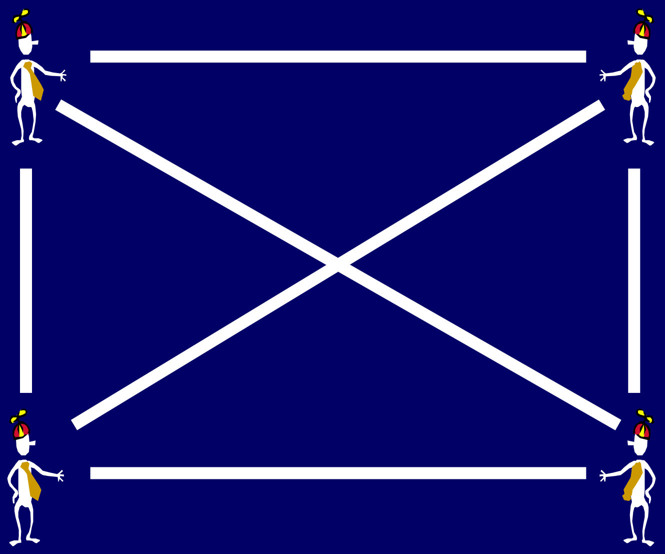
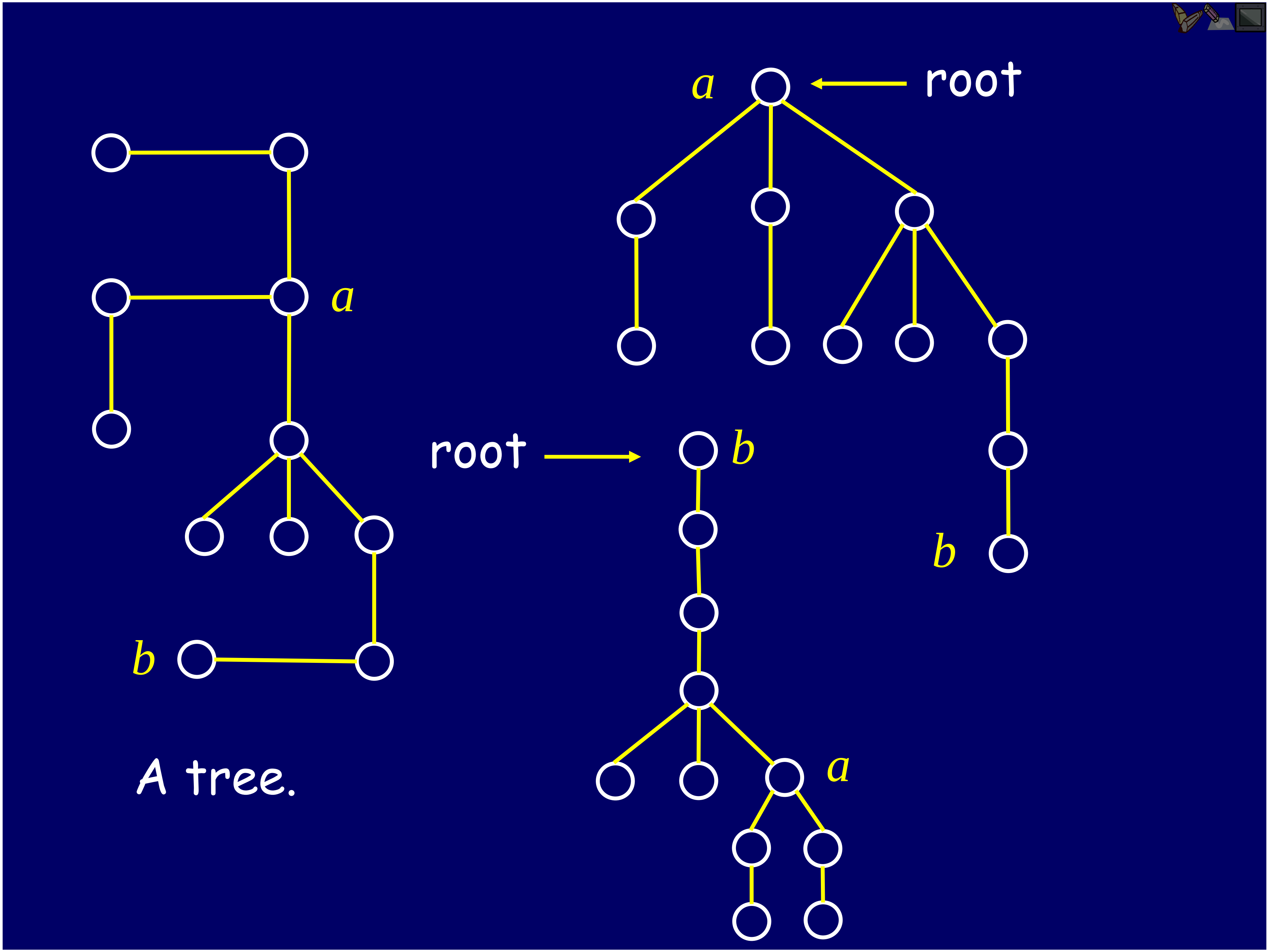
Great Theoretical Ideas In Computer Science

