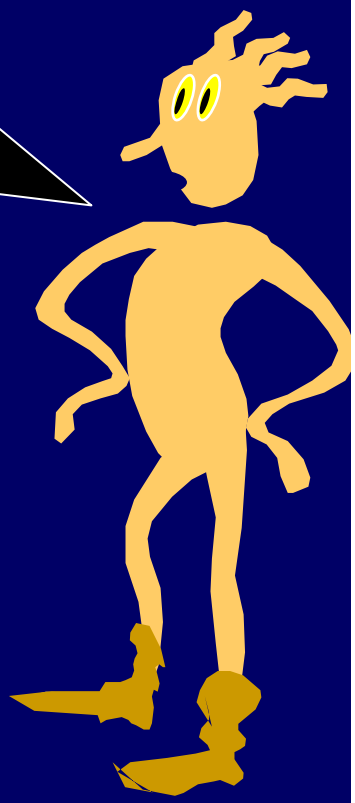
Great Theoretical Ideas In Computer Science

(Steven Rudich) John Lafferty — CS 15-251 — Fall 2005

Lecture 20 Nov. 3, 2005 — Carnegie Mellon University

Today, we will learn about a formidable tool in probability that will allow us to solve problems that seem <u>really really messy</u>…

If I randomly put 100 letters into 100 addressed envelopes, <u>on average</u> how many letters will end up in their correct envelopes?



Hmm…

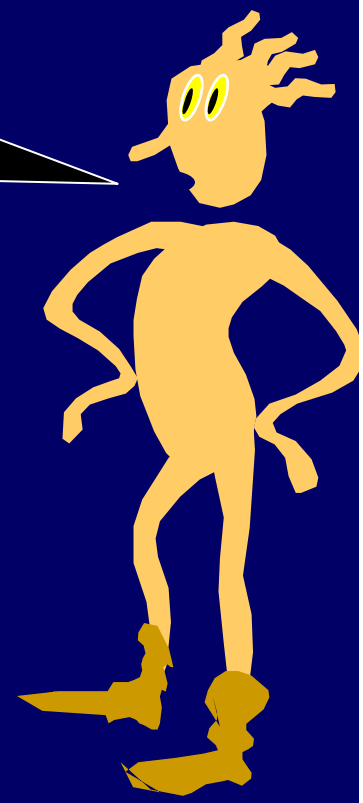
$\sum_k k \cdot \Pr(\textit{exactly}\ k$ letters end up in correct envelopes)

$= \sum_k k \cdot$ (…aargh!!…)

On average, in class of size m, how many pairs of people will have the same birthday?

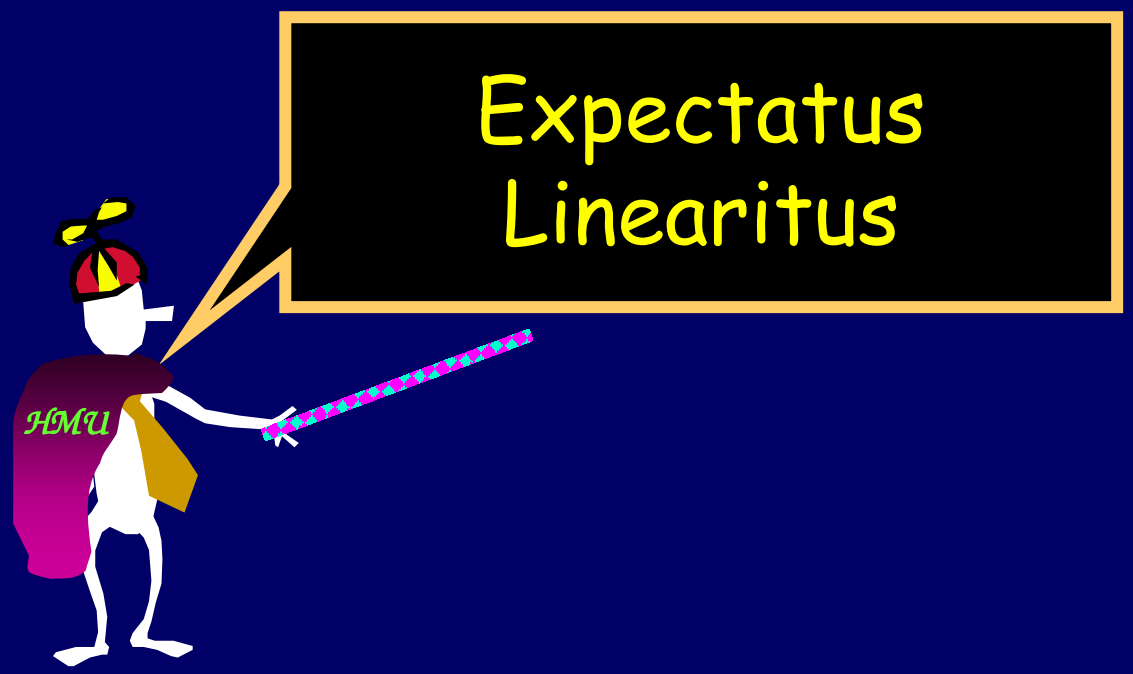
$\sum_k k \cdot \Pr(\text{exactly } k \text{ collisions})$

$= \sum_k k \cdot (\ldots\text{aargh!!!!}\ldots)$



The new tool is called

"Linearity of Expectation"



Random Variable

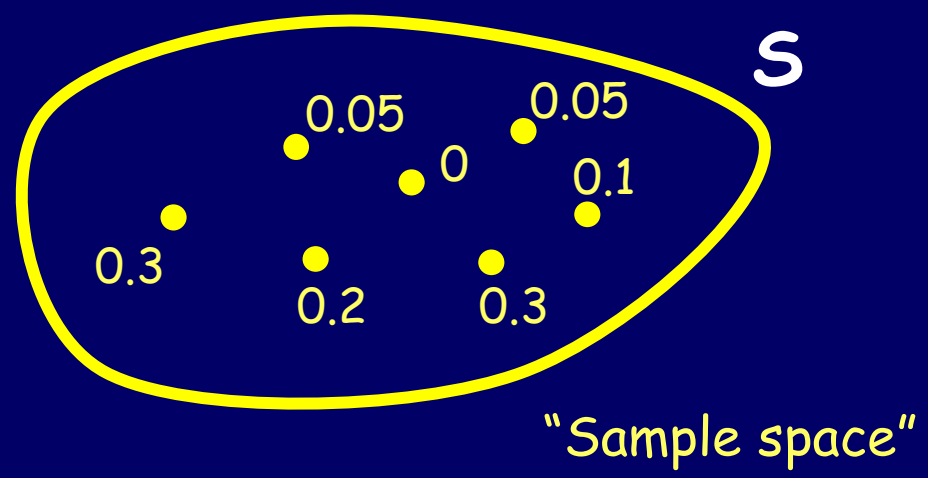
To use this new tool, we will also need to understand the concept of a Random Variable

Today's lecture: not too much material, but need to understand it well.

Probability Distribution

A (finite) <u>probability distribution</u> D

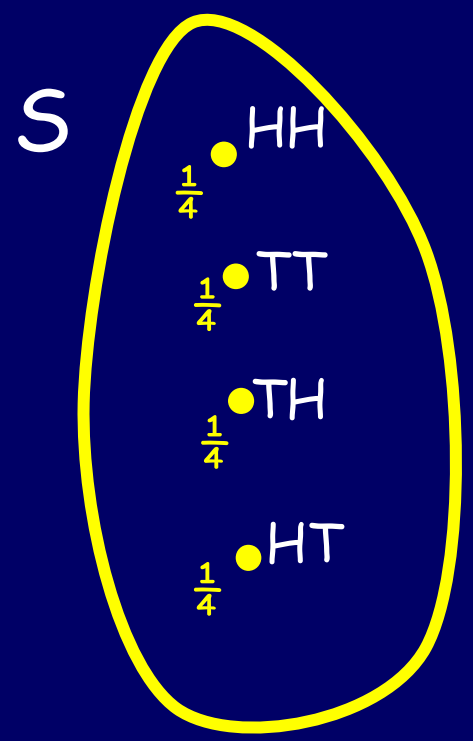
- a finite set S of elements (samples)
- each $x \in S$ has weight or <u>probability</u> $p(x) \in [0,1]$



weights must sum to 1

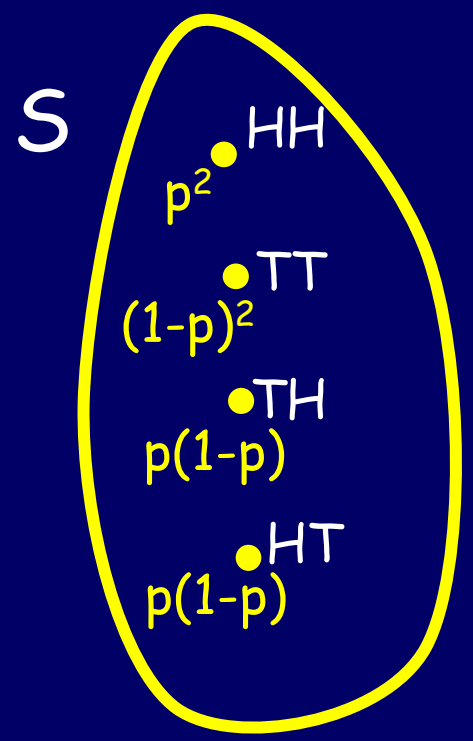
"Sample space"

Flip penny and nickel (unbiased)

