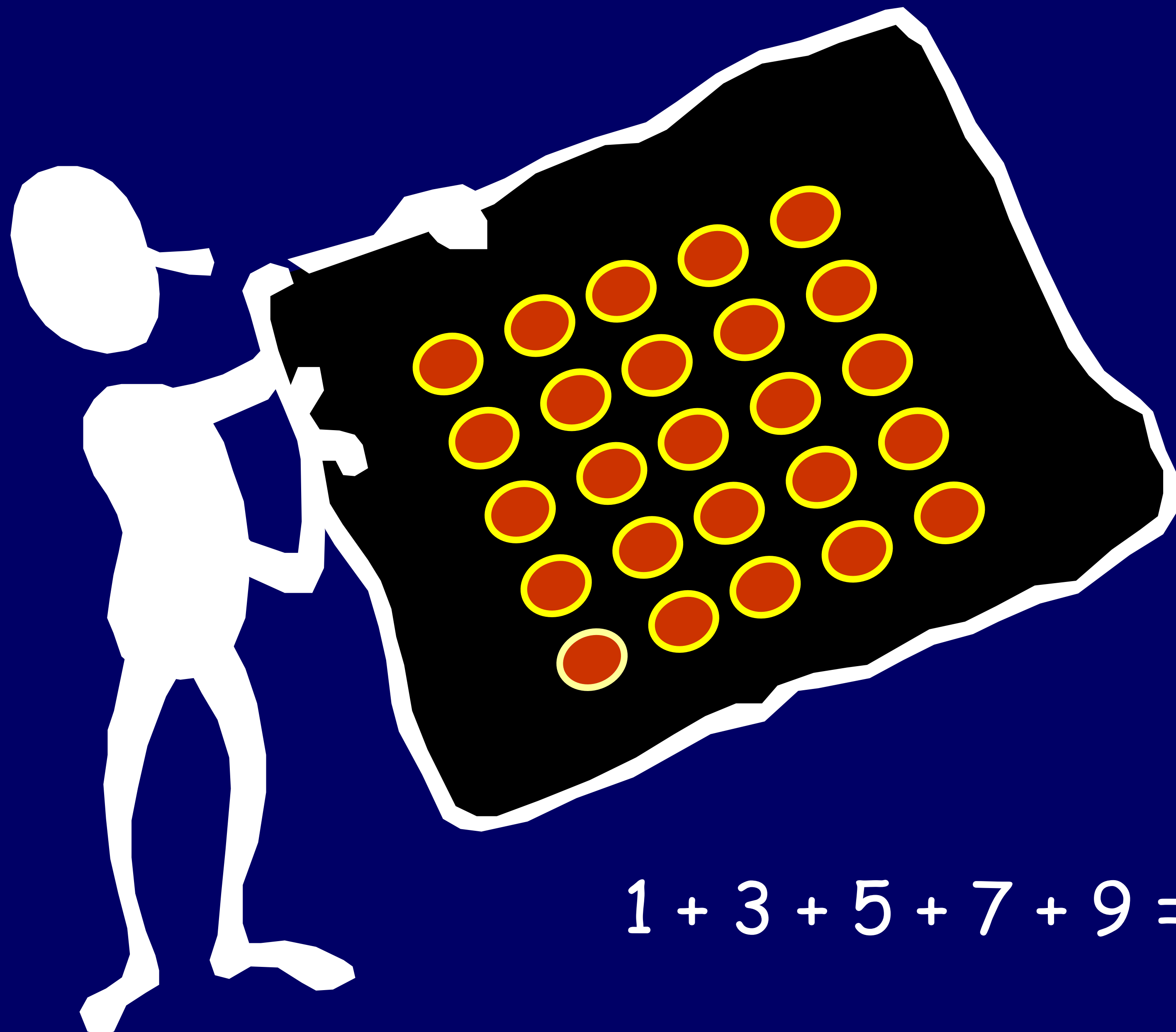
$1 + 3 + 5 + 7$

Breaking a square up in a new way



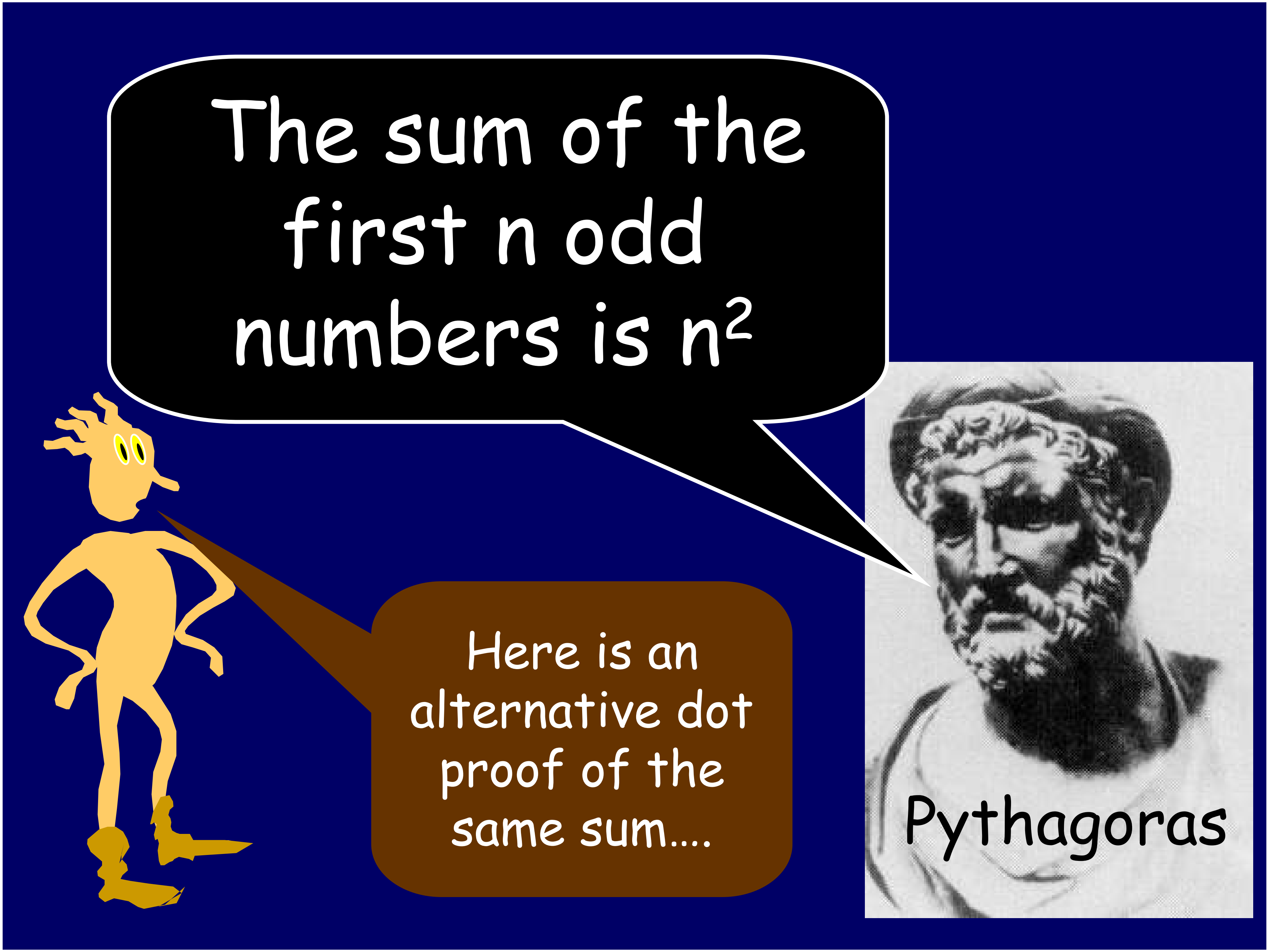
$1 + 3 + 5 + 7 + 9$

Breaking a square up in a new way



$$1 + 3 + 5 + 7 + 9 = 5^2$$

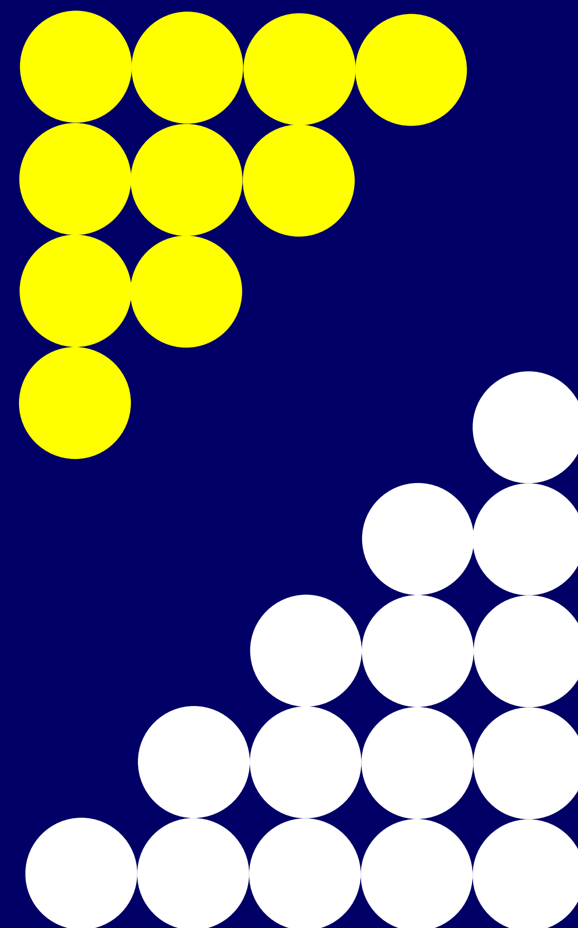
Breaking a square up in a new way


The sum of the first n odd numbers is n^2
Here is an alternative dot proof of the same sum....
Pythagoras

n^{th} Square Number

$$\square_n = \Delta_n + \Delta_{n-1}$$

$$= n^2$$



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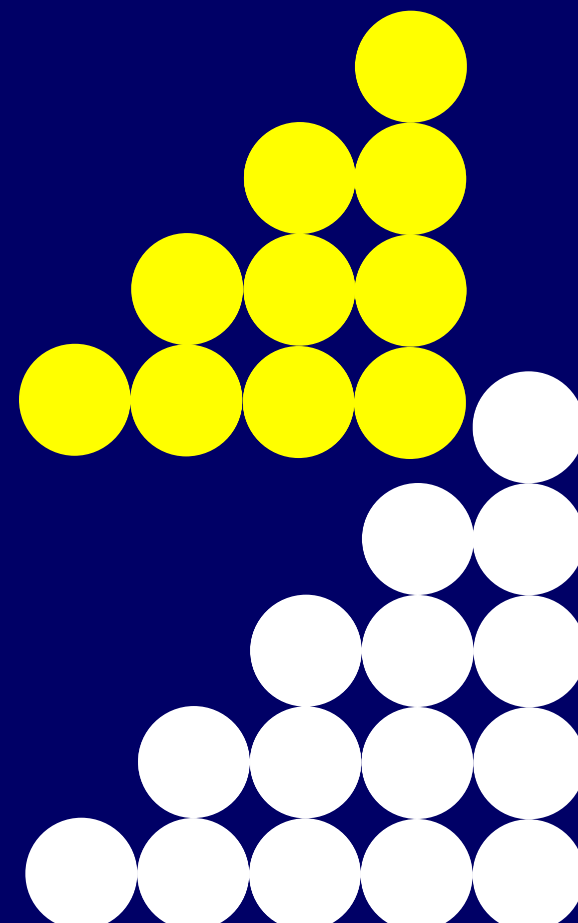
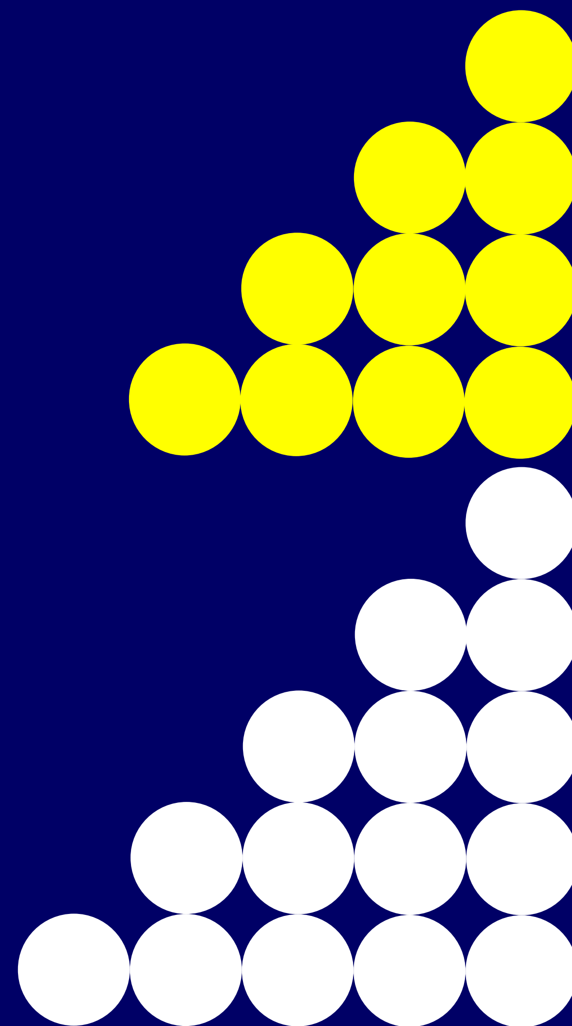