John Lafferty | CS 15-251 | Fall 2005

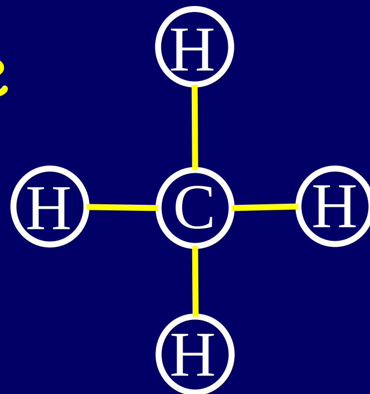
Lecture 14 | October 18, 2005 | Carnegie Mellon University

Graphs

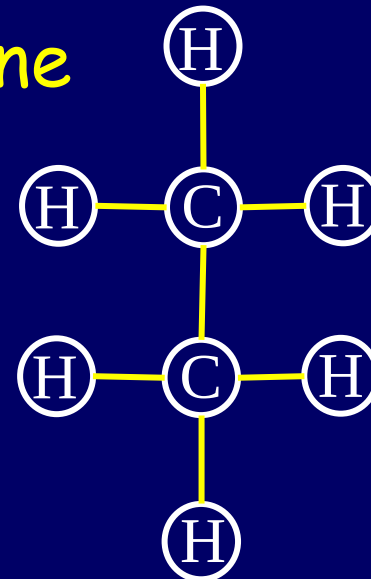



a
root
a
root
b
b
b
a
A tree.