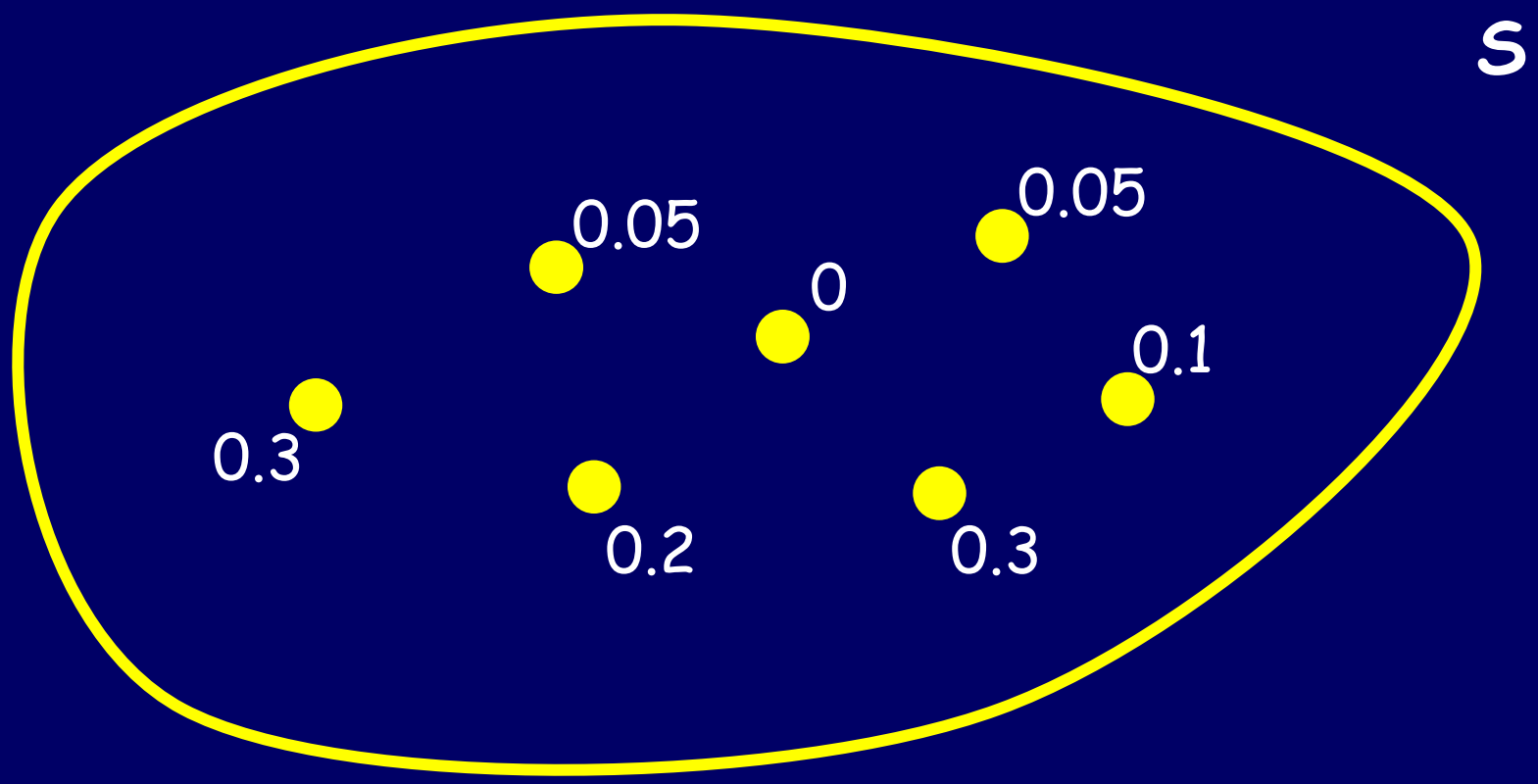


Flip penny and nickel (biased)

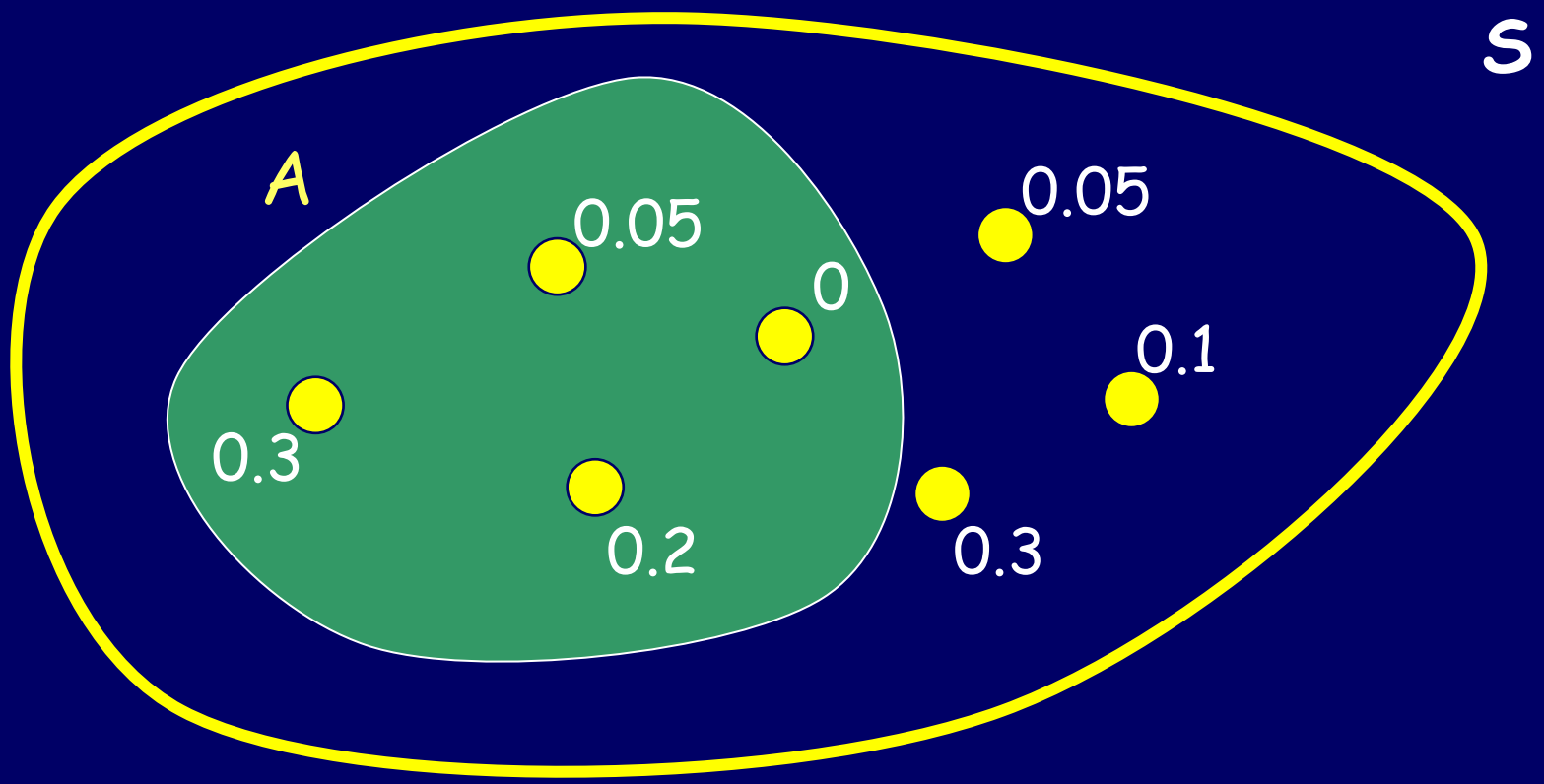
heads probability = p



Probability Distribution



An event is a subset



$$\Pr[A] = \sum_{x \in A} p(x) = 0.55$$

Running Example

I throw a white die and a black die.

Sample space S =

{ (1,1),	(1,2),	(1,3),	(1,4),	(1,5),	(1,6),
(2,1),	(2,2),	(2,3),	(2,4),	(2,5),	(2,6),
(3,1),	(3,2),	(3,3),	(3,4),	(3,5),	(3,6),
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(6,1),	(6,2),	(6,3),	(6,4),	(6,5),	(6,6) }

$Pr(x) = 1/36$
$\forall x \in S$

E = event that sum $\leq$ 3

Pr[E] = |E|/|S| = proportion of E in S = 3/36

Axioms:

1) $P(A) \geq 0$ (positivity)

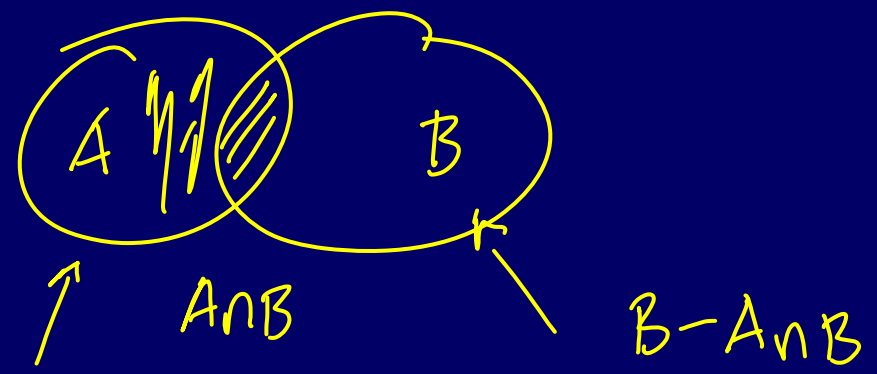
2) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$

(additivity)

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \qquad A_i \cap A_j = \emptyset$$

3) $P(S) = 1$ (normalization)

$P(A \cup B)$



By additivity

$$A - A \cap B = A \cap (A \cap B)^c$$

$$P(A \cup B) = P(A - A \cap B) + P(A \cap B) + P(B - A \cap B)$$

$$= P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) \leq P(A) + P(B)$$ "union bound"

New concept:
<u>Random Variables</u>

Random Variables

Random Variable: a (real-valued) function on S

Examples:

X = value of white die.

X(3,4) = 3, X(1,6) = 1 etc.

Y = sum of values of the two dice.

Y(3,4) = 7, Y(1,6) = 7 etc.

W = (value of white die)$^{\text{value of black die}}$

W(3,4) = 3^4 Y(1,6) = 1^6

Z = (1 if two dice are equal, 0 otherwise)

Z(4,4) = 1, Z(1,6) = 0 etc.

Toss a white die and a black die.

Sample space S =

{ (1,1),	(1,2),	(1,3),	(1,4),	(1,5),	(1,6),
(2,1),	(2,2),	(2,3),	(2,4),	(2,5),	(2,6),
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E.g., tossing a fair coin n times

S = all sequences of $\{H, T\}^n$

D = uniform distribution on S

$\Rightarrow D(x) = (\frac{1}{2})^n$ for all $x \in S$

Random Variables (say n = 10)

X = # of heads

$X(HHHTTTHTHTT) = 5$

Y = (1 if #heads = #tails, 0 otherwise)

$Y(HHHTTTHTHTT) = 1$, $Y(THHHHTTTTTT) = 0$

Notational conventions

Use letters like A, B, E for events.

Use letters like X, Y, f, g for R.V.'s.