n^{th} Square Number

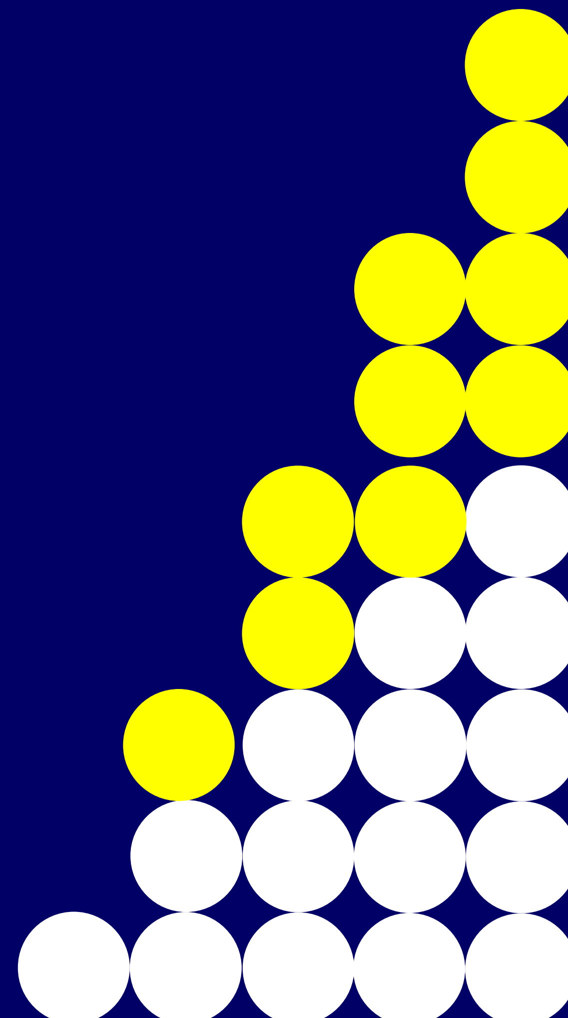
$$\square_n = \Delta_n + \Delta_{n-1}$$



n^{th} Square Number

$$\square_n = \Delta_n + \Delta_{n-1}$$

= Sum of first n odd numbers

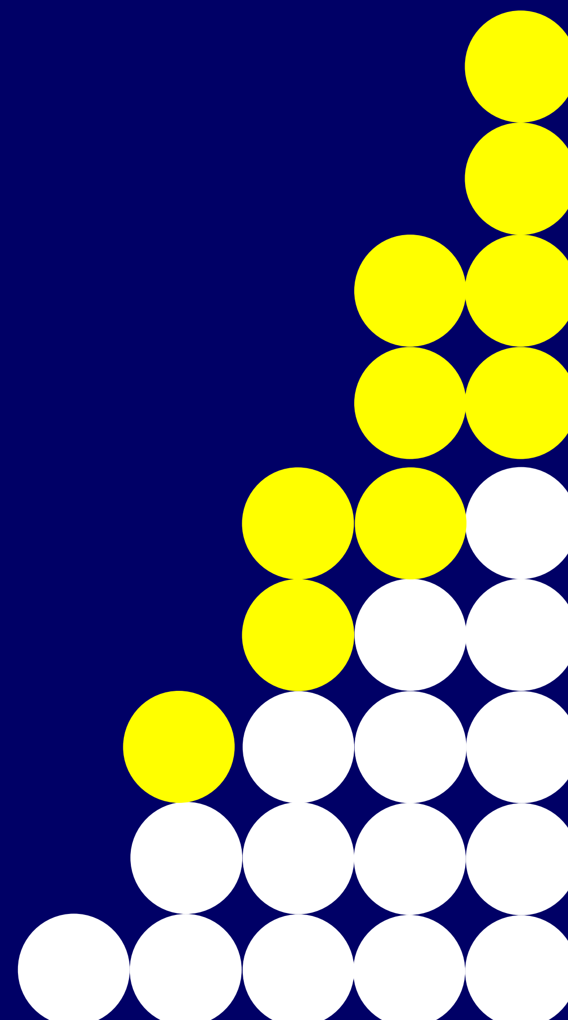


Same proof in high school notation

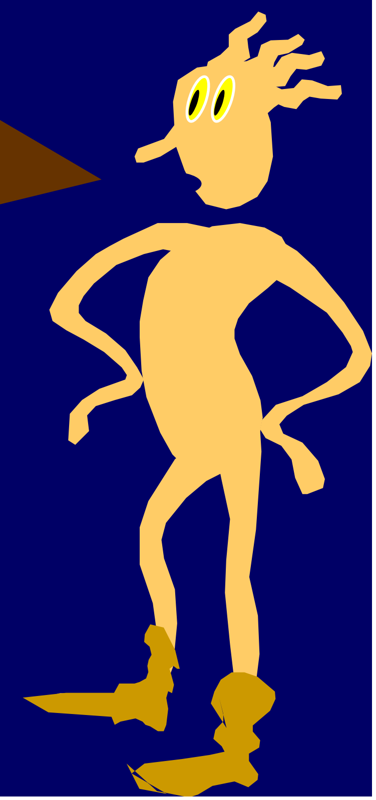
$$\Delta_n + \Delta_{n-1} =$$

$$\begin{array}{rl} & 1 + 2 + 3 + 4 \ldots \\ + & 1 + 2 + 3 + 4 + 5 \ldots \\ \hline & 1 + 3 + 5 + 7 + 9 \ldots \end{array}$$

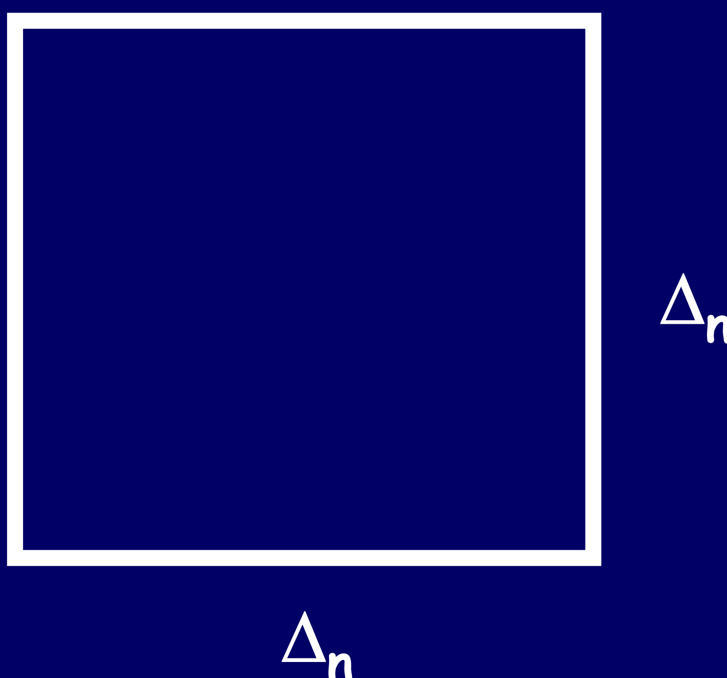
Sum of the first n odd numbers



Check the next one out…



Area of $\Delta_n \times \Delta_n$ square $= (\Delta_n)^2$



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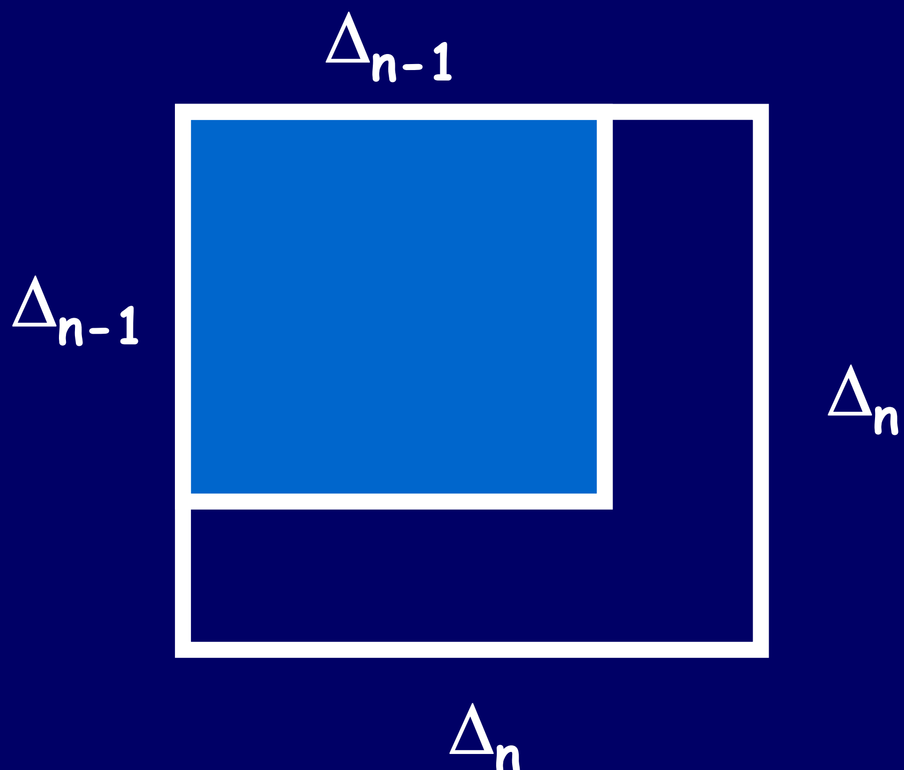

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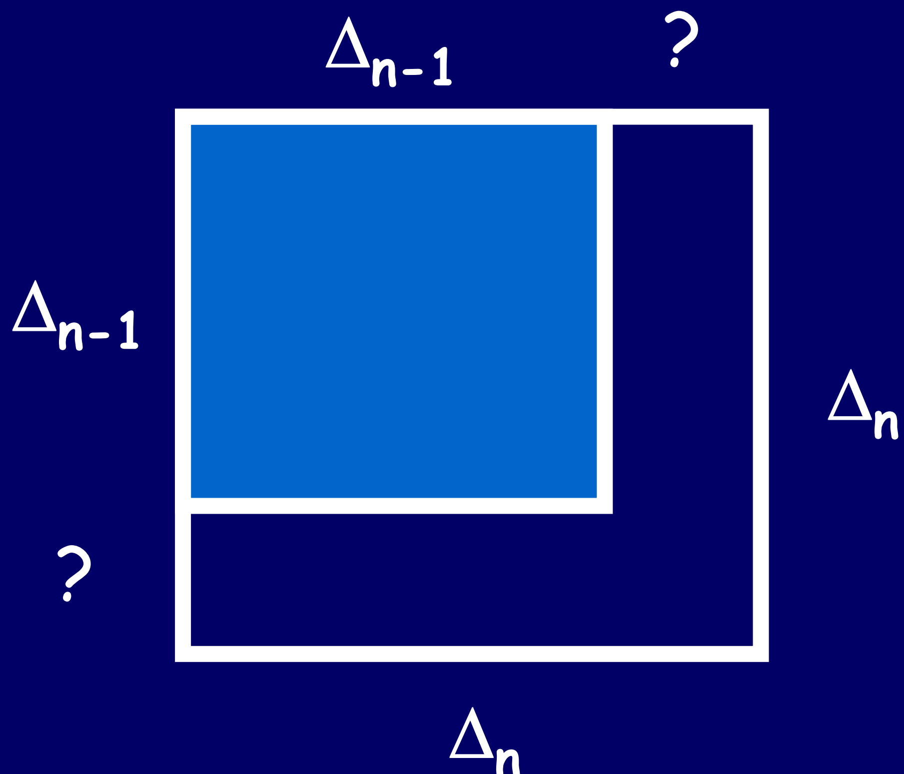