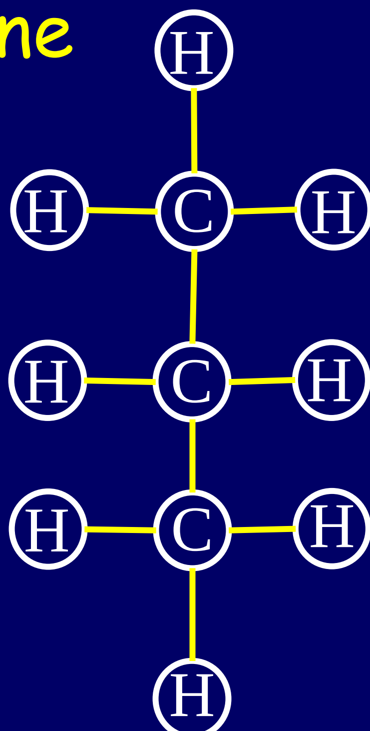
methane

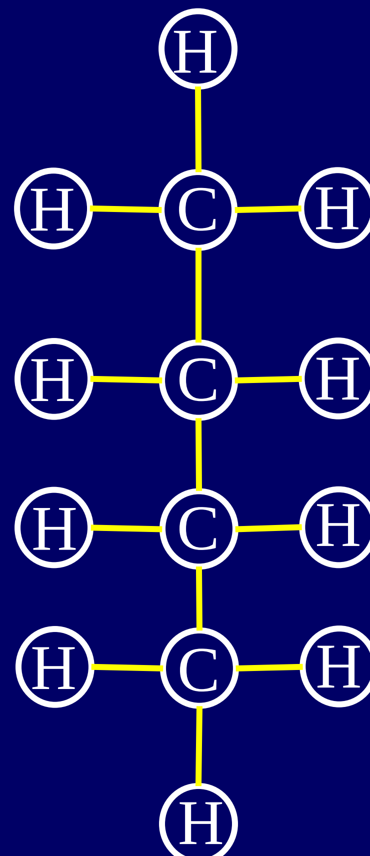


ethane



propane



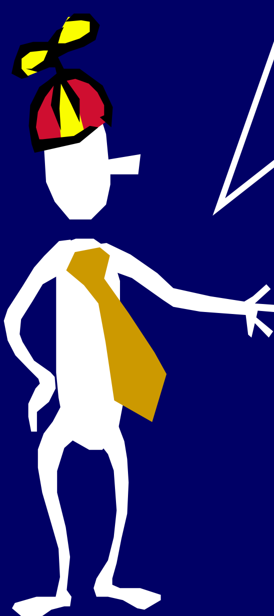


butane

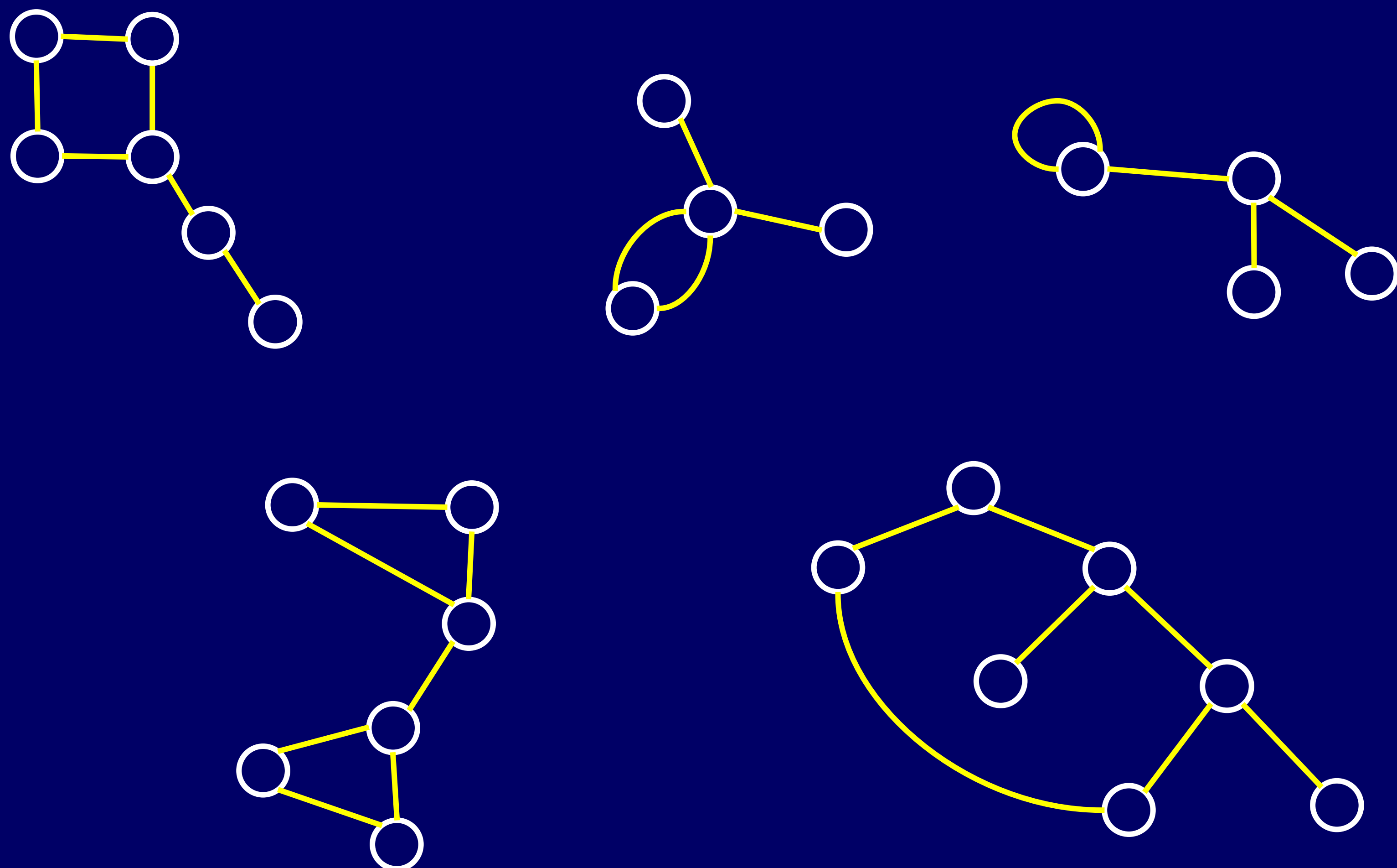
some saturated hydrocarbons

Putting a picture into words…

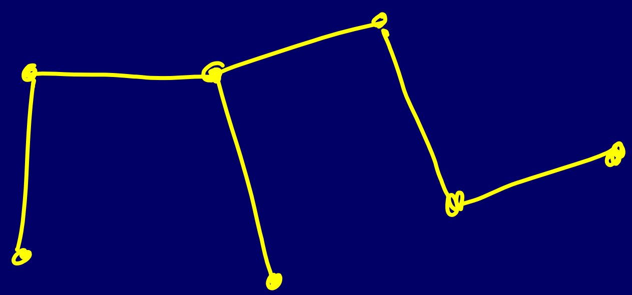
A ***tree*** is a connected graph with no cycles.



These are not trees...



Recall: The Shy People Party



no cycles can be formed

How many trees on 1-6 vertices?