R.V. = random variable

Two views of random variables

Think of a R.V. as

input to the function is random

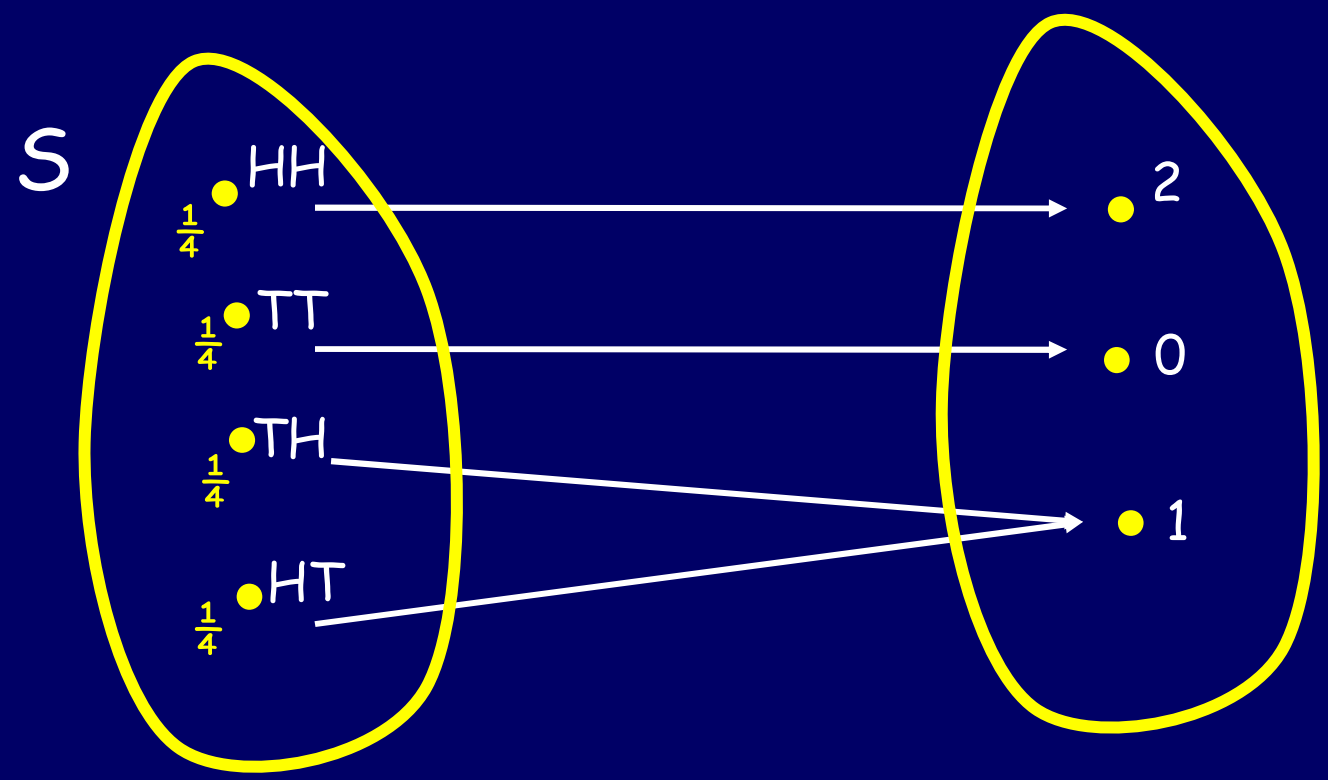
- a function from S to the reals $\mathbb{R}$

- or think of the induced distribution on $\mathbb{R}$

randomness is "pushed" to the values of the function

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads



Two views of random variables

Think of a R.V. as

- a function from S to the reals $\mathbb{R}$
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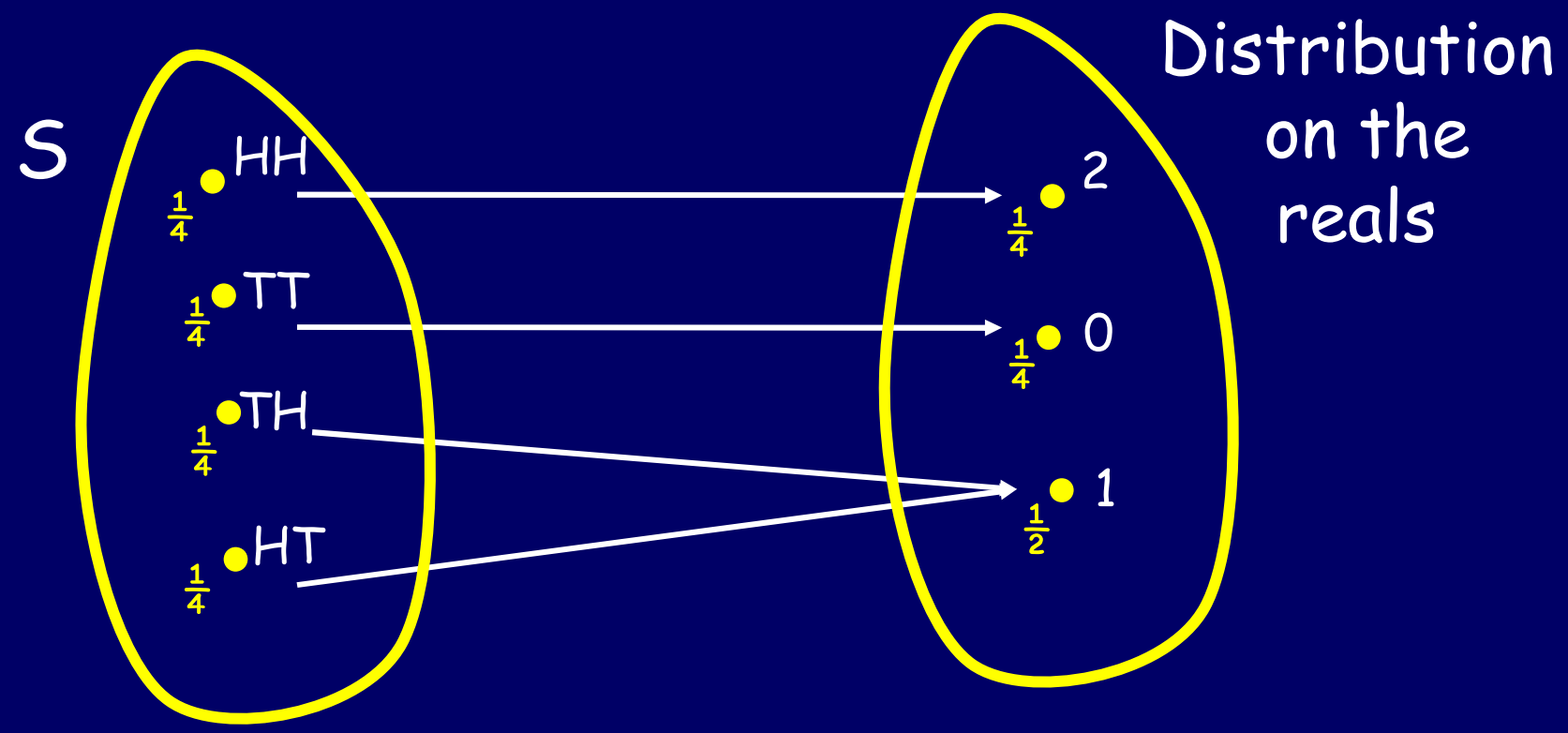
Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads



Two views of random variables

Think of a R.V. as

- a function from S to the reals $\mathbb{R}$
- or think of the induced distribution on $\mathbb{R}$

Two dice

I throw a white die and a black die.

Sample space S =

{ (1,1),	(1,2),	(1,3),	(1,4),	(1,5),	(1,6),
(2,1),	(2,2),	(2,3),	(2,4),	(2,5),	(2,6),
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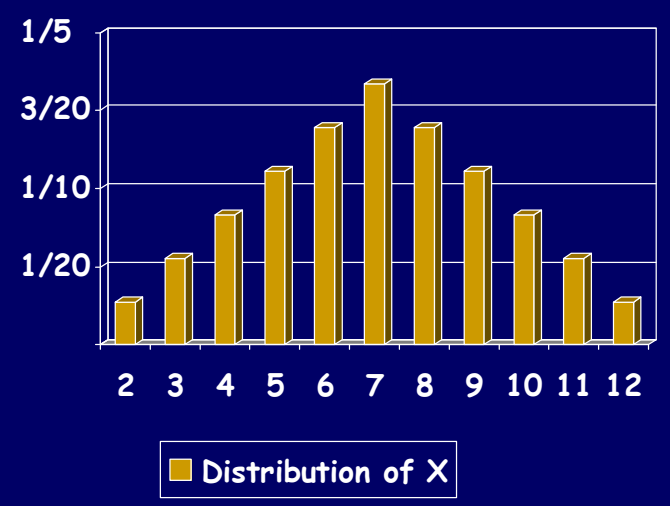
(6,1),	(6,2),	(6,3),	(6,4),	(6,5),	(6,6) }

0



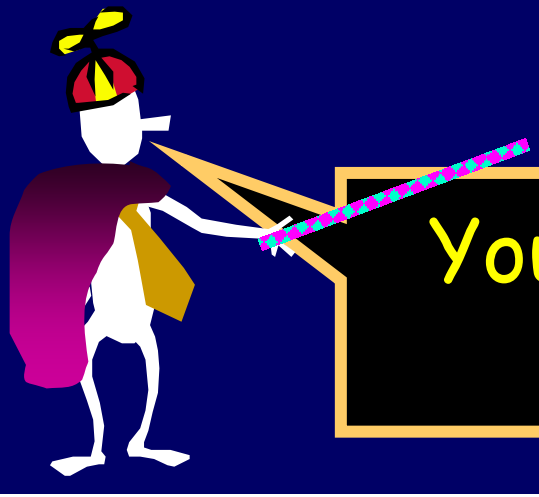
X = sum of both dice

function with X(1,1) = 2, X(1,2)=X(2,1)=3, …, X(6,6)=12

It's a floor wax *and* a dessert topping

It's a function on the sample space S.

It's a variable with a probability distribution on its values.



You should be comfortable with both views.

From Random Variables to Events

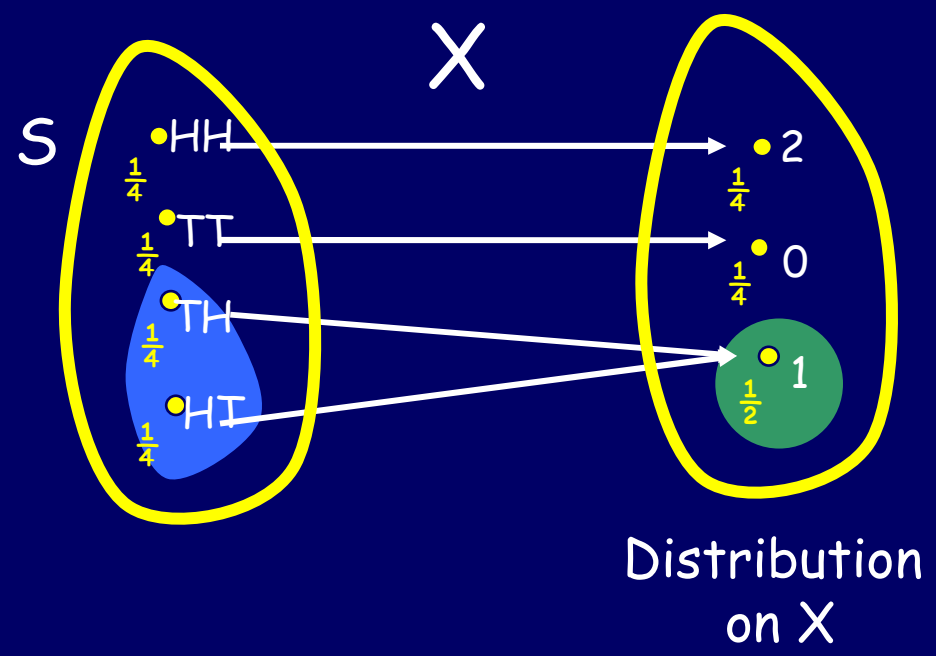
For any random variable X and value a,
we can define the <u>event</u> A that $X=a$.

