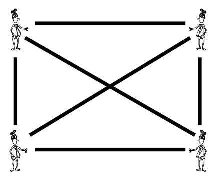
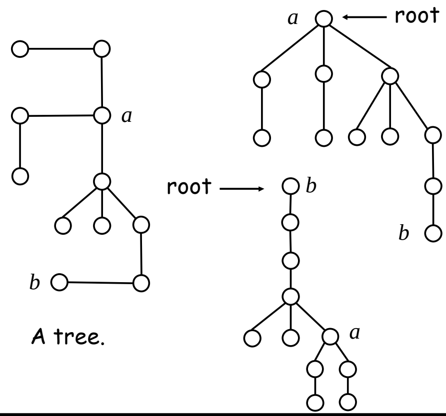
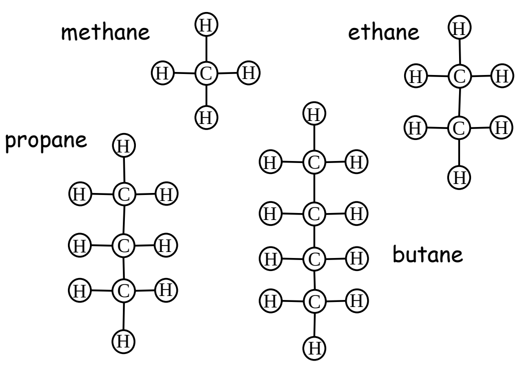
Great Theoretical Ideas In Computer Science		
John Lafferty		CS 15-251 Fall 2005
Lecture 14	October 18, 2005	Carnegie Mellon University

Graphs



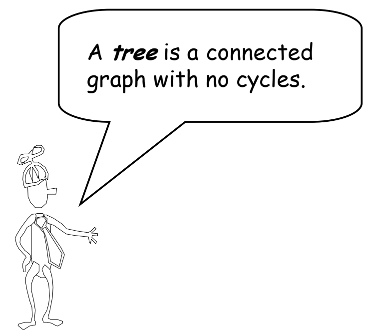


A tree.

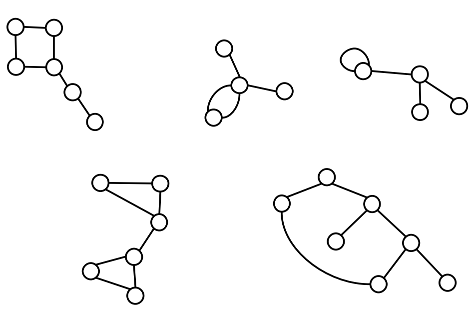


some saturated hydrocarbons

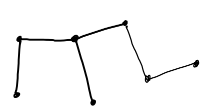
Putting a picture into words…



These are not trees…

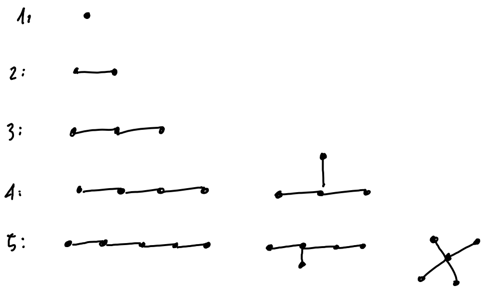


Recall: The Shy People Party



no cycles can be formed

How many trees on 1-6 vertices?



We'll pass around a piece of paper. Draw a new 8-node tree, and put your name next to it.

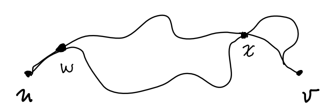
Theorem: Let G be a graph with n nodes and e edges.

The following are equivalent:

1. G is a tree (connected, acyclic)
2. Every two nodes of G are joined by a unique path
3. G is connected and n = e + 1
4. G is acyclic and n = e + 1
5. G is acyclic and if any two nonadjacent points are joined by a line, the resulting graph has exactly one cycle.

To prove this, it suffices to show
$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

$1 \Rightarrow 2$



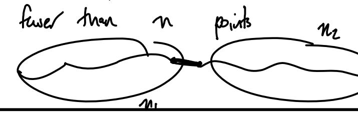
implies there is a cycle

By I.H.

$n_1 = e_1 + 1$

$2 \Rightarrow 3$ Pf. by induction. Assume true for fewer than n points

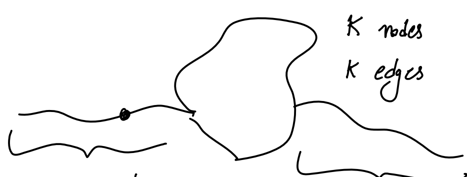
$n_2 = e_2 + 1$



$n = e_1 + e_2 + 2$
$= e - 1 + 2$
$= e + 1$

$3 \Rightarrow 4$

Suppose there is a cycle



So we have $\geq$ n-K edges outside cycle
$\Rightarrow$ at least n edges.

$4 \Rightarrow 5$. if there are K connected components, each is a tree.