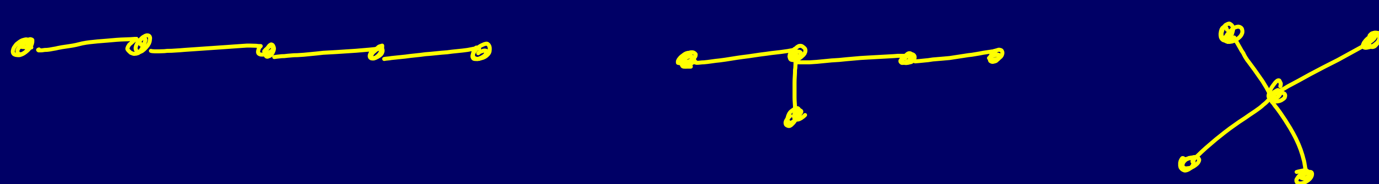
1:

2:

3:

4:

5:



We'll pass around a piece of paper. Draw a new 8-node tree, and put your name next to it.

Theorem: Let G be a graph with n nodes and e edges.

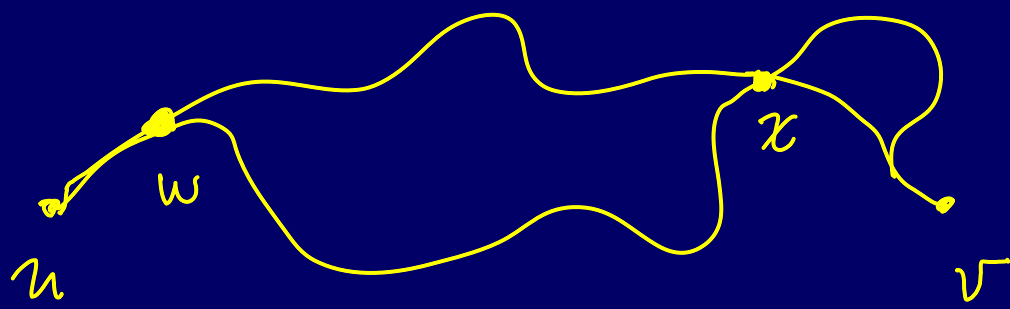
The following are equivalent:

1. G is a tree (connected, acyclic)
2. Every two nodes of G are joined by a unique path
3. G is connected and $n = e + 1$
4. G is acyclic and $n = e + 1$
5. G is acyclic and if any two nonadjacent points are joined by a line, the resulting graph has exactly one cycle.

To prove this, it suffices to show

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$$

$1 \Rightarrow 2$



implies there is a cycle

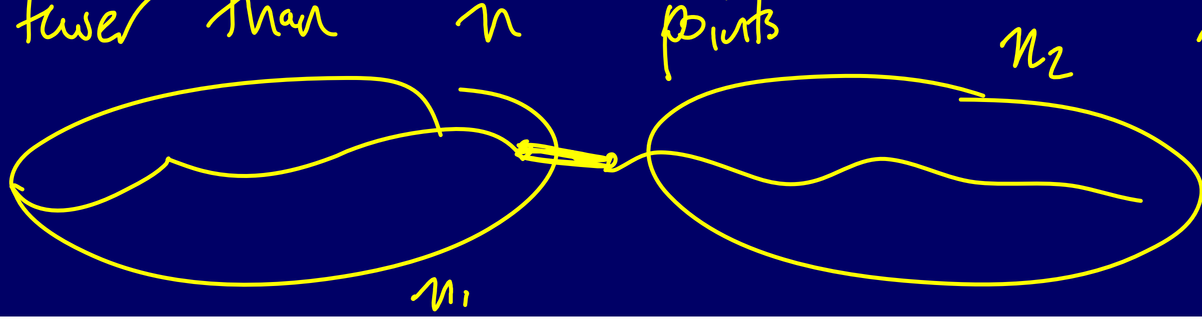
By I.H.