$$\Pr(A) = \Pr(X=a) = \Pr(\{x \in S \mid X(x)=a\}).$$

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads

$$\Pr(X = a) = \Pr(\{x \in S \mid X(x) = a\})$$



$\Pr(X = 1)$

$= \Pr(\{x \in S \mid X(x) = 1\})$

$= \Pr(\{TH, HT\}) = \frac{1}{2}.$

Two dice

I throw a white die and a black die. X = sum

Sample space S =

{ (1,1),	(1,2),	(1,3),	(1,4),	(1,5),	(1,6),
(2,1),	(2,2),	(2,3),	(2,4),	(2,5),	(2,6),
(3,1),	(3,2),	(3,3),	(3,4),	(3,5),	(3,6),
(4,1),	(4,2),	(4,3),	(4,4),	(4,5),	(4,6),
(5,1),	(5,2),	(5,3),	(5,4),	(5,5),	(5,6),
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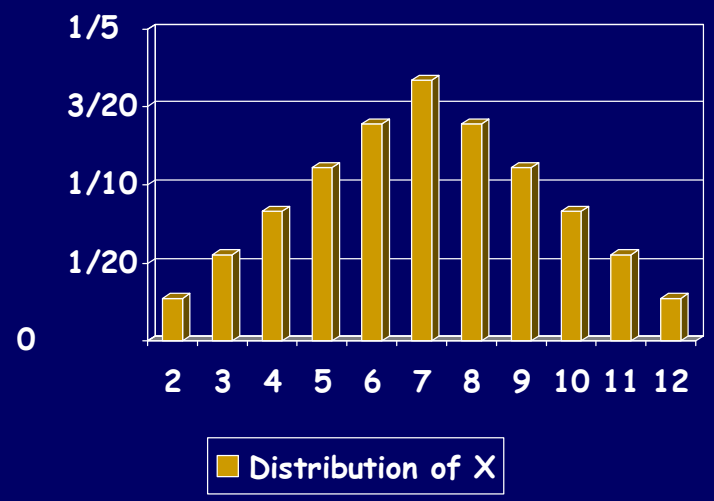


$\Pr(X = 6)$

$= \Pr(\{x \in S \mid X(x) = 6\})$

$= 5/36.$

$$\Pr(X = a) = \Pr(\{x \in S \mid X(x) = a\})$$

From Random Variables to Events

For any random variable X and value a,
we can define the <u>event</u> A that $X=a$.

$$\Pr(A) = \Pr(X=a) = \Pr(\{x \in S \mid X(x)=a\}).$$

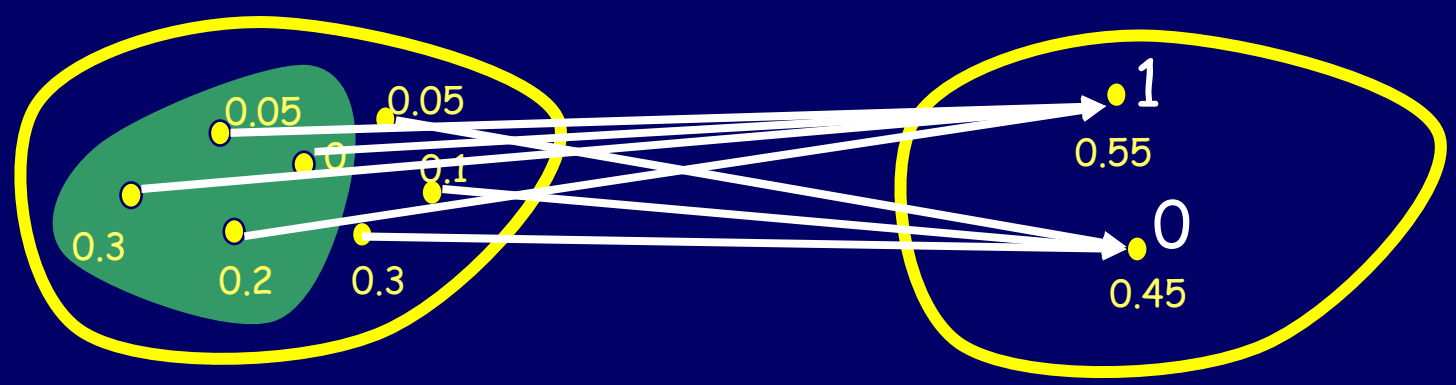
X has a distribution on its values

X is a function on the sample space S

From Events to Random Variables

For any event A, can define the <u>indicator random variable</u> for A:

$$X_A(x) = 1 \quad \text{if } x \in A$$
$$ = 0 \quad \text{if } x \notin A$$



Definition: expectation

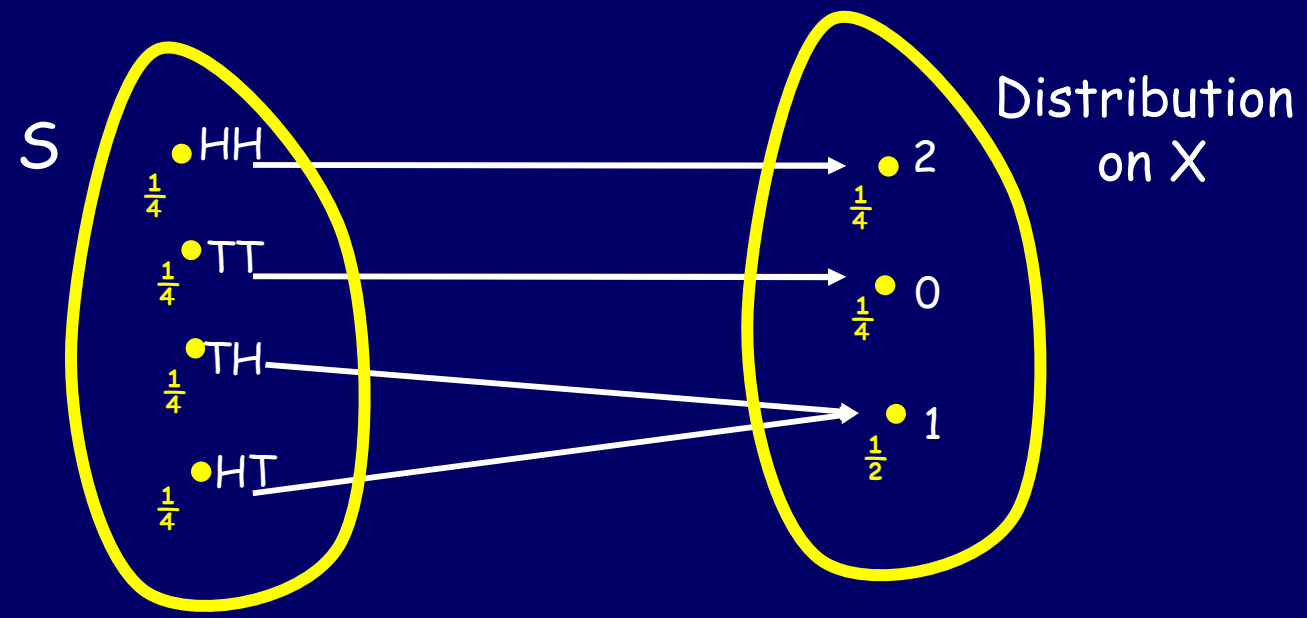
The <u>expectation</u>, or <u>expected value</u> of a random variable X is written as E[X], and is

$$E[X] = \sum_{x \in S} \Pr(x) X(x) = \sum_{k} k \Pr(X = k)$$

X is a function on the sample space S

X has a distribution on its values

Thinking about expectation



$\boxed{\sum_{x \in S} X(x)\, \Pr(x)}$ $\boxed{\sum_k k \Pr(X = k)}$

$E[X] = \frac{1}{4}*0 + \frac{1}{4}*1 + \frac{1}{4}*1 + \frac{1}{4}*2 = 1.$

$E[X] = \frac{1}{4}*0 + \frac{1}{2}*1 + \frac{1}{4}*2 = 1.$

A quick calculation…

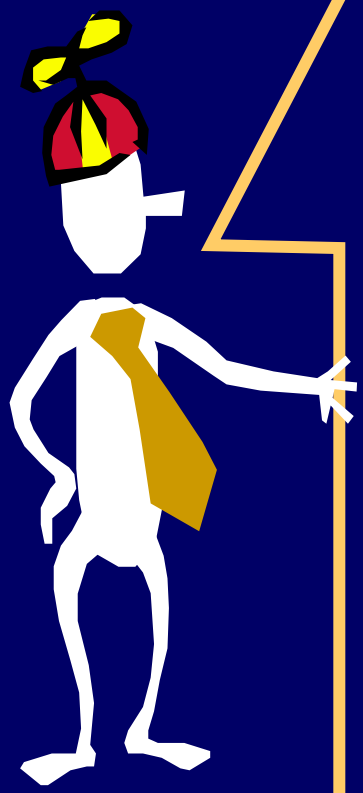
What if I flip a coin 3 times? Now what is the expected number of heads?

$$E[X] = (1/8) \times 0 + (3/8) \times 1 + (3/8) \times 2 + (1/8) \times 3 = 1.5$$

But $\Pr[\ X = 1.5\ ] = 0$…

Moral: don't always expect the expected.
$\Pr[\ X = E[X]\ ]$ may be 0 !

Type checking



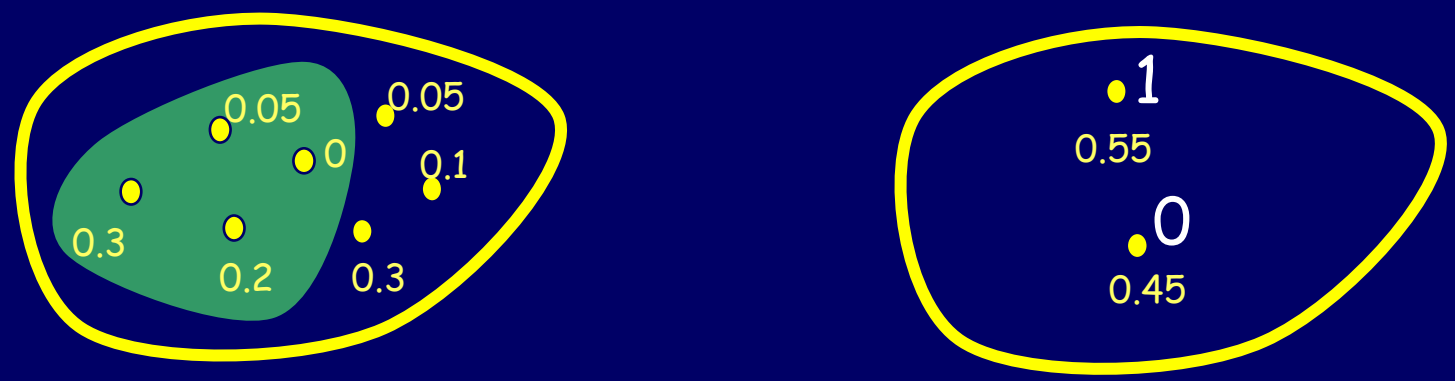
A Random Variable is the type of thing you might want to know an expected value of.