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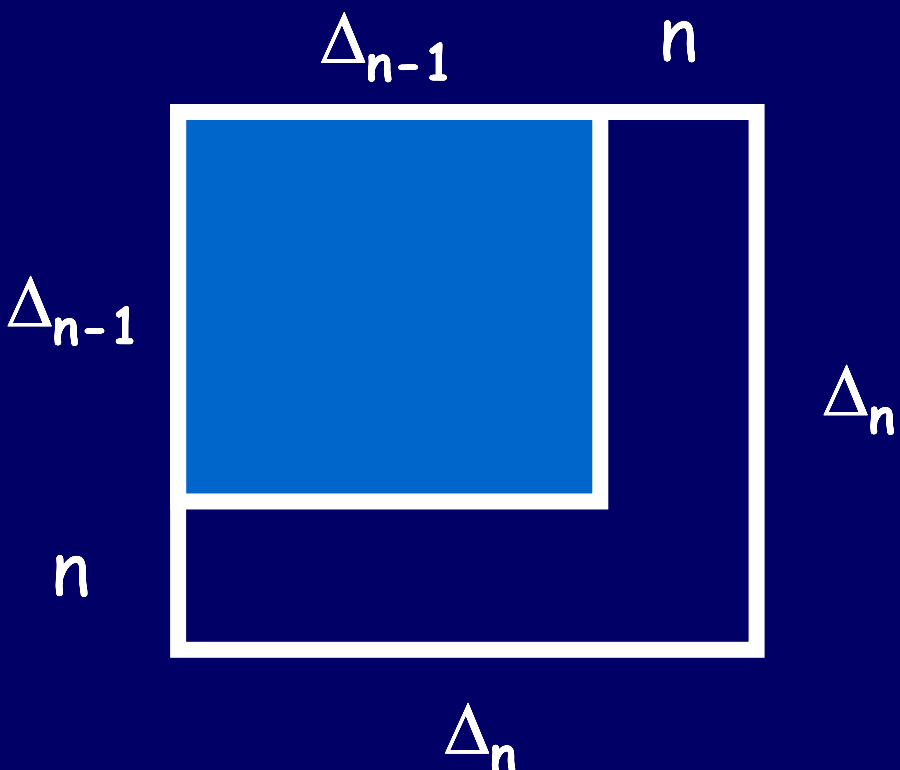

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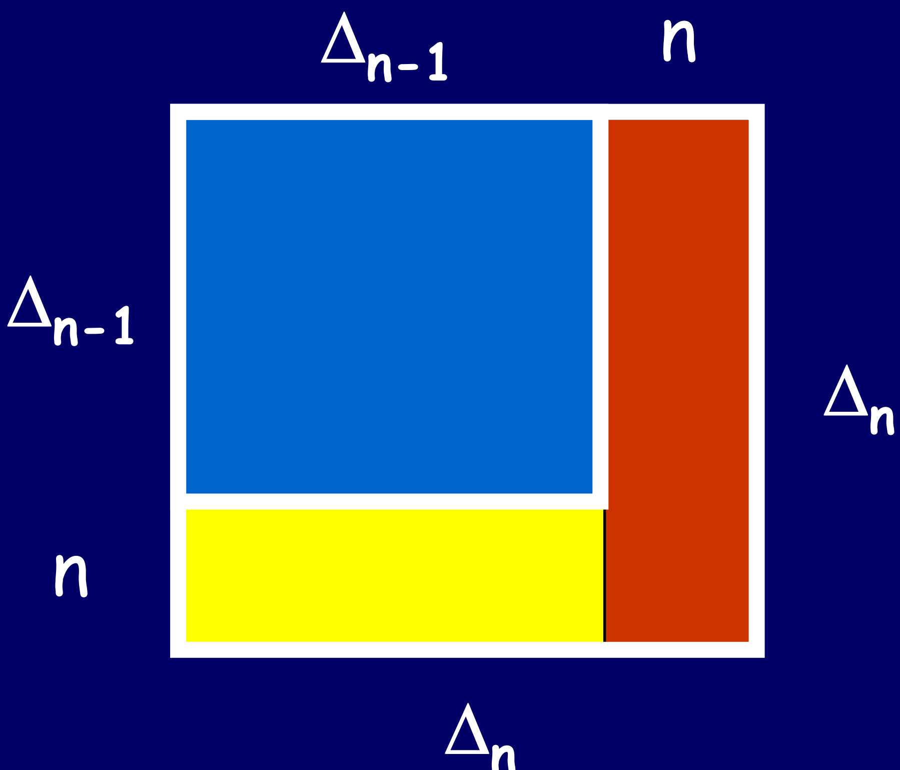


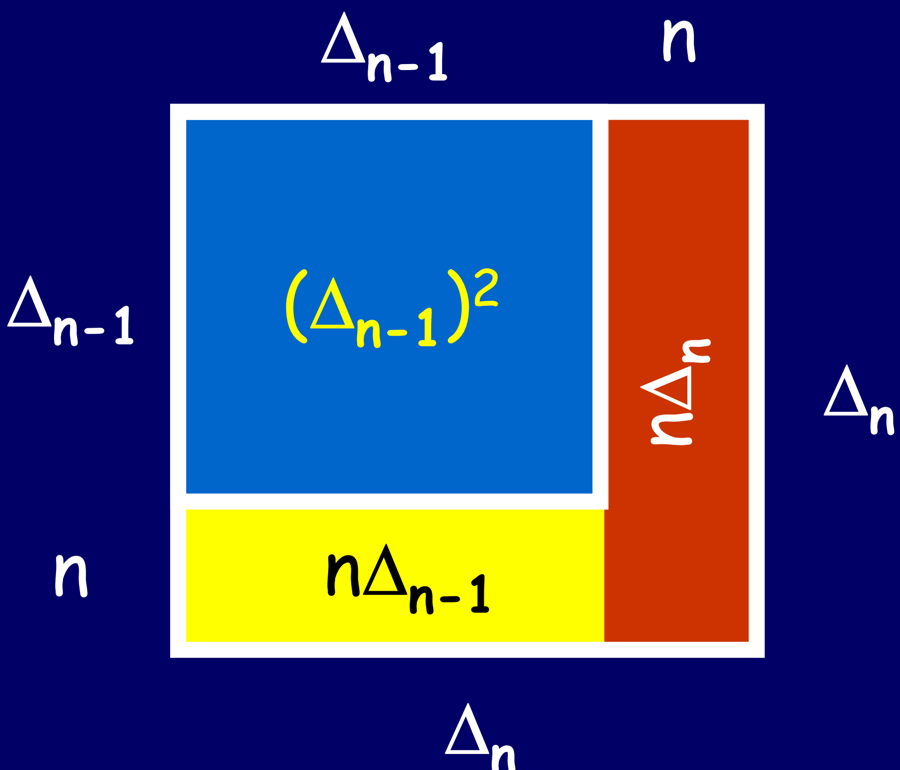
$$\text{Area of } \Delta_n \times \Delta_n \text{ square} = (\Delta_n)^2$$

$$= (\Delta_{n-1})^2 + n\Delta_{n-1} + n\Delta_n$$

$$= (\Delta_{n-1})^2 + n(\Delta_{n-1} + \Delta_n)$$

$$= (\Delta_{n-1})^2 + n(\square_n)$$

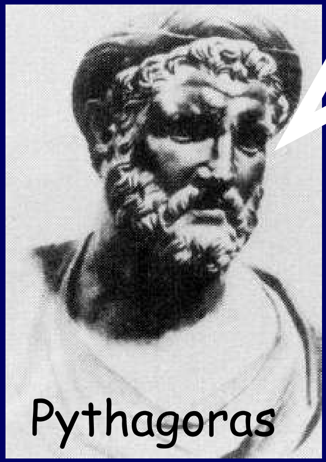
$$= (\Delta_{n-1})^2 + n^3$$



$$
\begin{aligned}
(\Delta_n)^2 &= n^3 + (\Delta_{n-1})^2 \\
&= n^3 + (n-1)^3 + (\Delta_{n-2})^2 \\
&= n^3 + (n-1)^3 + (n-2)^3 + (\Delta_{n-3})^2 \\
&= n^3 + (n-1)^3 + (n-2)^3 + \ldots + 1^3
\end{aligned}
$$