$n_1 = e_1 + 1$

$2 \Rightarrow 3$ Pf. by induction. Assume true for fewer than n points

$n_2 = e_2 + 1$

$n = e_1 + e_2 + 2$

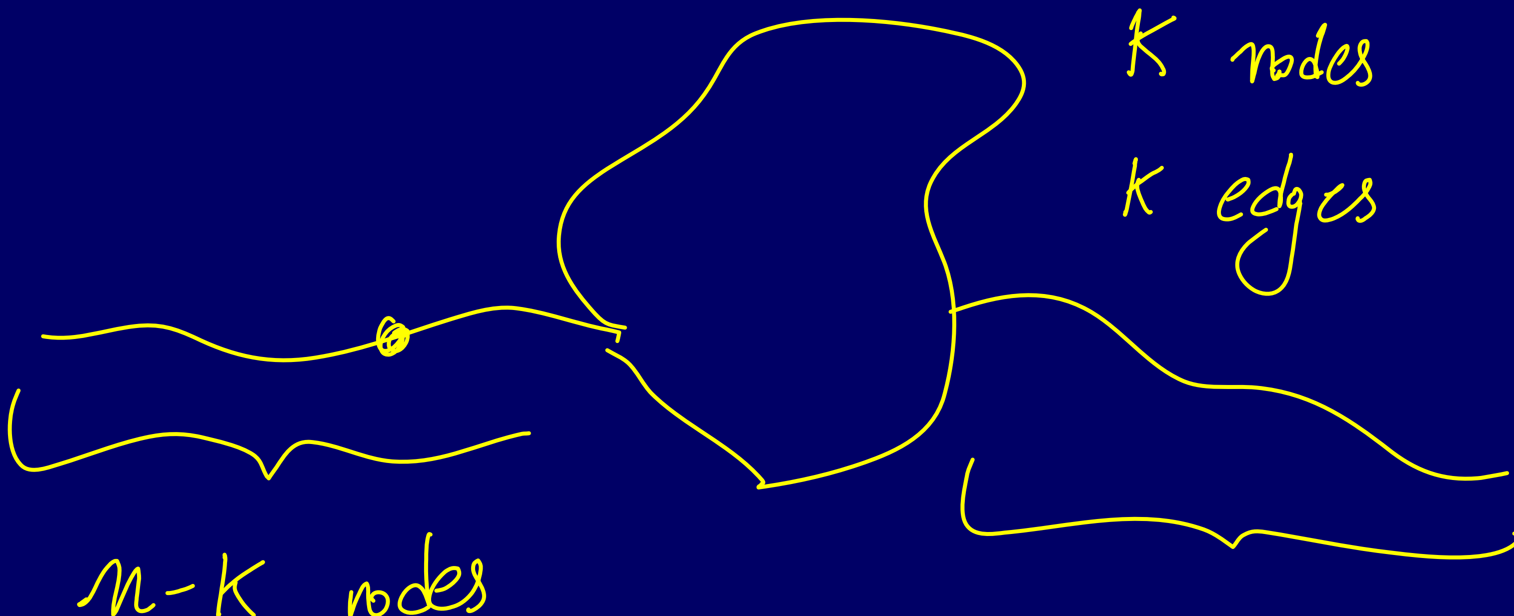


$= e - 1 + 2$

$= e + 1$

$3 \Rightarrow 4$

Suppose there is a cycle



So we have $\geq n-k$ edges outside cycle

$\Rightarrow$ at least n edges.

4 => 5. if there are K connected components, each is a tree.

$$n_i = e_i + 1$$

$$n = e + K \implies K = \underline{1}$$

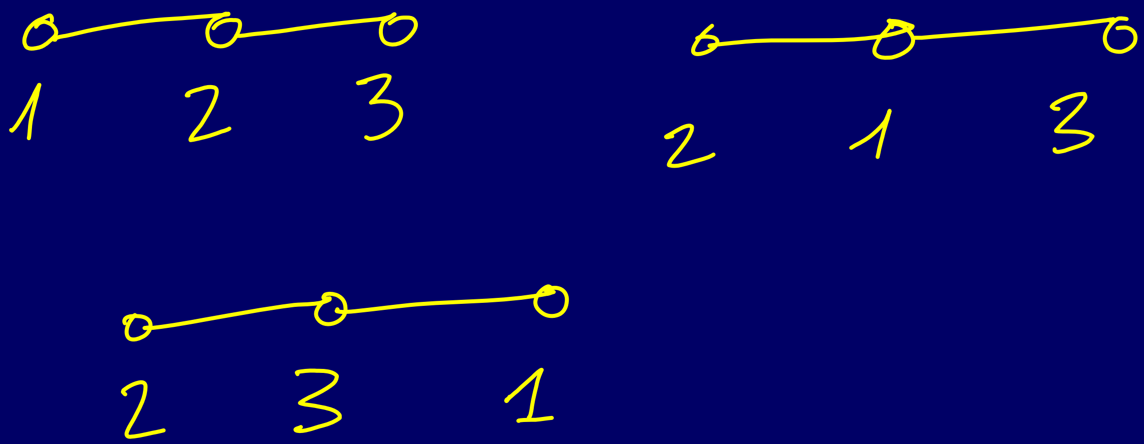
So, any two disconnected points, have a unique path between them, forming an edge between them creates a cycle.

Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

$$2e = 2p - 2$$

Question:

How many *labeled* trees are there with three nodes?