If you are computing an expectation, the thing whose expectation you are computing is a random variable.

Indicator R.V.s: $E[X_A] = Pr(A)$

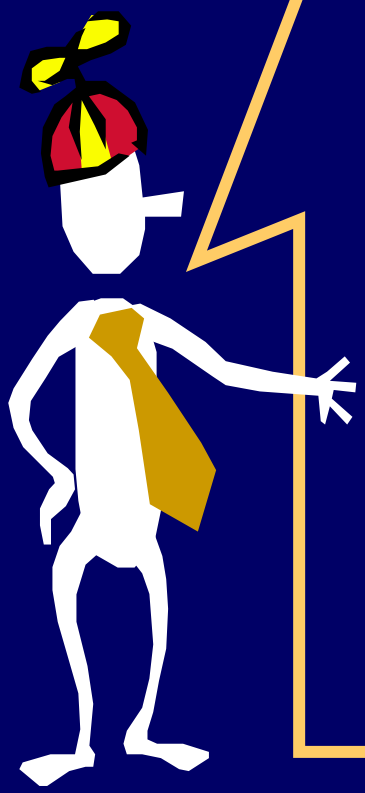
For event A, the <u>indicator random variable</u> for A:

$$X_A(x) = 1 \quad \text{if } x \in A$$
$$ = 0 \quad \text{if } x \notin A$$



$$E[X_A] = 1 \times Pr(X_A = 1) = Pr(A)$$

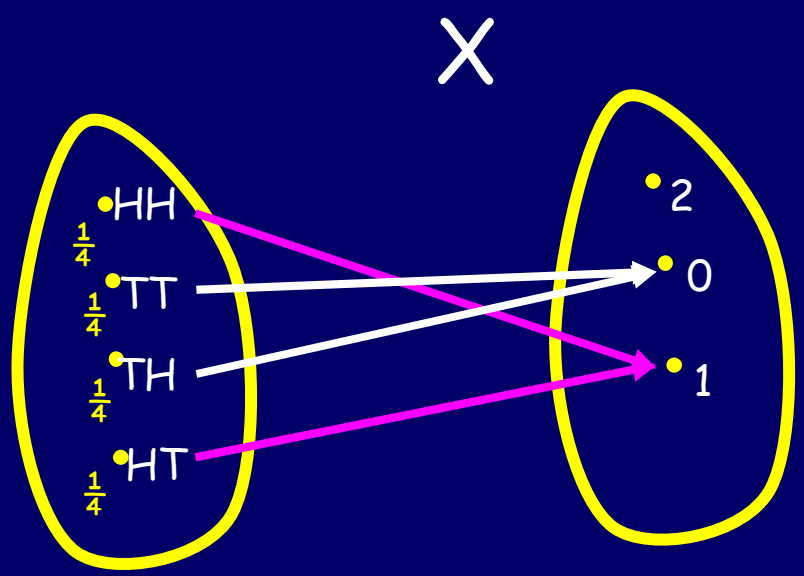
Adding Random Variables

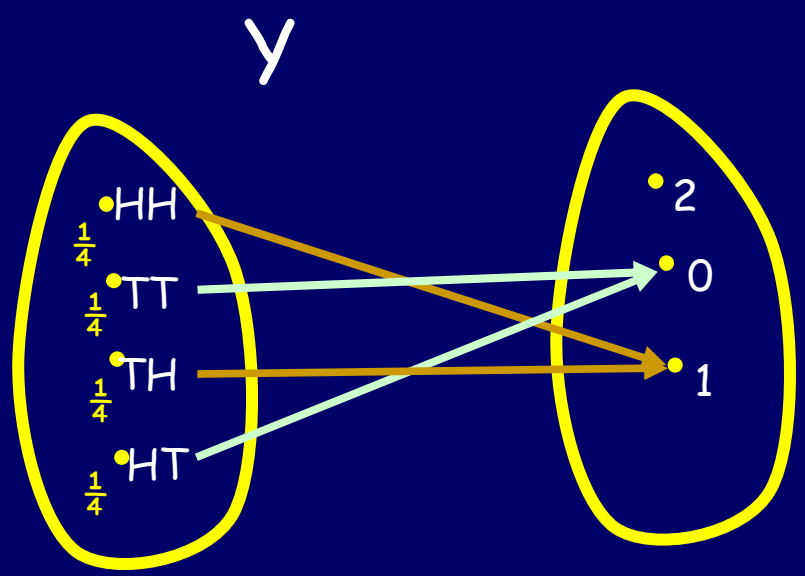


If X and Y are random variables (on the same set S), then $Z = X + Y$ is also a random variable.

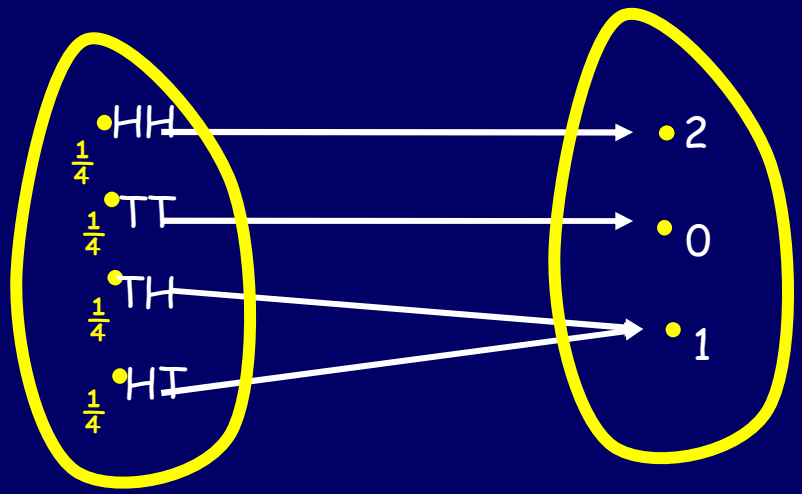
$$Z(x) \equiv X(x) + Y(x)$$

E.g., rolling two dice.
X = 1st die, Y= 2nd die,
Z = sum of two dice.


X
HH
TT
TH
HT
1/4
1/4
1/4
1/4
2
0
1


Y
HH
TT
TH
HT
1/4
1/4
1/4
1/4
2
0
1

$Z = X + Y$


HH
TT
TH
HT
1/4
1/4
1/4
1/4
2
0
1

Adding Random Variables

Example: Consider picking a random person in the world. Let X = length of the person's left arm in inches. Y = length of the person's right arm in inches. Let Z = X+Y. Z measures the combined arm lengths.

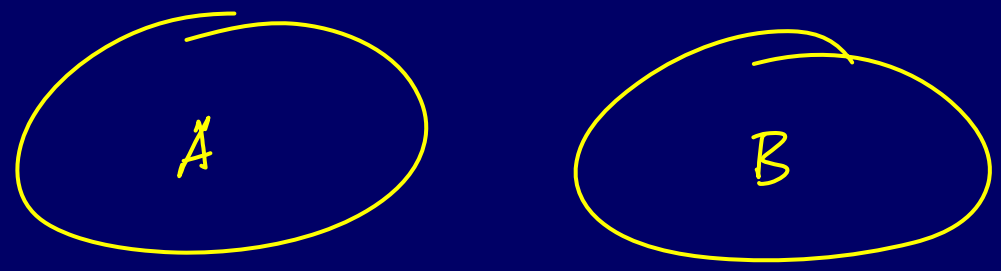
Formally, S = {people in world}, D = uniform distribution on S.

Independence

Two random variables X and Y are <u>independent</u> if for every a,b, the events X=a and Y=b are independent.

How about the case of
X=1st die, Y=2nd die?
X = left arm, Y=right arm?

$A \cap B = \phi$

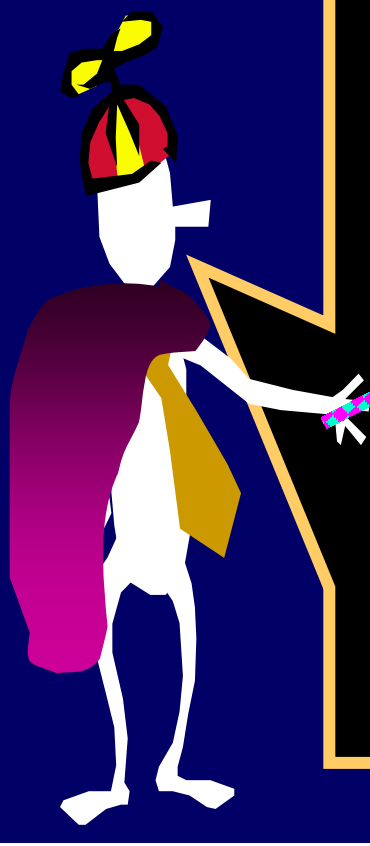


independent

$$P(A \cap B) = P(A)\,P(B)$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

Linearity of Expectation



If $Z = X+Y$, then

$$E[Z] = E[X] + E[Y]$$

Even if X and Y are not independent.