$$(\Delta_n)^2 = 1^3 + 2^3 + 3^3 + \ldots + n^3$$

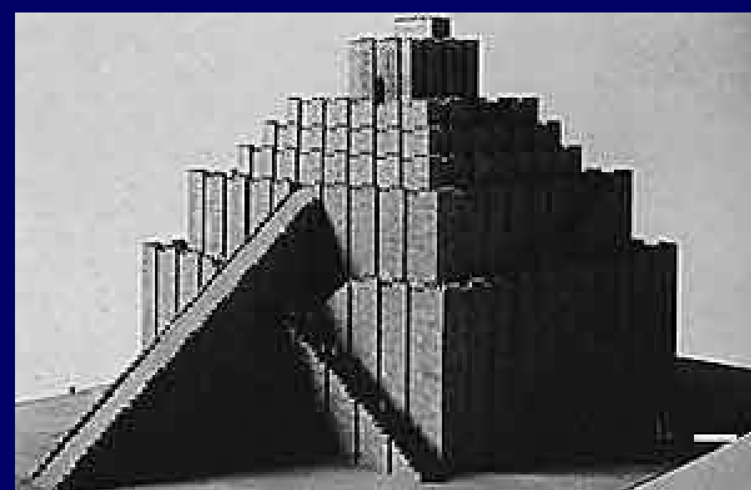
$$= [\ n(n+1)/2\ ]^2$$

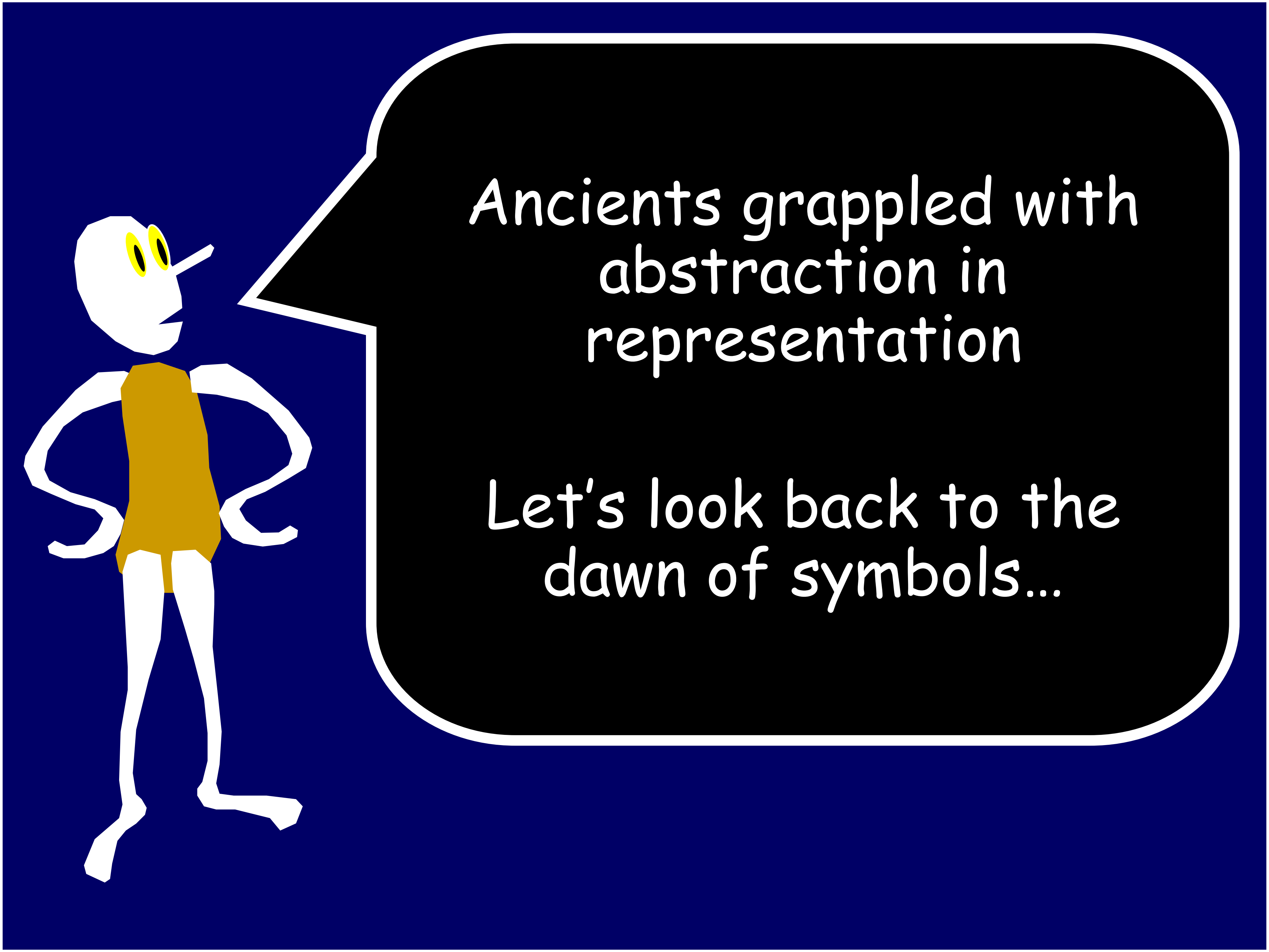


Can you find a formula for the sum of the first n squares?

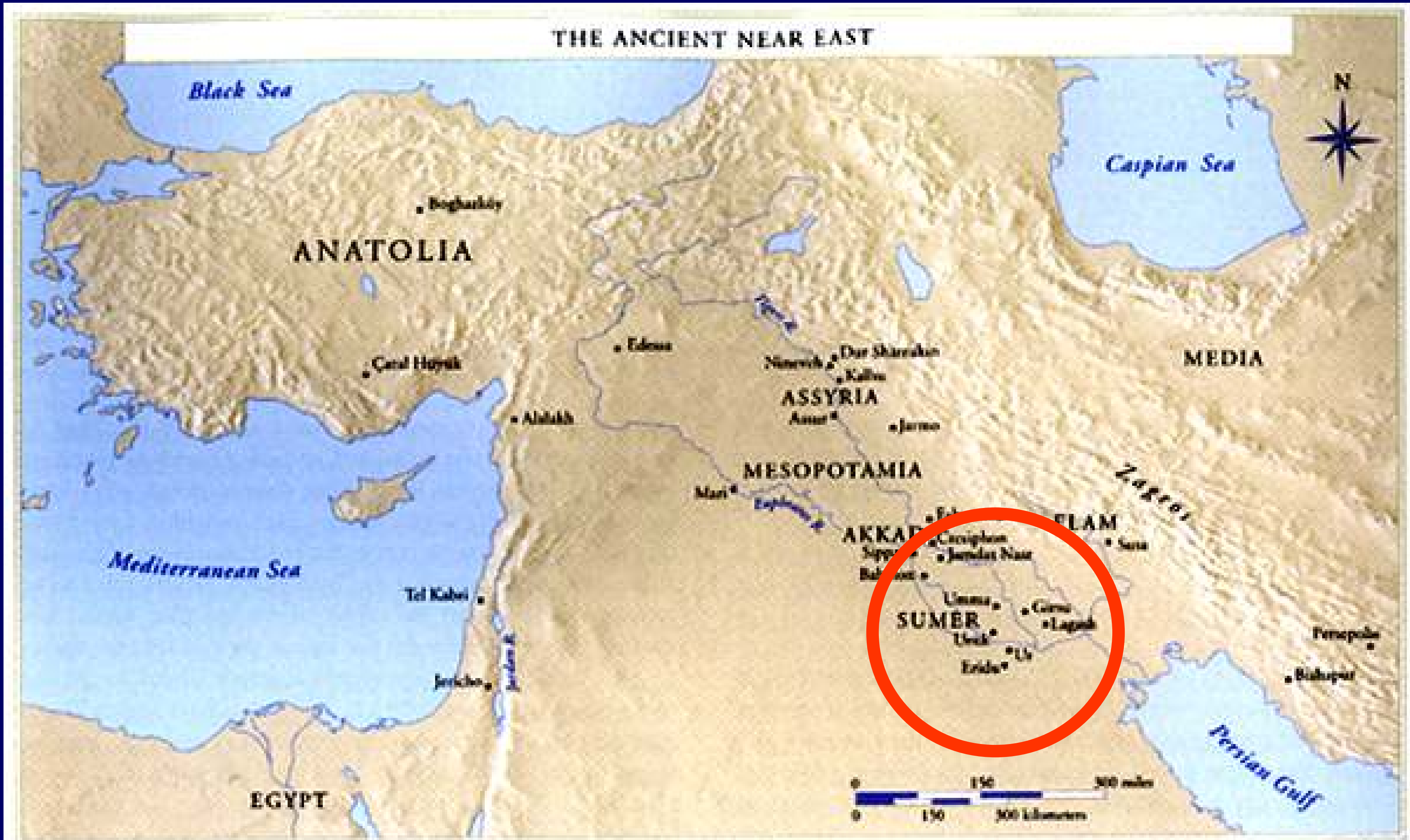
Babylonians needed this sum to compute the number of blocks in their pyramids

$$\boxed{\frac{n(n+1)(2n+1)}{6}}$$




Ancients grappled with abstraction in representation
Let's look back to the dawn of symbols…

Sumerians [modern Iraq]



Sumerians [modern Iraq]

8000 BC	Sumerian tokens use multiple symbols to represent numbers
3100 BC	Develop Cuneiform writing
2000 BC	Sumerian tablet demonstrates base 10 notation (no zero), solving linear equations, simple quadratic equations

Biblical timing: Abraham born in the Sumerian city of Ur in 2000 BC

Babylonians Absorb Sumerians

1900 BC Sumerian/Babylonian Tablet:
Sum of first n numbers
Sum of first n squares
"Pythagorean Theorem"
"Pythagorean Triples"
some bivariate equations