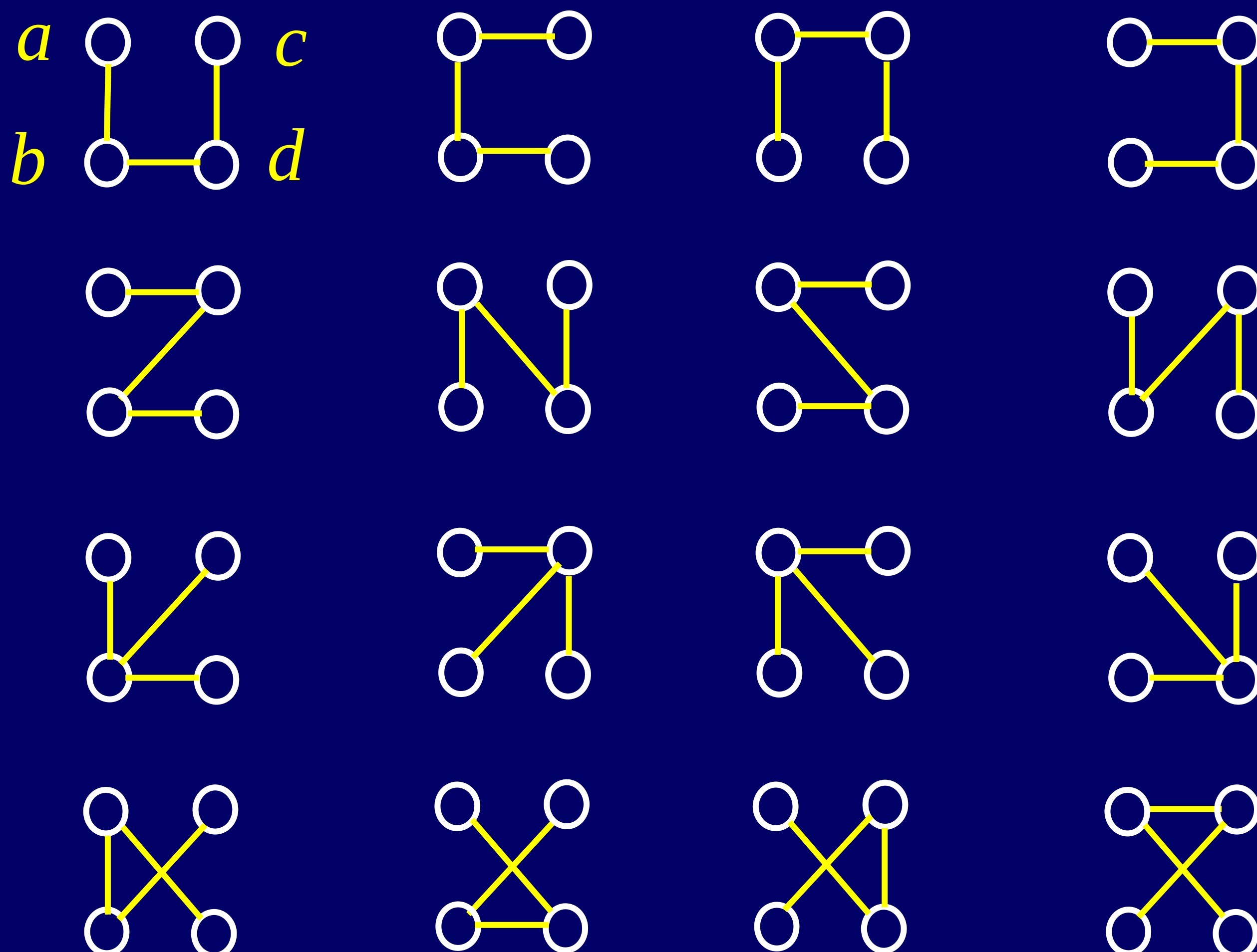


Question:

How many *labeled* trees are there with four nodes?

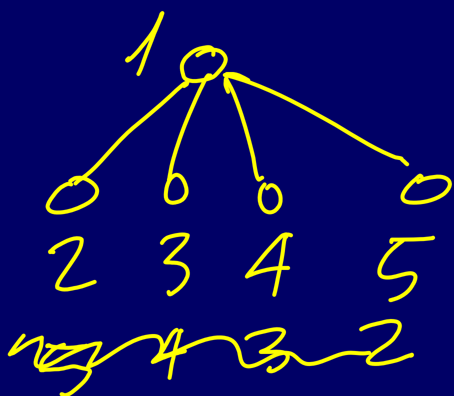
2 trees

16 labeled trees

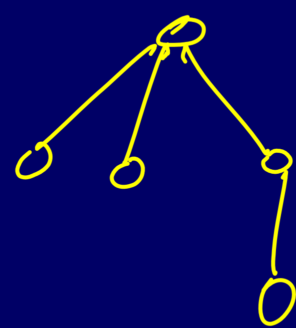


Question:

How many labeled trees are there with five nodes?

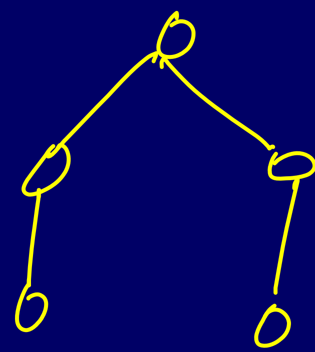


5 labelings



$5 \times \binom{4}{3} \times 3$

60



$5 \times \binom{4}{2} \times 2$

60

125 labeled trees.