$$\mathbb{E}(Z) = \sum_{x \in S} \Pr(x)\, Z(x)$$

$$= \sum_{x \in S} \Pr(x) \left(X(x) + Y(x) \right)$$

$$= \sum_{x \in S} \Pr(x)\, X(x) + \sum_{x \in S} \Pr(x)\, Y(x)$$

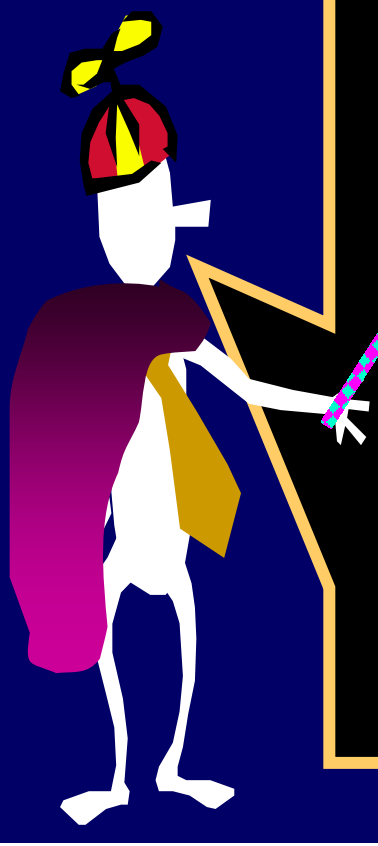
$$= \mathbb{E}(X) + \mathbb{E}(Y)$$

$$\mathbb{E}(aX) = a\,\mathbb{E}(X)$$

Linearity of Expectation

If $Z = X+Y$, then

$$E[Z] = E[X] + E[Y]$$



Proof:

$$E[X] = \sum_{x \in S} \Pr(x)\, X(x)$$
$$E[Y] = \sum_{x \in S} \Pr(x)\, Y(x)$$
$$E[Z] = \sum_{x \in S} \Pr(x)\, Z(x)$$

but $Z(x) = X(x) + Y(x)$

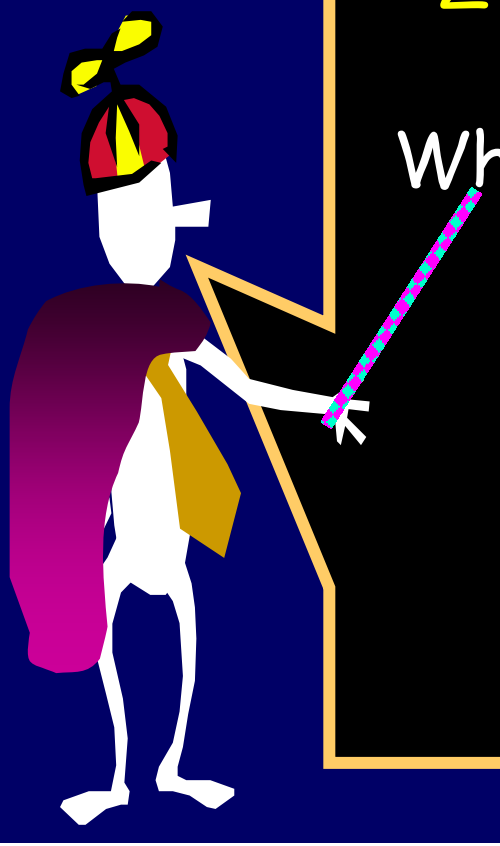
Linearity of Expectation

E.g., 2 fair flips:

X = 1st coin, Y = 2nd coin.

$Z = X+Y$ = total # heads.

What is E[X] ? E[Y] ? E[Z] ?



	1,1,2 HH	
1,0,1 HT		0,1,1 TH
	0,0,0 TT	

Linearity of Expectation

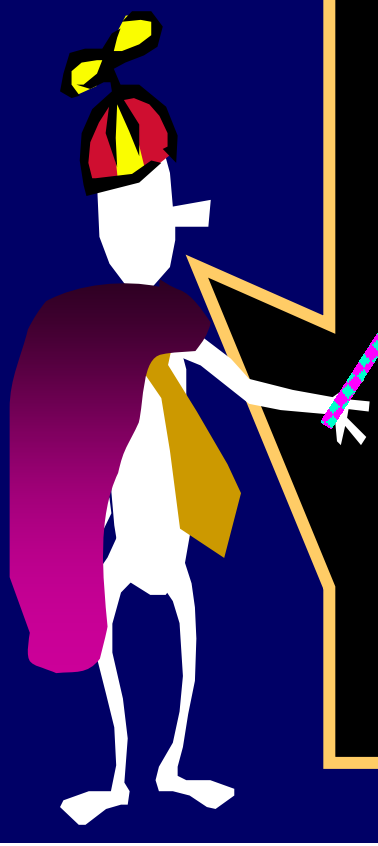
E.g., 2 fair flips:

X = at least one coin heads,

Y = both coins are heads, Z = X+Y

Are X and Y independent?

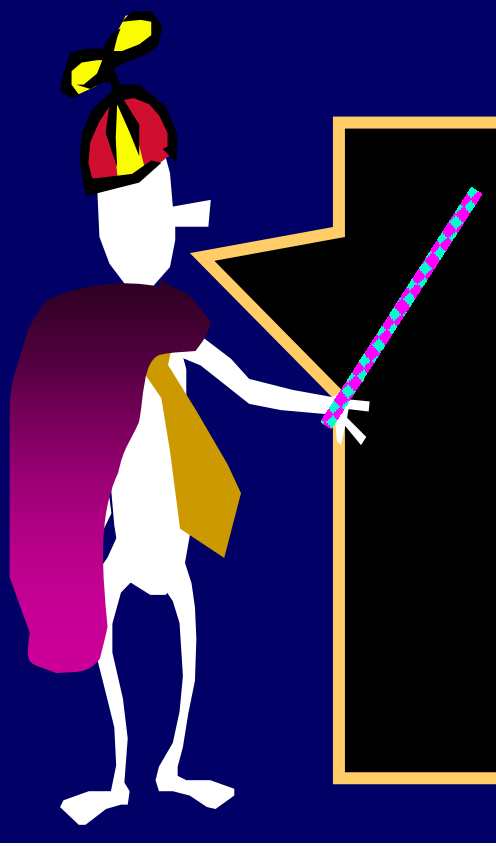
What is E[X] ? E[Y] ? E[Z] ?



	1,1,2	
1,0,1	HH	1,0,1
HT		TH
	0,0,0	
	TT	

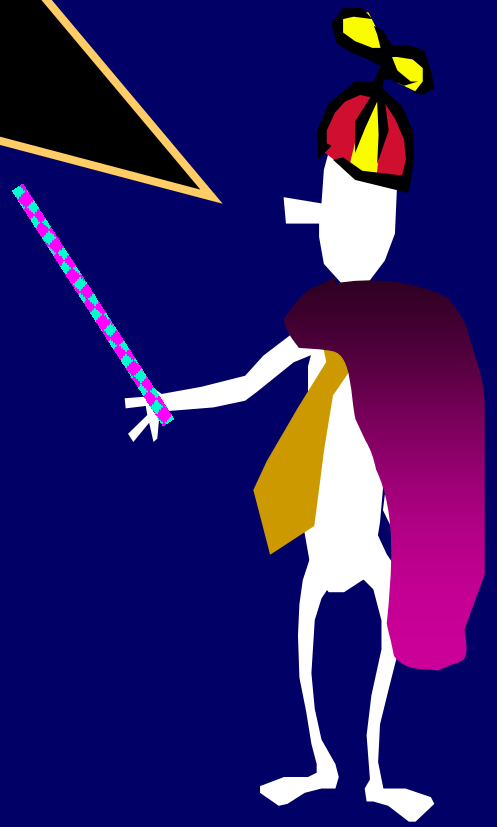
By induction

$$E[X_1 + X_2 + \ldots + X_n] =$$
$$E[X_1] + E[X_2] + \ldots. + E[X_n]$$

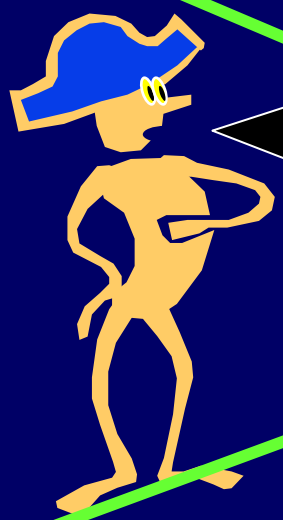


the expectation
of the sum
=
the sum of the
expectations

It is finally time to show off our probability prowess…

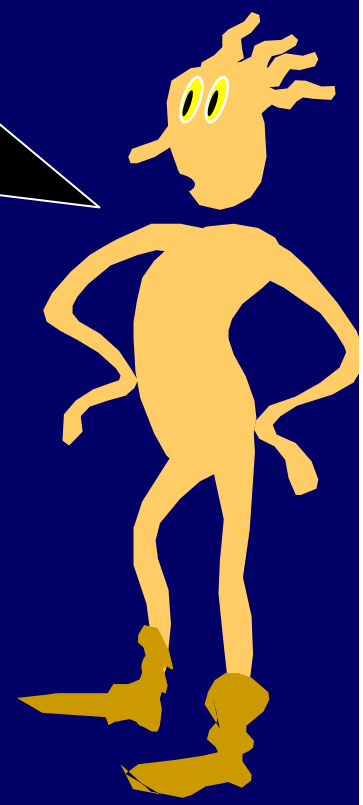


If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?



Hmm…

$\sum_k k \cdot \Pr(\textit{exactly } k \text{ letters end up in correct envelopes})$
$= \sum_k k \cdot$ (…aarghh!…)



Use Linearity of Expectation

Let A_i be the event the ith letter ends up in its correct envelope.

Let X_i be the indicator R.V. for A_i.

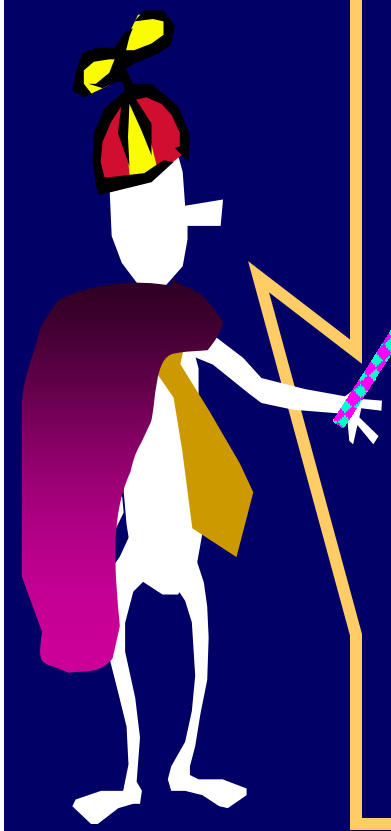
$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = X_1 + \ldots + X_{100}$.

We are asking for $E[Z]$.

Use Linearity of Expectation

Let A_i be the event the ith letter ends up in the correct envelope.
Let X_i be the indicator R.V. for A_i.
Let $Z = X_1 + \ldots + X_n$. We are asking for $E[Z]$.