1600 BC Babylonian Tablet:
Take square roots
Solve system of n linear equations

Egyptians

6000 BC Multiple symbols for numbers

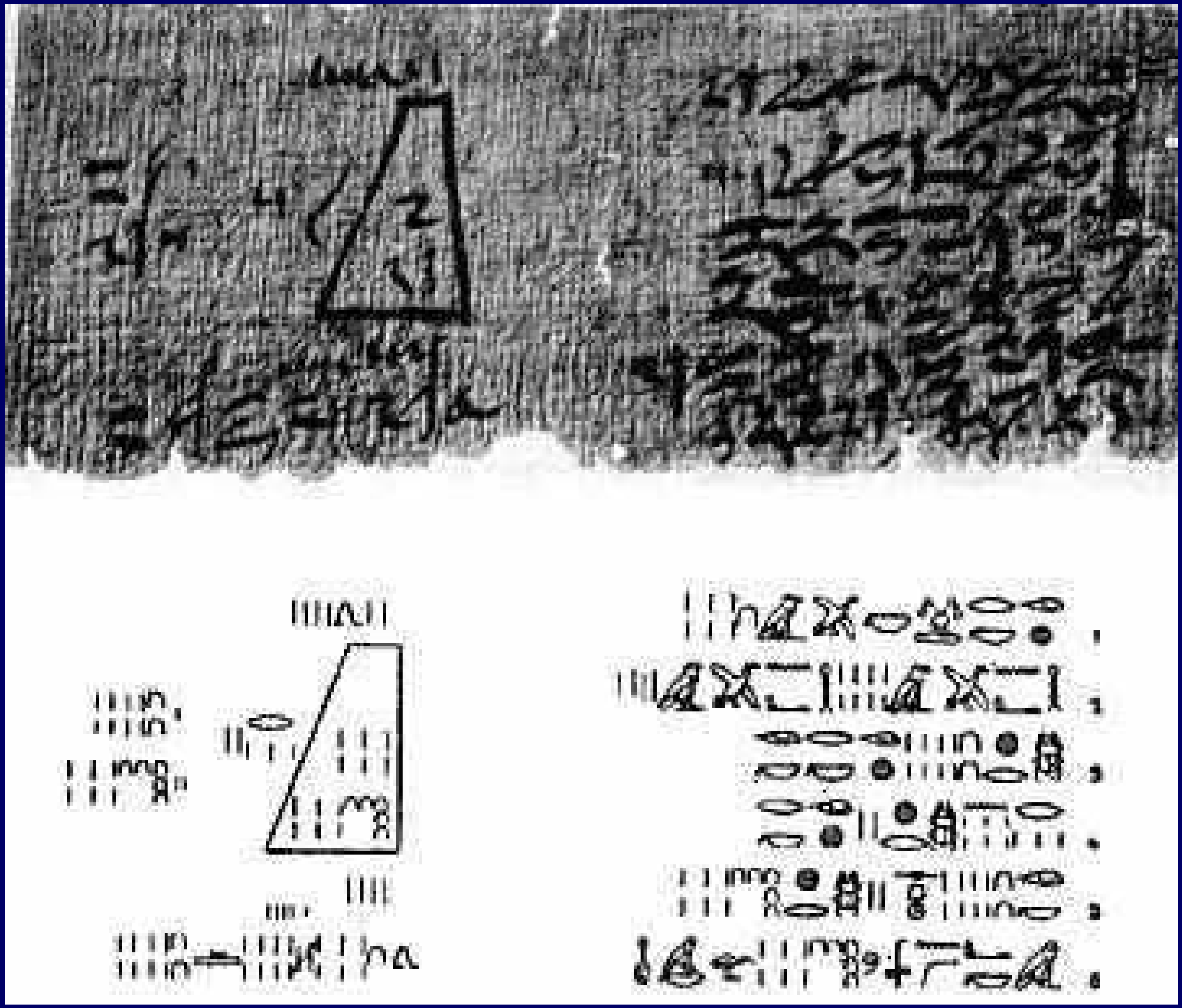
3300 BC Developed Hieroglyphics

1850 BC Moscow Papyrus:
Volume of truncated pyramid

1650 BC Rhind Papyrus [Ahmes/Ahmose]:
Binary Multiplication/Division
Sum of 1 to n
Square roots
Linear equations

Biblical timing: Joseph Governor is of Egypt

Moscow Papyrus



Harrappans

[Indus Valley Culture] Pakistan/India

3500 BC Perhaps the first writing system?!

2000 BC Had a uniform decimal system of weights and measures

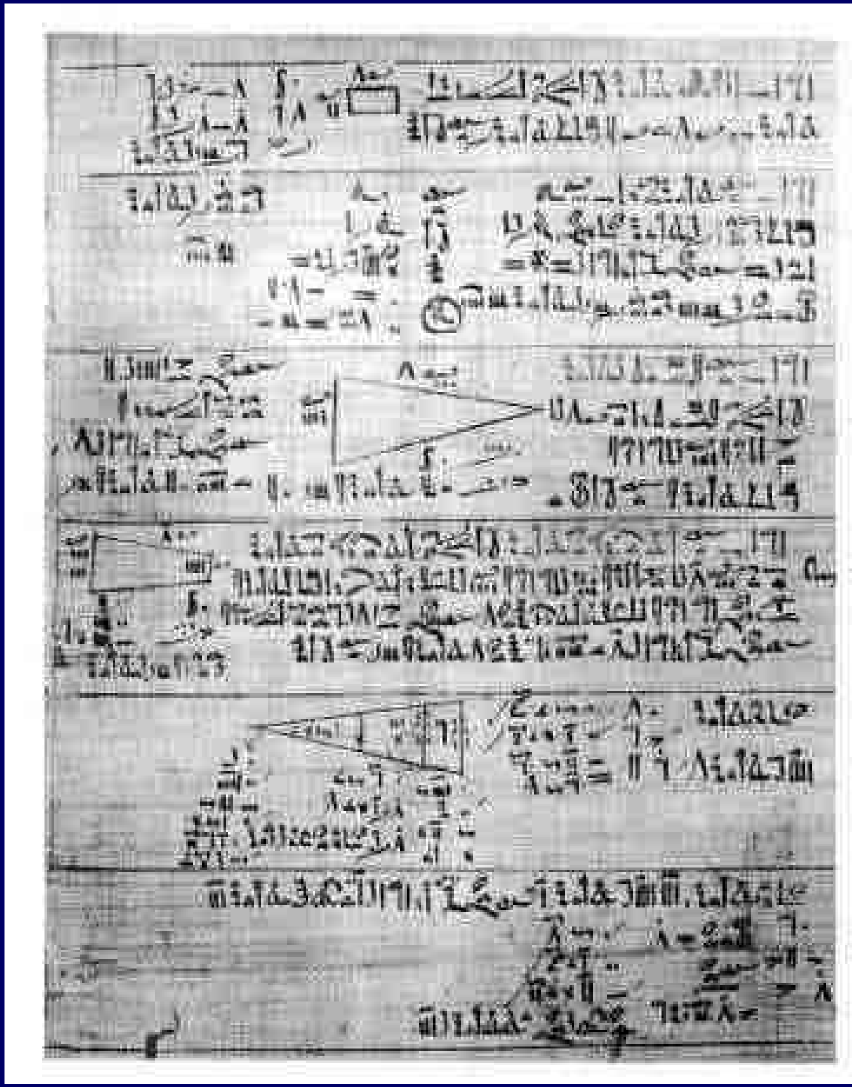
China

1200 BC Independent writing system
(Surprisingly late)

1200 BC I Ching [Book of changes]:
Binary system developed to do numerology

Rhind Papyrus

Scribe Ahmes was Martin Gardener of his day!



Rhind Papyrus

Scribe Ahmes was Martin Gardener of his day!

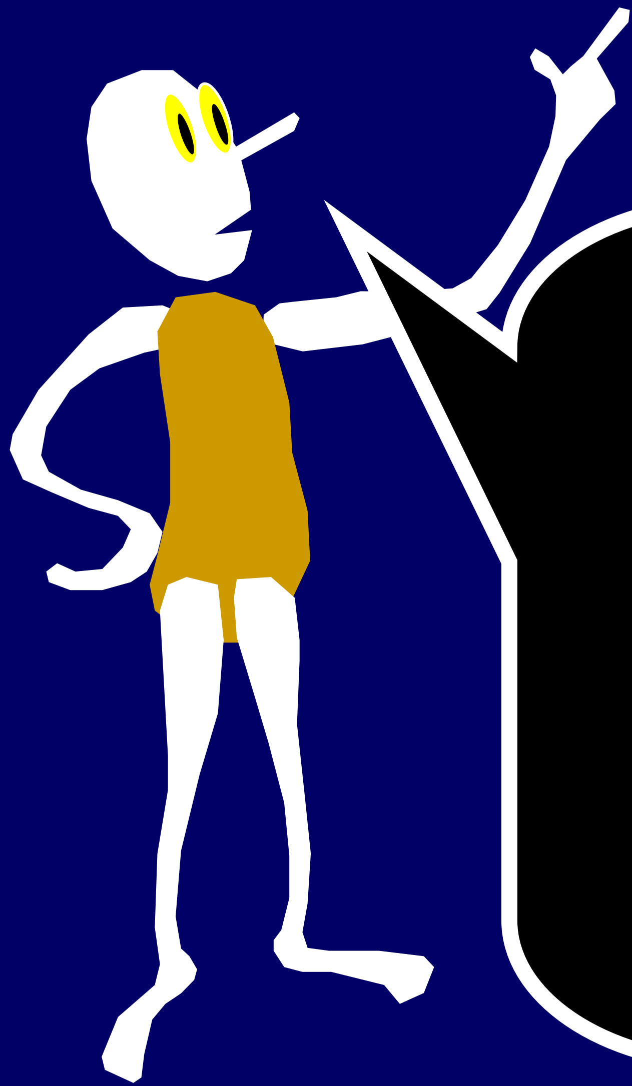
Rhind Papyrus

Scribe Ahmes was Martin Gardener of his day!

A man has 7 houses,
Each house contains 7 cats,
Each cat has killed 7 mice,
Each mouse had eaten 7 ears of spelt,
Each ear had 7 grains on it.
What is the total of all of these?

Sum of powers of 7

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$



We'll use this fundamental sum again and again:

The Geometric Series

A Frequently Arising Calculation

$$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1})$$

$$= \quad \cancel{X^1} + \cancel{X^2} + \cancel{X^3} + \ldots + \cancel{X^{n-2}} + \cancel{X^{n-1}} + \boxed{X^n}$$

$$\boxed{-\ 1} - \cancel{X^1} - \cancel{X^2} - \cancel{X^3} - \ldots - \cancel{X^{n-2}} - \cancel{X^{n-1}}$$

$$= \quad X^n - 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

(when $x \neq 1$)

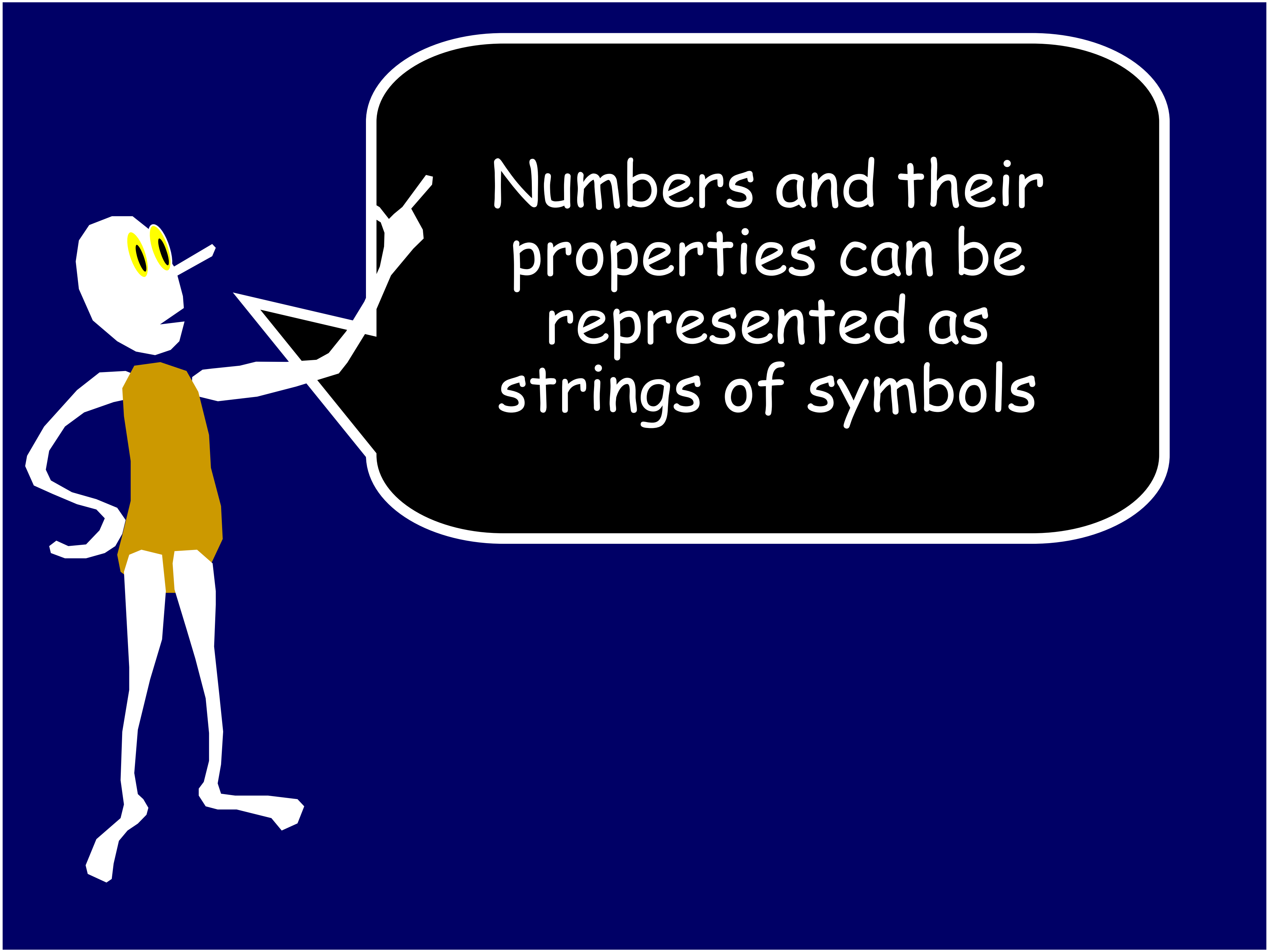
Geometric Series for X=2

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^{n-1} = \frac{2^n - 1}{2 - 1}$$

$$1 + X^1 + X^2 + X^3 + \dots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

(when $x \neq 1$)

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}} = \frac{\left(\frac{1}{2}\right)^n - 1}{1/2 - 1}$$


Numbers and their properties can be represented as strings of symbols

Strings Of Symbols

We take the idea of symbol and sequence of symbols as primitive

Let Σ be any fixed finite set of symbols.
Σ is called an alphabet, or a set of symbols

Examples:

$\Sigma = \{0,1,2,3,4\}$

$\Sigma = \{a,b,c,d, \ldots, z\}$

Σ = all typewriter symbols

$\Sigma = \{ \alpha , \beta , \partial , \delta , \ldots, z \}$

Strings Over the Alphabet Σ

A string is a sequence of symbols from Σ

Let s and t be strings

Then st denotes the concatenation of s and t
i.e., the string obtained by the string s followed by the string t

Now define Σ^+ by these inductive rules:

$$x \in \Sigma \Rightarrow x \in \Sigma^+$$

$$s,t \in \Sigma^+ \Rightarrow st \in \Sigma^+$$

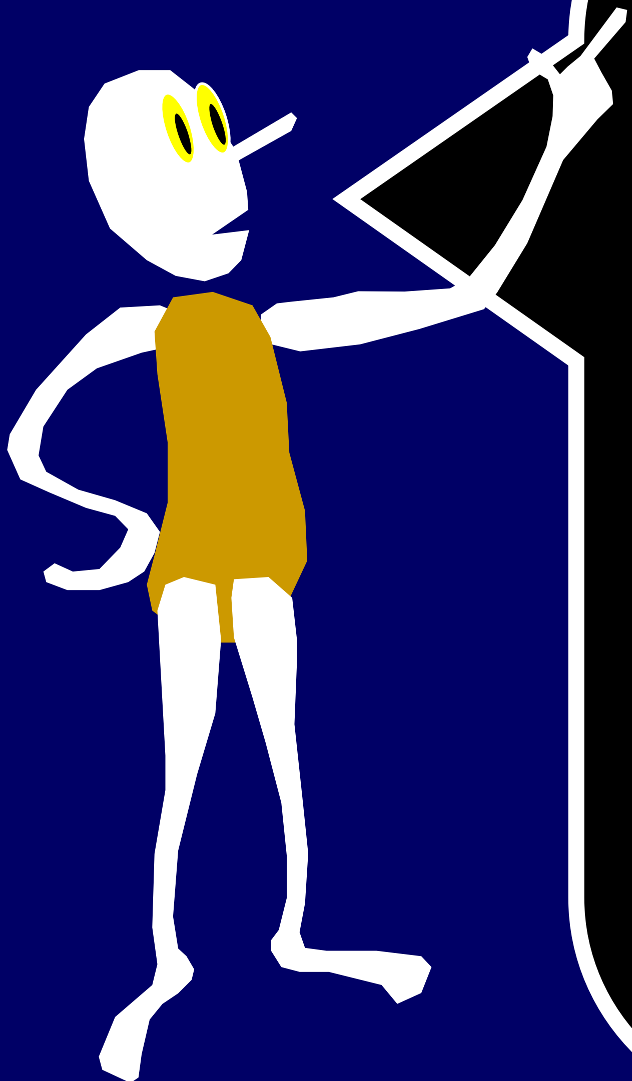
The Set Σ^*

Define ε to be the empty string

i.e., $X\varepsilon Y = XY$ for all strings X and Y

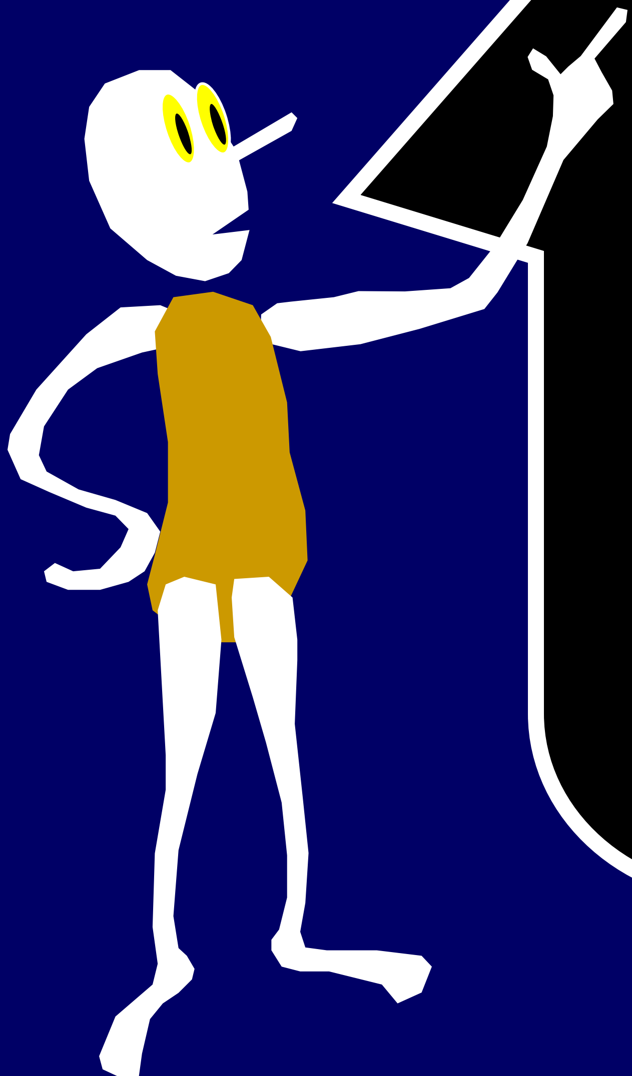
ε is also called the string of length 0

Define $\Sigma^* = \Sigma^+ \cup \{\varepsilon\}$



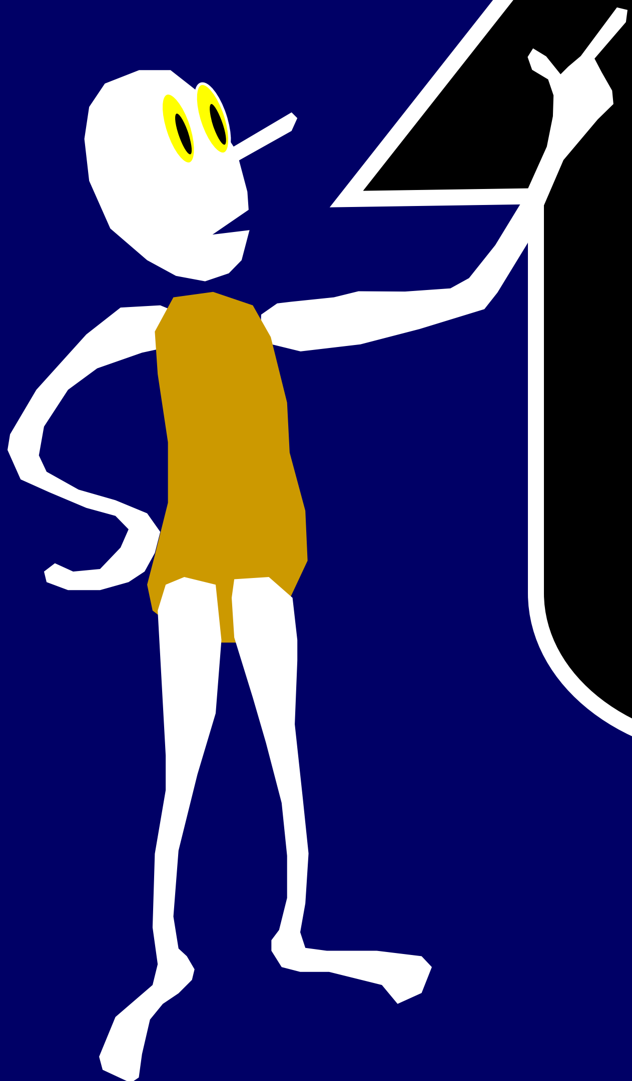
Σ^{+} is set of all finite strings that we can make using (at least one) letters from Σ

Σ^{*} is the set of all finite strings that we can make using letters from Σ, including the empty string.



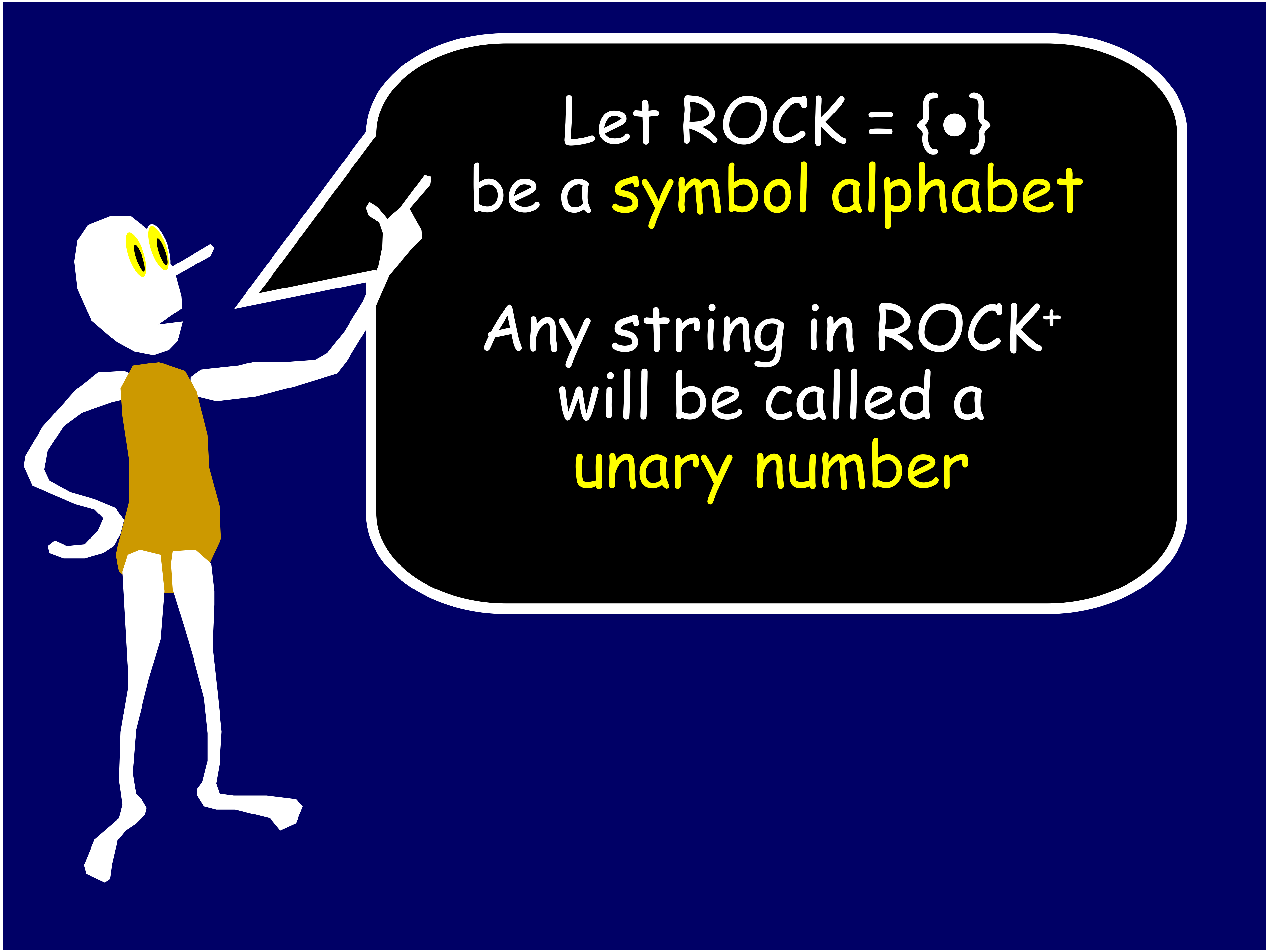
Let DIGITS =
{0,1,2,3,4,5,6,7,8,9}
be a symbol alphabet

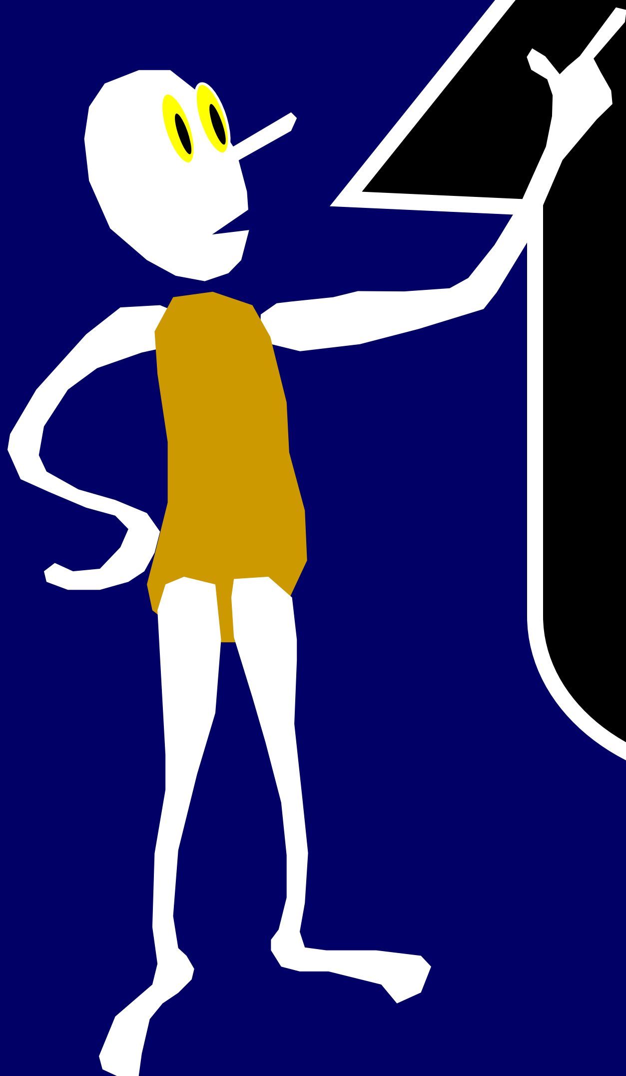
Any string in
DIGITS^+ will be
called a
decimal number