Question:

How many labeled trees on n nodes are there?

3: 3 trees

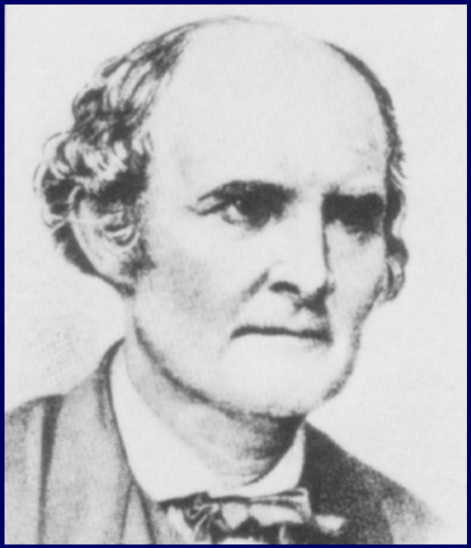
4: 16 trees

5: 125 trees

n^{n-2}

Cayley's formula

The number of labeled trees on n nodes is



$$n^{n-2}$$

The proof will use the correspondence principle.

Each labeled tree on n nodes

corresponds to

A sequence in $\{1,2,\ldots,n\}^{n-2}$ that is, (n-2) numbers, each in the range [1..n]

How to make a sequence from a tree.

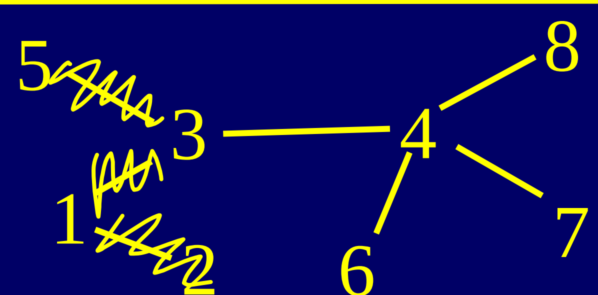
Loop through i from 1 to n-2

Let l be the degree-1 node with the lowest label.

Define the i^{th} element of the sequence as the label of the node adjacent to l.

Delete the node l from the tree.

Example:



$$\langle 1, 3, 3, 4, 4, 4 \rangle$$

$\langle 1, 3, 3, 4, 4, 4 \rangle$

How to reconstruct the unique tree from a sequence S.

Let $I = \{1, 2, 3, \ldots, n\}$

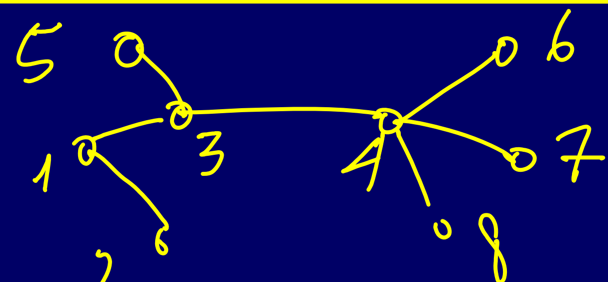
Loop until $S = \varepsilon$ ← empty sequence

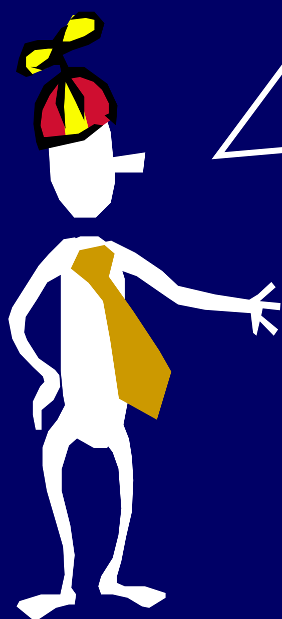
- Let l = smallest # in I but not in S
- Let s = first label in sequence S
- •Add edge $\{l, s\}$ to the tree.
- •Delete l from I.
- •Delete s from S.