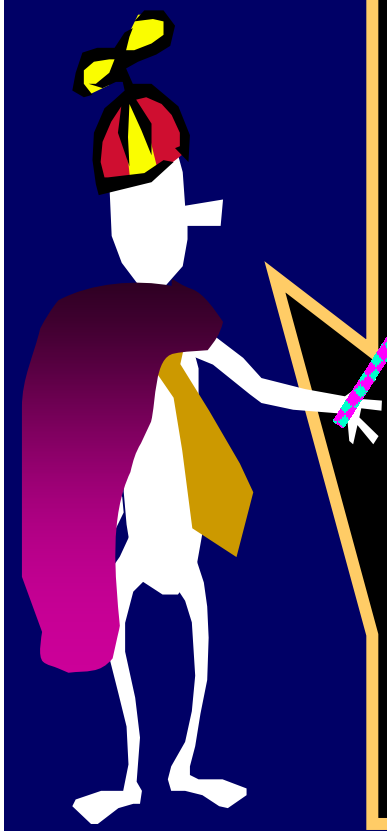
What is $E[X_i]$?

$$E[X_i] = Pr(A_i) = 1/100.$$

What is $E[Z]$?

$$\begin{aligned} E[Z] &= E[X_1 + \ldots + X_{100}] \\ &= E[X_1] + \ldots + E[X_{100}] \\ &= 1/100 + \ldots + 1/100 = 1. \end{aligned}$$

Use Linearity of Expectation



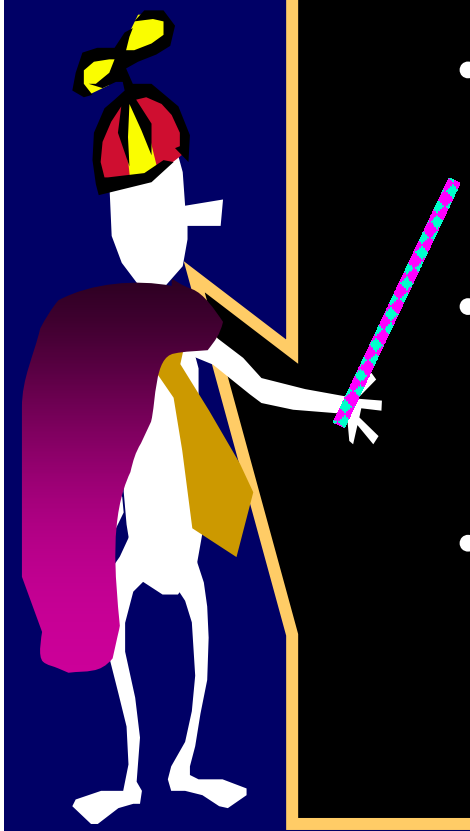
So, in expectation, 1 card will be in the same position as it started.

Pretty neat: it doesn't depend on how many cards!

Question: were the X_i independent?

No! E.g., think of n=2.

Use Linearity of Expectation



General approach:

- View thing you care about as expected value of some RV.
- Write this RV as sum of simpler RVs (typically indicator RVs).
- Solve for their expectations and add them up!

Example

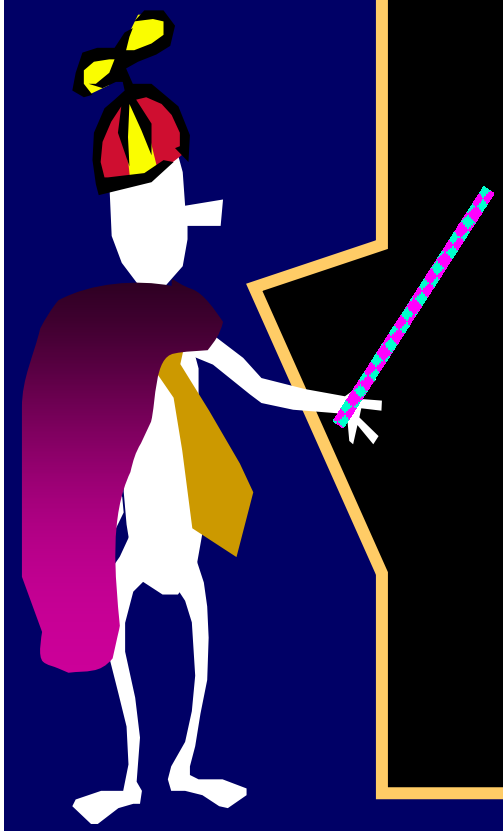
We flip n coins of bias p. What is the expected number of heads?

We <u>could</u> do this by summing

$$\sum_k k \Pr(X = k)$$

$$= \sum_k k \binom{n}{k} p^k (1-p)^{n-k}$$

But now we know a better way

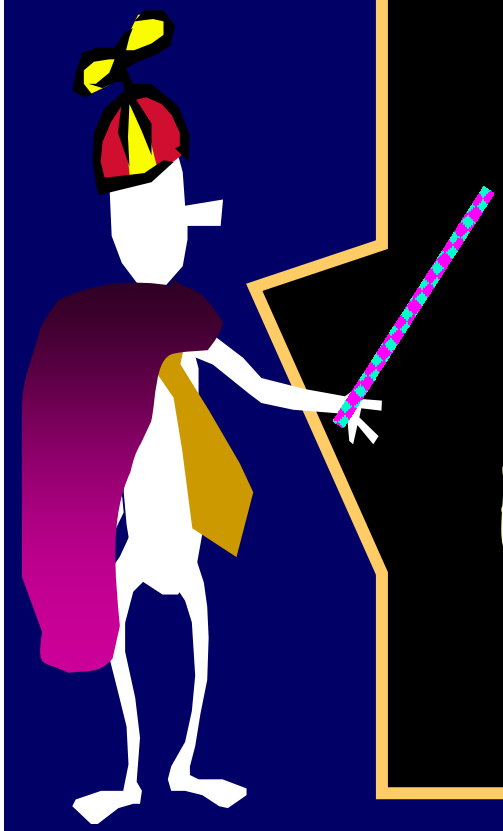


Let X = number of heads when n independent coins of bias p are flipped.

Break X into n simpler RVs,

$$X_i = \begin{cases} 0, & \text{if the } i^{th} \text{ coin is tails} \\ 1, & \text{if the } i^{th} \text{ coin is heads} \end{cases}$$

$$E[X] = E[\Sigma_i X_i] = ?$$



Let X = number of heads when n independent coins of bias p are flipped.

Break X into n simpler RVs,

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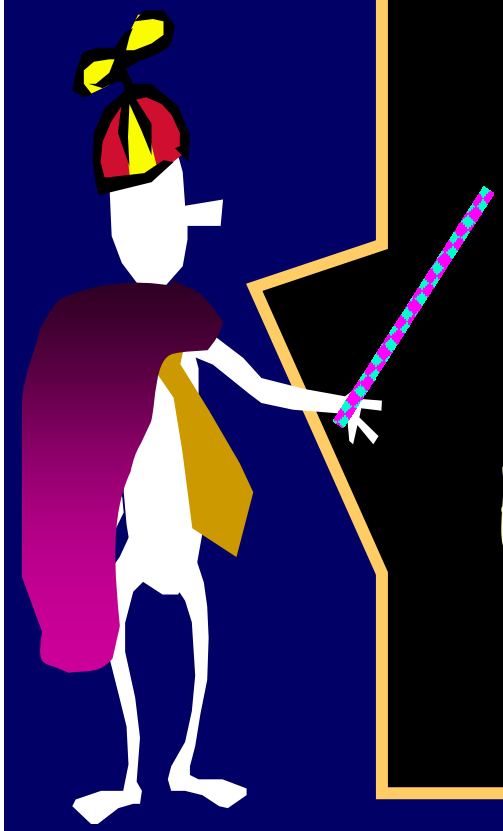
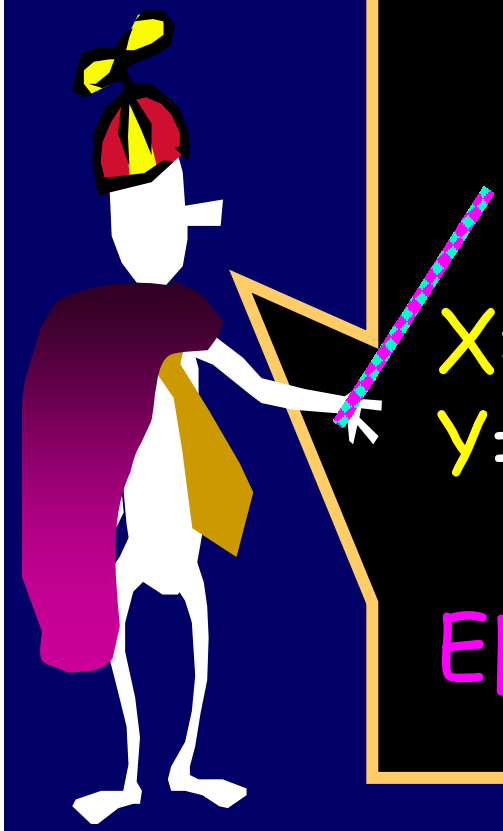


$$E[X] = E[\Sigma_i X_i] = np$$

What about Products?



If $Z = XY$, then
$E[Z] = E[X] \times E[Y]$?

No!

X=indicator for "1st flip is heads"
Y=indicator for "1st flip is tails".

$E[XY]=0$.

But it is true if RVs are independent

$$\mathbb{E}(Z) = \sum_{c} c \, \mathbb{P}(Z=c)$$

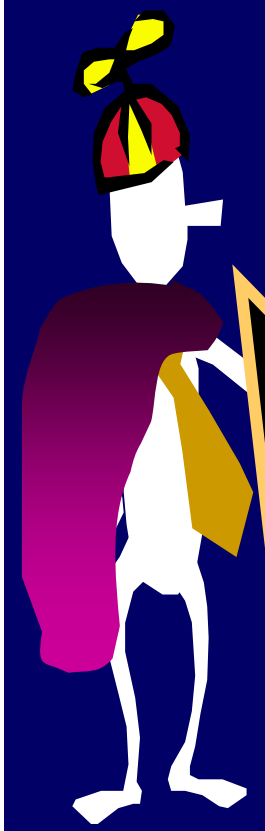
$$= \sum_{a,b} a.b \, \mathbb{P}(XY=c)$$

$$= \sum_{a,b} ab \, \mathbb{P}(X=a \cap Y=b)$$

$$= \sum_{a,b} ab \, \mathbb{P}(X=a)\,\mathbb{P}(Y=b)$$

$$= \left(\sum_{a} a \, \mathbb{P}(X=a)\right)\left(\sum_{b} b \, \mathbb{P}(Y=b)\right)$$

But it is true if RVs are independent



Proof:

$$E[X] = \sum_a a \times \Pr(X=a)$$
$$E[Y] = \sum_b b \times \Pr(Y=b)$$

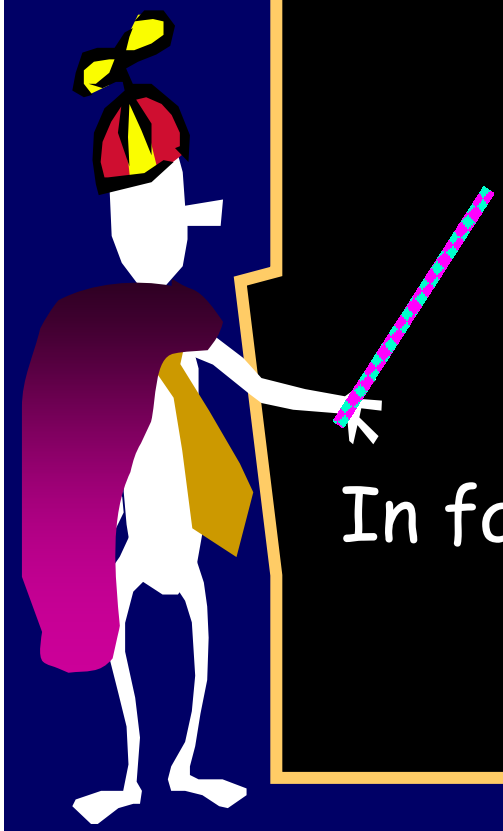
$$\begin{aligned} E[XY] &= \sum_c c \times \Pr(XY = c) \\ &= \sum_c \sum_{a,b:ab=c} c \times \Pr(X=a \cap Y=b) \\ &= \sum_{a,b} ab \times \Pr(X=a \cap Y=b) \\ &= \sum_{a,b} ab \times \Pr(X=a)\Pr(Y=b) \\ &= E[X]\,E[Y] \end{aligned}$$

E.g., 2 fair flips.
X = indicator for 1st coin being heads,
Y = indicator for 2nd coin being heads.
XY = indicator for "both are heads".