Let BITS = {0,1}
be a symbol alphabet

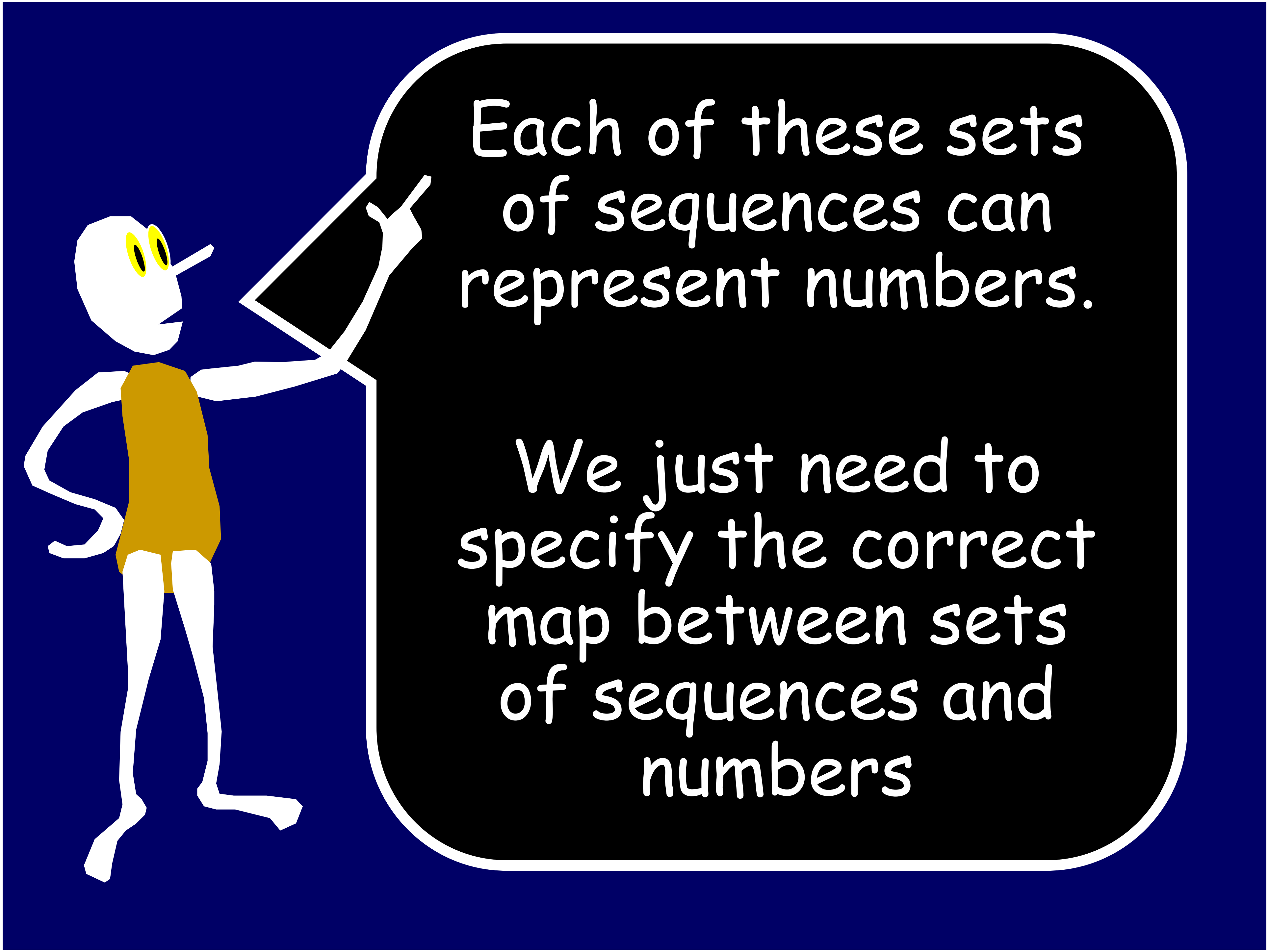
Any string in BITS$^+$
will be called a
binary number

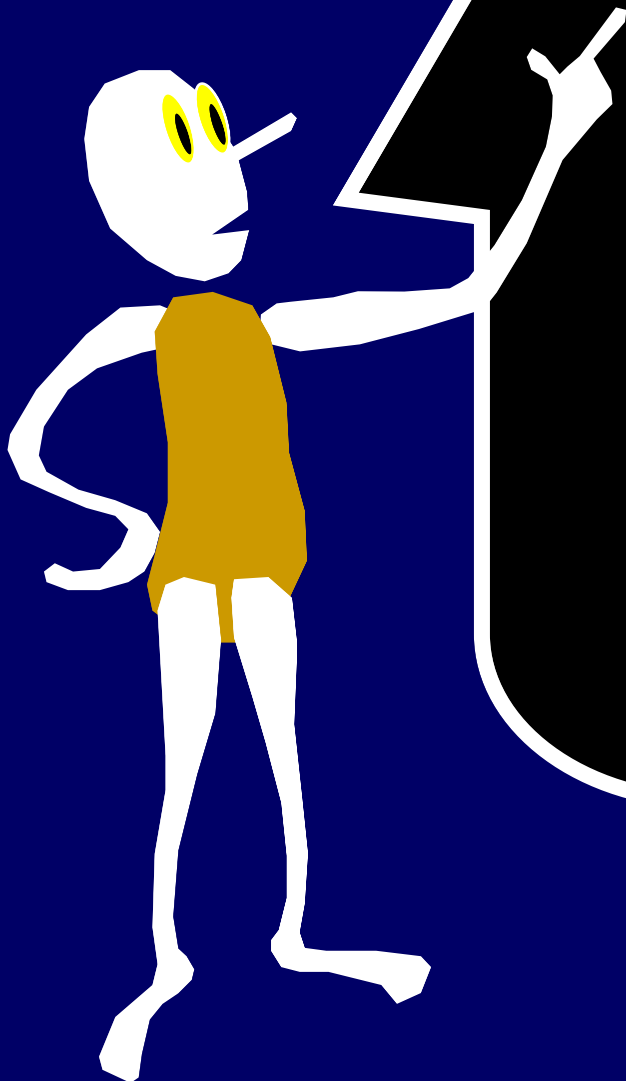

Let ROCK = {•}
be a symbol alphabet
Any string in ROCK+
will be called a
unary number



Let BASE-X = {0,1,2,…,X-1} be a **symbol alphabet**

Any string in BASE-X^+ will be called a **base-X number**


Each of these sets of sequences can represent numbers.
We just need to specify the correct map between sets of sequences and numbers

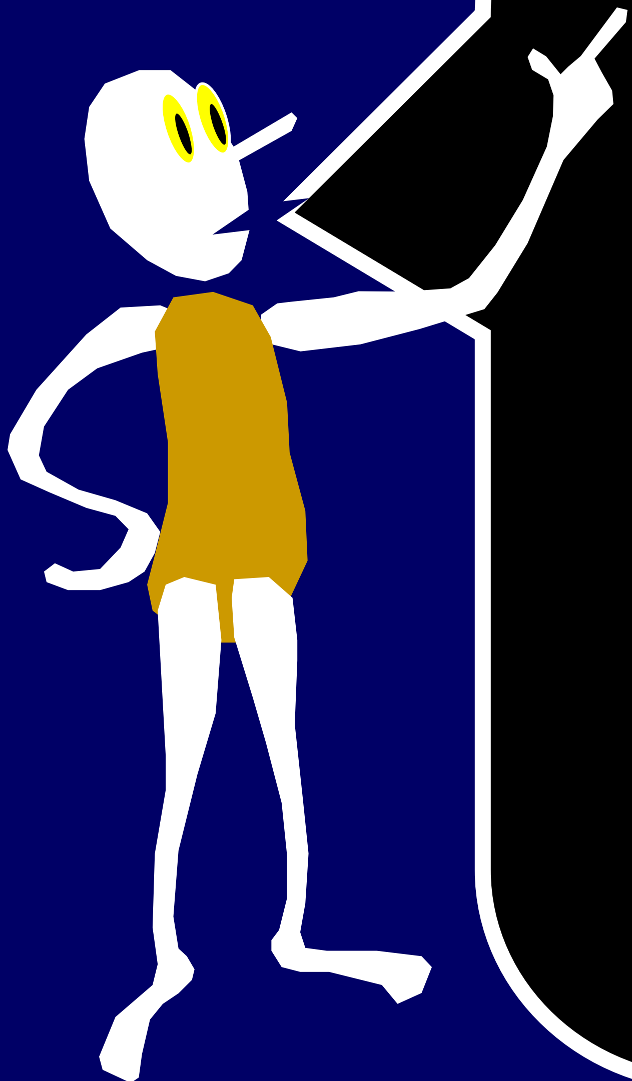


Inductively defined function

$f : \mathrm{ROCK}^{+} \rightarrow \mathbb{N}$

$f(\bullet) = 1$

$f(\bullet\ X) = f(X) + 1$

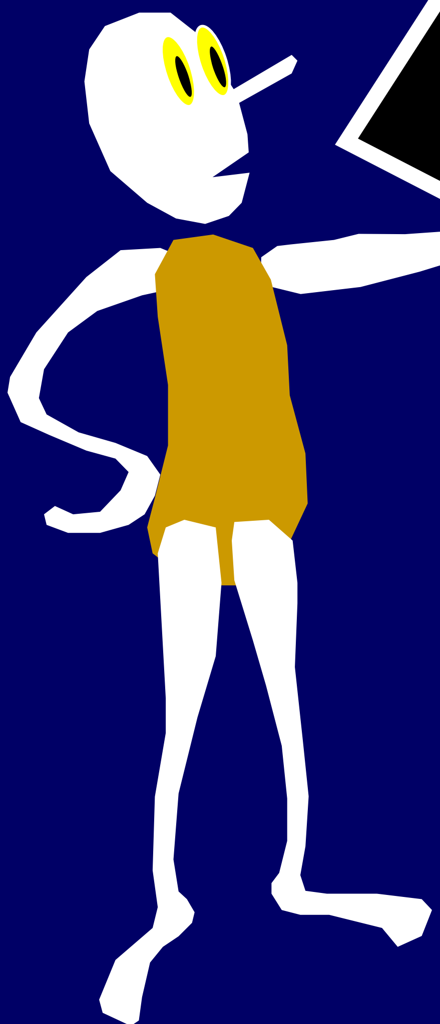


Inductively defined function

$f : \text{BITS}^{+} \rightarrow \mathbb{N}$

$f(0) = 0$; $f(1) = 1$

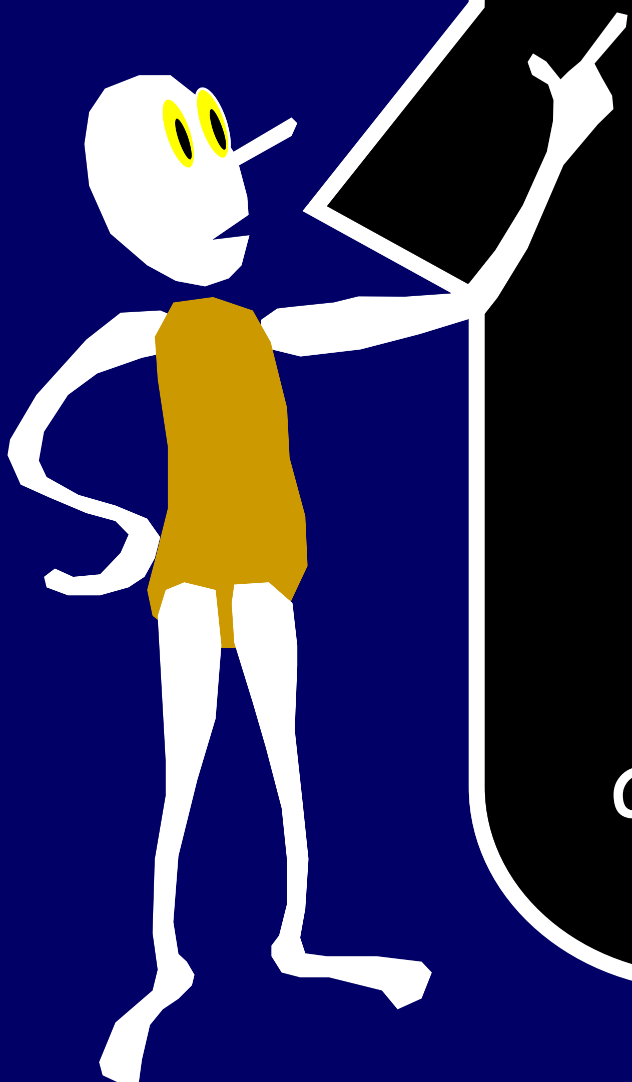
If $|W| > 1$ then
$W = Xb$
(b is a bit)
$f(Xb) = 2f(X) + b$



Non-inductive representation of f:

$$f(a_{n-1}\, a_{n-2}\, \dots\, a_0) =$$

$$a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \dots + a_0 2^0$$



Two identical maps from sequences to numbers:

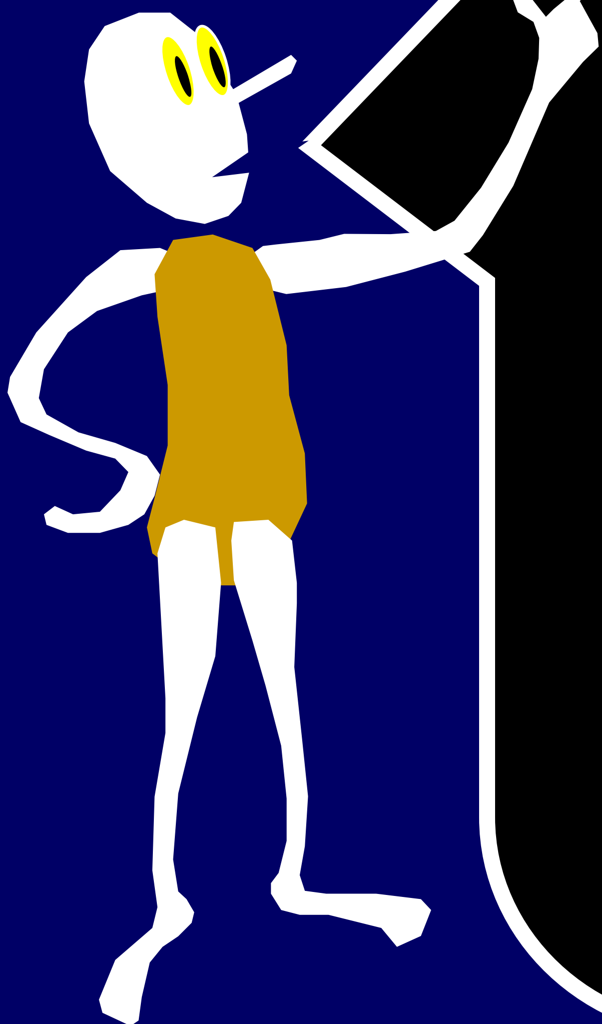
$$f(0) = 0;\ f(1) = 1$$

$$f(Xb) = 2f(X) + b$$

and

$$f(a_{n-1}\, a_{n-2} \ldots a_0) =$$

$$a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \ldots + a_0 2^0$$



The symbol a_0 is called the Least Significant Bit or the Parity Bit

$$a_0 = 0$$

iff

$$f(a_{n-1}a_{n-2}\ldots a_0) = a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \ldots + a_0 2^0$$

is an even number

Theorem: Each natural has a binary representation

Base Case: 0 and 1 do

Induction hypothesis: Suppose all natural numbers less than n have a binary representation

Induction Step: Note that $n = 2m+b$ for some $m < n$, with $b = 0$ or 1

Represent n as the left-shifted sequence for m concatenated with the symbol for b

No Leading Zero Binary (NLZB)

A binary string that is either 0 or 1,
Or has length > 1, and does not have a leading zero

1	Is in NLZB
000001101001	Is NOT in NLZB
0	Is in NLZB
01	Is NOT in NLZB
10000001	Is in NLZB

Theorem: Each natural has a unique NLZB representation

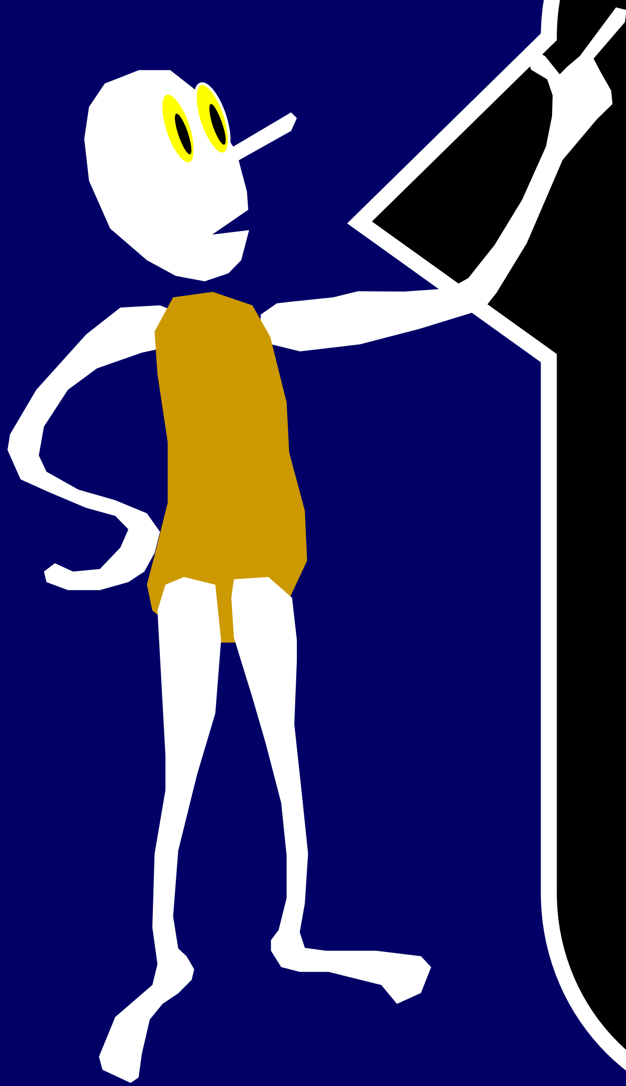
Base Case: 0 and 1 do

Induction hypothesis: Suppose all natural numbers less than n have a unique NLZB representation

Induction Step: Suppose $n = 2m+b$ has 2 NLZB representations

Their parity bit b must be identical

Hence, m also has two distinct NLZB representations, which contradicts the induction hypothesis. So n must have a unique representation



Inductive definition is great for showing UNIQUE representation:

$$f(Xb) = 2f(X) + b$$

Let n be the smallest number reprinted by two different binary sequences. They must have the same parity bit, thus we can make a smaller number that has distinct representations


Each natural number has a unique representation as a (No Leading Zeroes) Binary number!

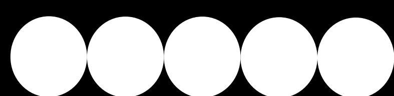
BASE X Representation

$S = a_{n-1}\ a_{n-2} \ldots a_1\ a_0$ represents the number:

$$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$$

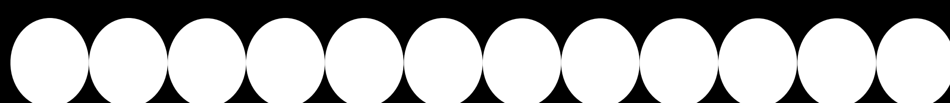
Base 2 [Binary Notation]

101 represents: $1\,(2)^2 + 0\,(2^1) + 1\,(2^0)$

= 

Base 7

015 represents: $0\,(7)^2 + 1\,(7^1) + 5\,(7^0)$

= 

Bases In Different Cultures

Sumerian-Babylonian: 10, 60, 360
Egyptians: 3, 7, 10, 60
Maya: 20
Africans: 5, 10
French: 10, 20
English: 10, 12, 20

BASE 10 Representation

$S = (a_{n-1}\ a_{n-2} \ldots a_1\ a_0)_{10}$ represents the number:

$$a_{n-1}\, 10^{n-1} + a_{n-2}\, 10^{n-2} + \ldots + a_0\, 10^0$$

Largest number representable in base-10 with n "digits"

$= (999999\ldots9)_X$ [with n 9's]

$= 9 \times (10^{n-1} + 10^{n-2} + \ldots + 10^0)$

$= (10^n - 1)$

BASE X Representation

$S = (a_{n-1}\ a_{n-2} \ldots a_1\ a_0)_X$ represents the number:

$$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$$

Largest number representable in base-X with n "digits"

$$= (X\text{-}1 \quad X\text{-}1 \quad X\text{-}1 \quad X\text{-}1 \quad X\text{-}1 \quad \ldots \quad X\text{-}1)_X$$

$$= (X-1)(X^{n-1} + X^{n-2} + \ldots + X^0)$$

$$= (X^n - 1)$$

Fundamental Theorem For Binary

Each of the numbers from 0 to 2^n-1 is uniquely represented by an n-bit number in binary

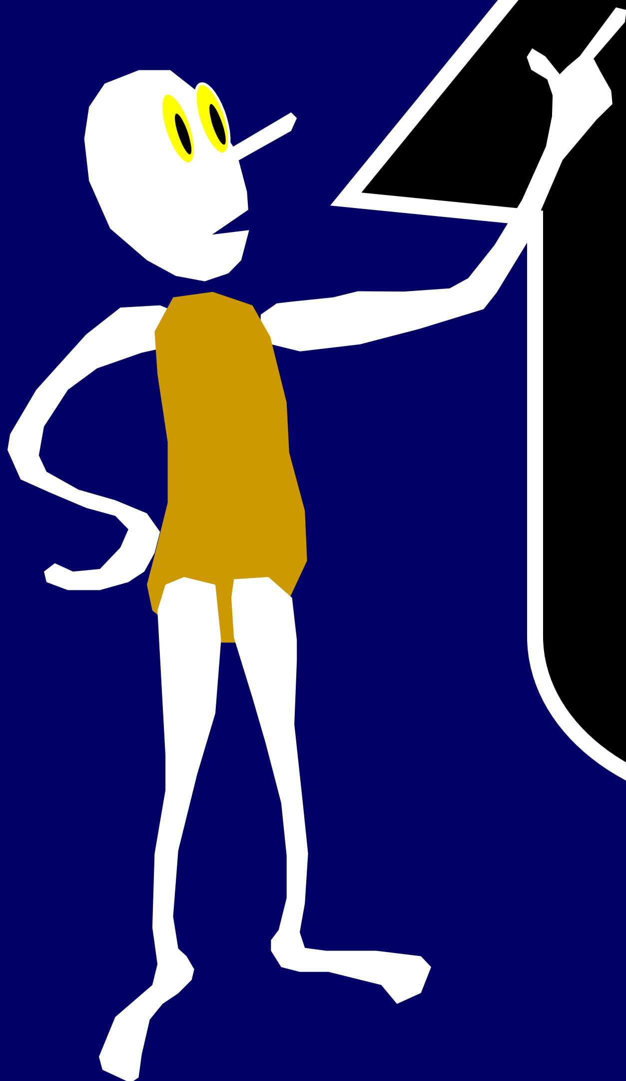
k uses $\lfloor \log_2 k \rfloor + 1$ digits in base 2

$\lceil \log_2(k+1) \rceil$

Fundamental Theorem For Base-X

Each of the numbers from 0 to X^n-1 is uniquely represented by an n-"digit" number in base X

k uses $\lfloor \log_X k \rfloor + 1$ digits in base X



n has length n in unary,
but has length
$\lfloor \log_2 n \rfloor + 1$ in binary

Unary is exponentially
longer than binary

Other Representations: Egyptian Base 3

Conventional Base 3:
Each digit can be 0, 1, or 2

$a_2 a_1 a_0$

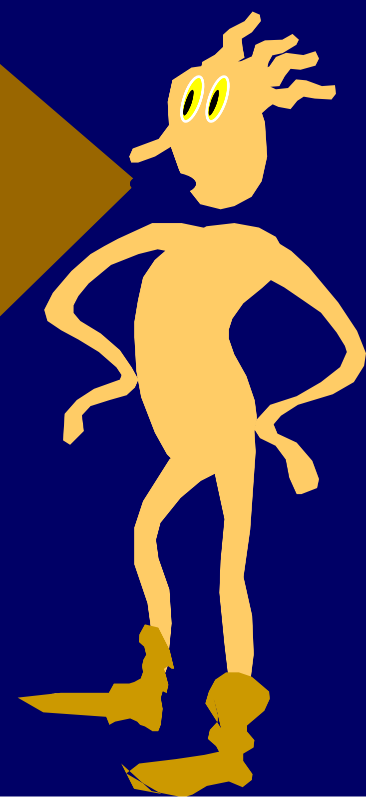
$a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0$

Here is a strange new one:
Egyptian Base 3 uses -1, 0, 1

Example: 1 -1 -1 = 9 - 3 - 1 = 5

We can prove a unique representation theorem

How could this be Egyptian? Historically, negative numbers first appear in the writings of the Indian mathematician Brahmagupta (628 AD)



One weight for each power of 3
Left = “negative”. Right = “positive”

Study Bee

Unary and Binary

Triangular Numbers

Dot proofs

Geometric sum

$(1+x+x^2 + \ldots + x^{n-1}) = (x^n - 1)/(x-1)$

Base-X representations

unique binary representations

proof for no-leading zero binary

k uses $\lfloor \log_2 k \rfloor + 1 = \lceil \log_2 (k+1) \rceil$ digits in base 2

Largest length n number in base X