$n_i = e_i + 1$

$n = e + K \Rightarrow K = 1$

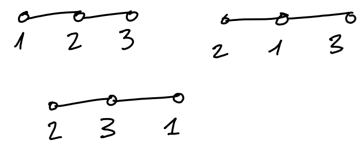
So, any two disconnected points, have a unique path between them, forming an edge between them creates a cycle.

Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

$2e = 2p - 2$

Question:

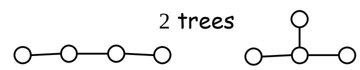
How many *labeled* trees are there with three nodes?



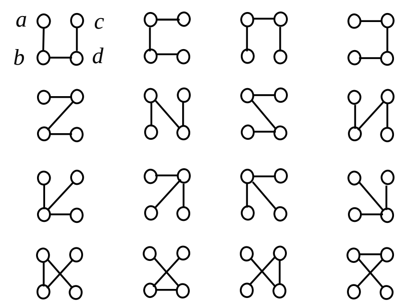
Question:

How many *labeled* trees are there with four nodes?

2 trees

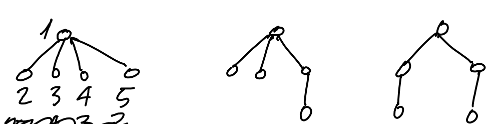


16 labeled trees



Question:

How many labeled trees are there with five nodes?



5 labelings

$5 \times \binom{4}{3} \times 3$

60

$5 \times \binom{4}{2} \times 2$

60

125 labeled trees.

Question:

How many labeled trees on n nodes are there?

3: 3 trees

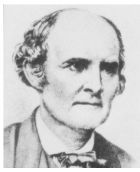
4: 16 trees

5: 125 trees

n^{n-2}

Cayley's formula

The number of labeled trees on n nodes is



n^{n-2}

The proof will use the correspondence principle.

Each labeled tree on n nodes

corresponds to

A sequence in $\{1,2,\ldots,n\}^{n-2}$ that is, (n-2) numbers, each in the range [1..n]

How to make a sequence from a tree.

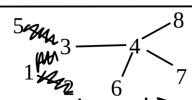
Loop through i from 1 to n-2

Let l be the degree-1 node with the lowest label.

Define the i^{th} element of the sequence as the label of the node adjacent to l.

Delete the node l from the tree.

Example:



$\langle 1,3,3,4,4,4 \rangle$

$\langle 1, 3, 3, 4, 4, 4 \rangle$

How to reconstruct the unique tree from a sequence S.

Let $I = \{1,2,3,\ldots,n\}$

Loop until $S=\varepsilon$ ← empty sequence

Let l = smallest # in I but not in S

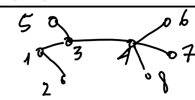
Let s = first label in sequence S

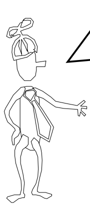
- Add edge $\{l, s\}$ to the tree.
- Delete l from I.
- Delete s from S.

Add edge $\{l, s\}$ to the tree, where $I = \{l, s\}$

$\langle 1,3,3,4,4,4 \rangle$

$I = \cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, \cancel{7}, 8$
4





A graph is *planar* if it can be drawn in the plane without crossing edges. A *plane graph* is any such drawing, which breaks up the plane into a number f of *faces* or regions

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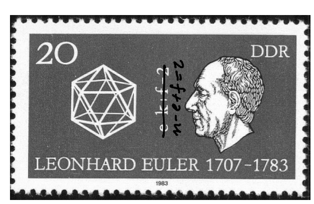
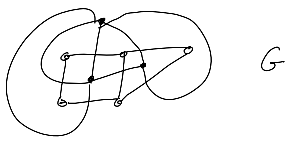
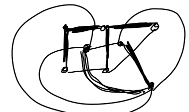
Euler's Formula

If G is a connected plane graph with n vertices, e edges and f faces, then n - e + f = 2



Rather than using induction, we'll use the important notion of the *dual graph*





Let T be a spanning tree of G

Let T^* be the graph where there is an edge in dual graph for each edge in $G-T$

T^* is a spanning tree of G^*

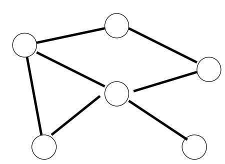
$$\begin{aligned} n &= e_T + 1 \\ f &= e_{T^*} + 1 \end{aligned} \Rightarrow \begin{aligned} n+f &= e_T + e_{T^*} + 2 \\ &= e + 2 \end{aligned}$$

Corollary: Let G be a plane graph with $n > 2$ vertices. Then

a) G has a vertex of degree at most 5.
b) G has at most $3n - 6$ edges

Graph Coloring

A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color.

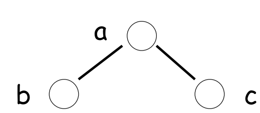


Graph Coloring

Arises surprisingly often in CS.

Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register.