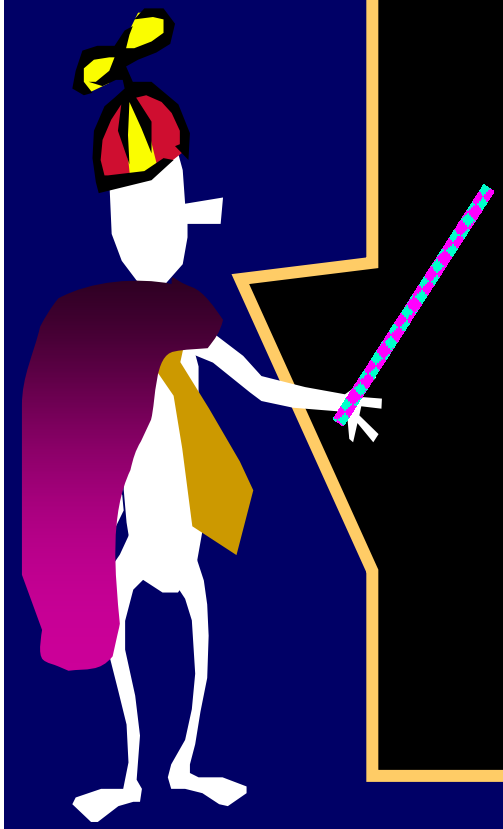
$$E[X] = \tfrac{1}{2},\ E[Y] = \tfrac{1}{2},\ E[XY] = \tfrac{1}{4}.$$

$$E[X*X] = E[X]^2?$$

No: $E[X^2] = \tfrac{1}{2},\ E[X]^2 = \tfrac{1}{4}.$



In fact, $E[X^2] - E[X]^2$ is called the *variance* of X.



Most of the time, though, power will come from using sums.

Mostly because **Linearity of Expectations** holds even if RVs are <u>not independent</u>.

Another problem

On average, in class of size m, how many pairs of people will have the same birthday?

$\sum_k k \cdot \Pr(\text{exactly } k \text{ collisions})$

$= \sum_k k \cdot (\ldots\text{aargh!!!!}\ldots)$

Use linearity of expectation.

Suppose we have m people each with a uniformly chosen birthday from 1 to 366.

X = number of pairs of people with the same birthday.

E[X] = ?

X = number of pairs of people with the same birthday.

$E[X] = ?$

Use $m(m-1)/2$ indicator variables, one for each pair of people.

$X_{jk} = 1$ if person j and person k have the same birthday; else 0.

$$E[X_{jk}] = (1/366)\,1 + (1 - 1/366)\,0$$
$$= 1/366$$

X = number of pairs of people with the same birthday.

$$E[X] = E[\Sigma_{j \le k \le m} X_{jk}]$$

There are many dependencies among the indicator variables. E.g., X_{12} and X_{13} and X_{23} are dependent.

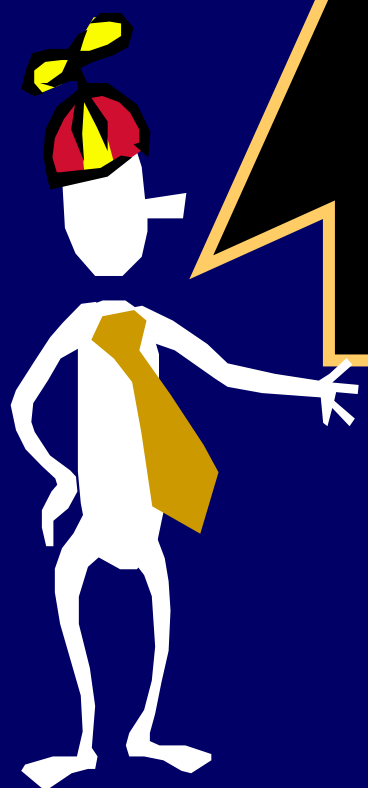
But we don't care!

X = number of pairs of people with the same birthday.

$$E[X] = E[\Sigma_{j \le k \le m} X_{jk}]$$

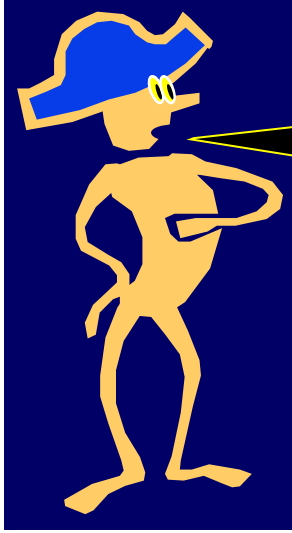
$$= \Sigma_{j \le k \le m} E[X_{jk}]$$

$$= m(m-1)/2 \times 1/366$$

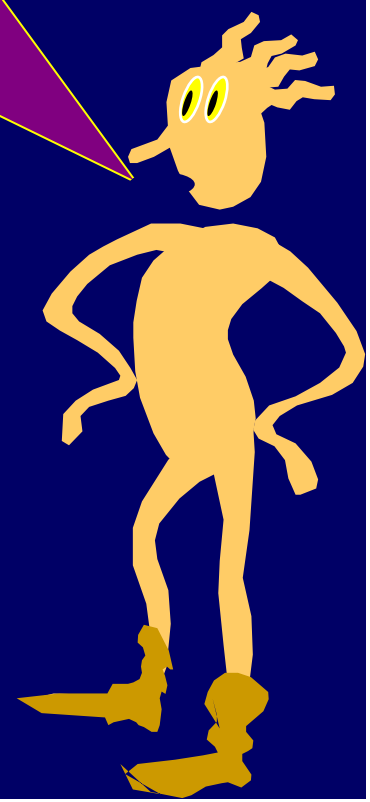


Step right up…

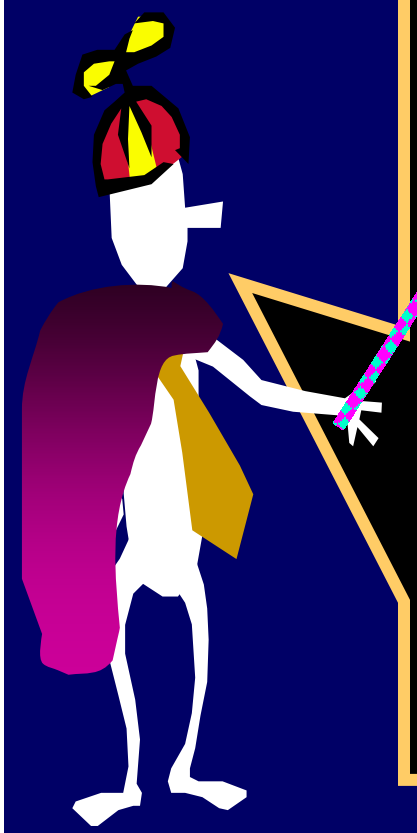
You pick a number $n \in [1..6]$. You roll 3 dice. If any match n, you win \$1. Else you pay me \$1. Want to play?



Hmm… let's see



Analysis



A_i = event that i-th die matches.

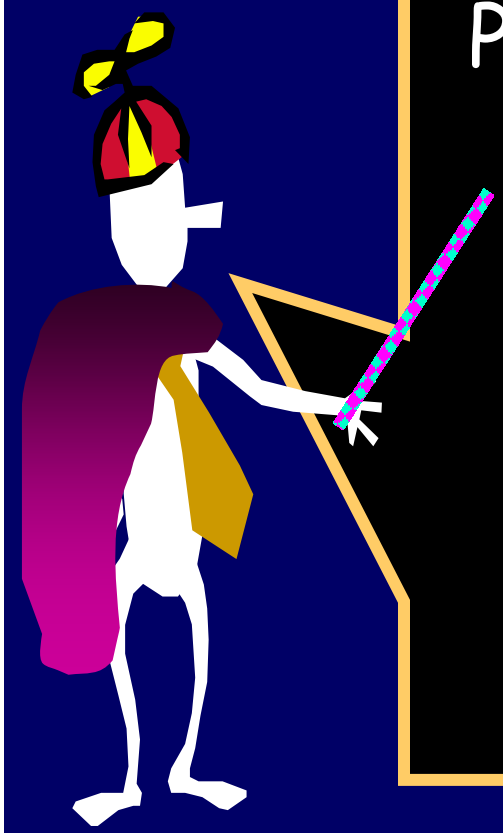
X_i = indicator RV for A_i.

Expected number of dice that match:

$$E[X_1+X_2+X_3] = 1/6+1/6+1/6 = \tfrac{1}{2}.$$

But this is not the same as Pr(at least one die matches).

Analysis



Pr(at least one die matches)

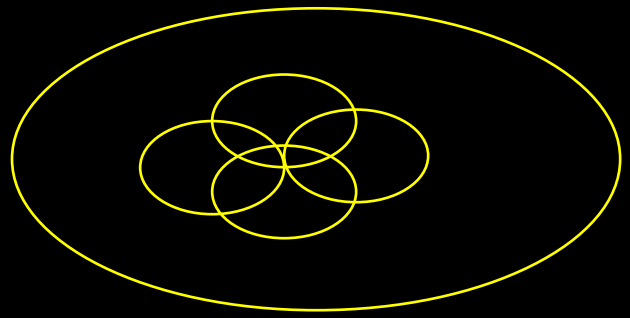
$= 1 - \Pr(\text{none match})$

$= 1 - (5/6)^3 = 0.416.$

What's going on?

Say we have a collection of events A_1, A_2, ….

How does E[# events that occur] compare to Pr(at least one occurs)?



What's going on?

E[# events that occur]

$= \sum_k \Pr(k \text{ events occur}) \times k$

$= \sum_{(k > 0)} \Pr(k \text{ events occur}) \times k$

Pr(at least one event occurs)

$= \sum_{(k > 0)} \Pr(k \text{ events occur})$

What's going on?

Moral #1: be careful you are modeling problem correctly.

Moral #2: watch out for carnival games.