Instructions	Live variables
	a
b = a+2	
	a,b
c = b*b	
	a,c
b = c+1	
	a,b
return a*b	

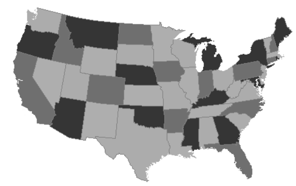


Every plane graph can be 6-colored

By induction. Assume true for $< n$ nodes.
If have a plane graph on n.
There is some node v with deg. ≤ 5.
Remove v and color by I.H.

Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree <= 5.

4-color theorem remains challenging



http://www.math.gatech.edu/~thomas/FC/fourcolor.html

Graph Spectra

We now move to a different *representation* of graphs that is extremely powerful, and useful in many areas of computer science: AI, information retrieval, computer vision, machine learning, CS theory,…

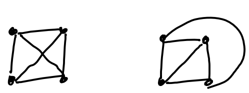
Adjacency matrix

Suppose we have a graph G with n vertices and edge set E. The *adjacency matrix* is the nxn matrix $A=[a_{ij}]$ with

$a_{ij} = 1$ if (i,j) is an edge
$a_{ij} = 0$ if (i,j) is not an edge

Example

K_4



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Counting Paths

The number of paths of length k from node i to node j is the entry in position (i,j) in the matrix A^k

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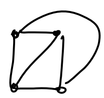
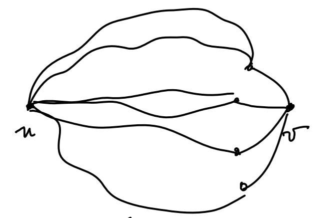
$$A^2 = \begin{pmatrix} 0&1&1&1\\ 1&0&1&1\\ 1&1&0&1\\ 1&1&1&0 \end{pmatrix}\begin{pmatrix} 0&1&1&1\\ 1&0&1&1\\ 1&1&0&1\\ 1&1&1&0 \end{pmatrix} = \begin{pmatrix} 3&2&2&2\\ 2&3&2&2\\ 2&2&3&2\\ 2&2&2&3 \end{pmatrix}$$



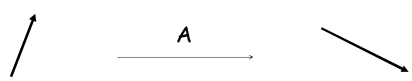
By induction on k



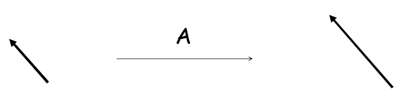
$$(A^k)_{uv} = \sum_j (A^{k-1})_{uj} A_{jv} = \sum_{j \in N(v)} (A^{k-1})_{uj}$$

Eigenvalues

An nxn matrix A is a linear transformation from n-vectors to n-vectors



An *eigenvector* is a vector *fixed* (up to length) by the transformation. The associated *eigenvalue* is the scaling of the vector.



Eigenvalues

Vector x is an eigenvector of A with eigenvalue λ if

$$Ax = \lambda x$$

A symmetric nxn matrix has at most n distinct real eigenvalues

Characteristic Polynomial

The *determinant* of A is the product of its eigenvalues:

$$\det A = \lambda_1 \lambda_2 \dots \lambda_n$$

The *characteristic polynomial* of A is the polynomial

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n)$$

Example: K_4

Example: K_4

$$A\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\3\\3\end{pmatrix} \qquad \lambda = 3$$

$$A\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} \qquad \lambda = \cancel{-3}\ -1$$

$$p_A(\lambda) = (\lambda-3)(\lambda+1)^3$$
$$= \lambda^4 - 6\lambda^2 - 8\lambda - 3$$

If graph G has adjacency matrix A with characteristic polynomial

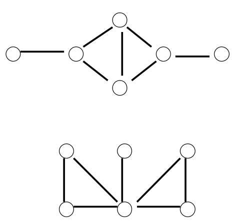
$$p_A(\lambda) = \lambda^n + c_1\,\lambda^{n-1} + c_2\,\lambda^{n-2} + \ldots + c_n$$

then

$c_1 = 0$
$-c_2$ = # of edges in G
$-c_3$ = twice # of triangles in G

Two different graphs with the same spectrum

$$p_A(\lambda) = \lambda^6 - 7\,\lambda^4 - 4\lambda^3 + 7\lambda^2 + 4\lambda - 1$$



Let your spectrum do the counting…

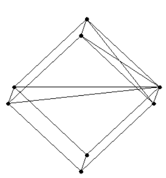
A *closed walk* or *loop* in a graph is a path whose initial and final vertices are the same. We easily get

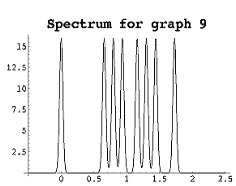
$\mathrm{trace}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n = 0$

$\mathrm{trace}(A^2) = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2$
= twice # of edges

$\mathrm{trace}(A^3) = \lambda_1^3 + \lambda_2^3 + \ldots + \lambda_n^3$
= six times # of triangles

Graph Muzak





http://math.ucsd.edu/~fan/hear/