Add edge $\{l, s\}$ to the tree, where $I = \{l, s\}$

$\langle 1, 3, 3, 4, 4, 4 \rangle$

$I = 1, 2, 3, 4, 5, 6, 7, 8$



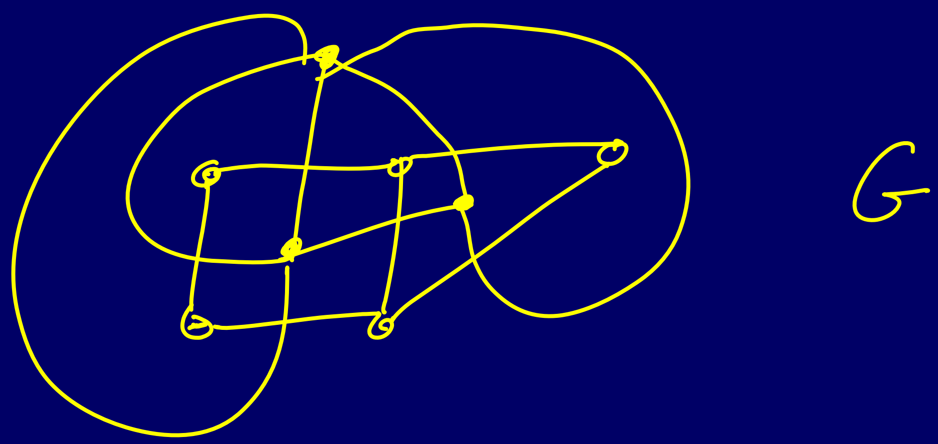


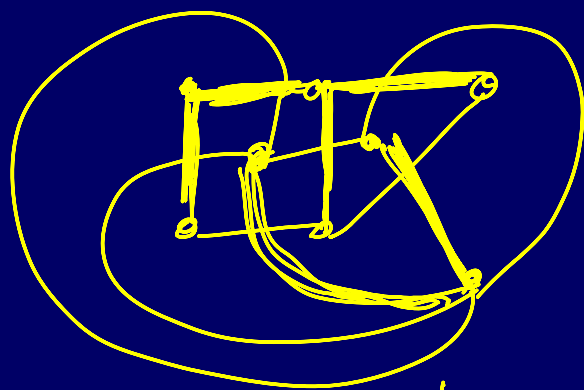
A graph is *planar* if it can be drawn in the plane without crossing edges. A *plane graph* is any such drawing, which breaks up the plane into a number f of *faces* or regions

Euler's Formula

If G is a connected plane graph with n vertices, e edges and f faces, then $n - e + f = 2$

Rather than using induction, we'll use the important notion of the *dual graph*





Let T be a spanning tree of G

Let T^* be the graph where there is an edge in dual graph for each edge in $G-T$

T^* is a spanning tree of G^*

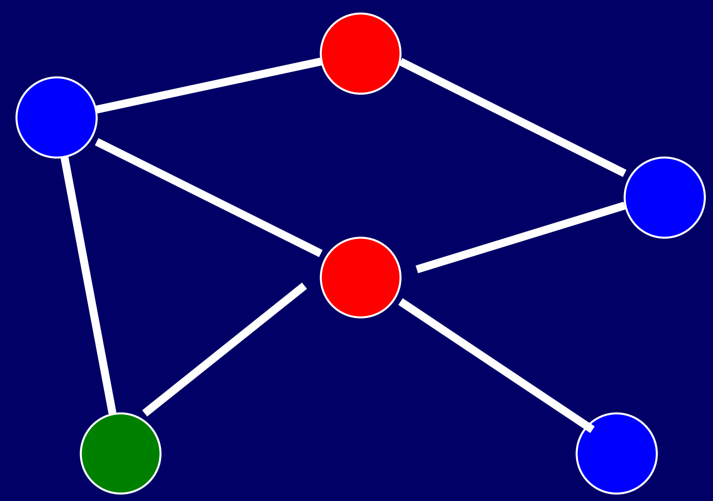
$$n = e_T + 1$$
$$f = e_{T^*} + 1$$
$$\Longrightarrow \quad n + f = e_T + e_{T^*} + 2$$
$$= e + 2$$

Corollary: Let G be a plane graph with $n > 2$ vertices. Then

a) G has a vertex of degree at most 5.
b) G has at most $3n - 6$ edges

Graph Coloring

A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color.

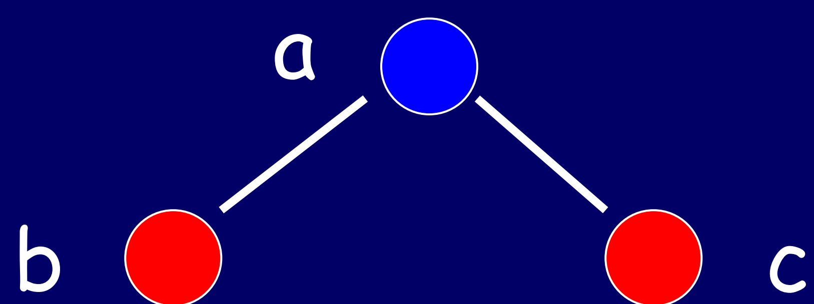


Graph Coloring

Arises surprisingly often in CS.

Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register.

Instructions	Live variables
	a
b = a+2	
	a,b
c = b*b	
	a,c
b = c+1	
	a,b
return a*b	



Every plane graph can be 6-colored

By induction. Assume true for < n nodes.

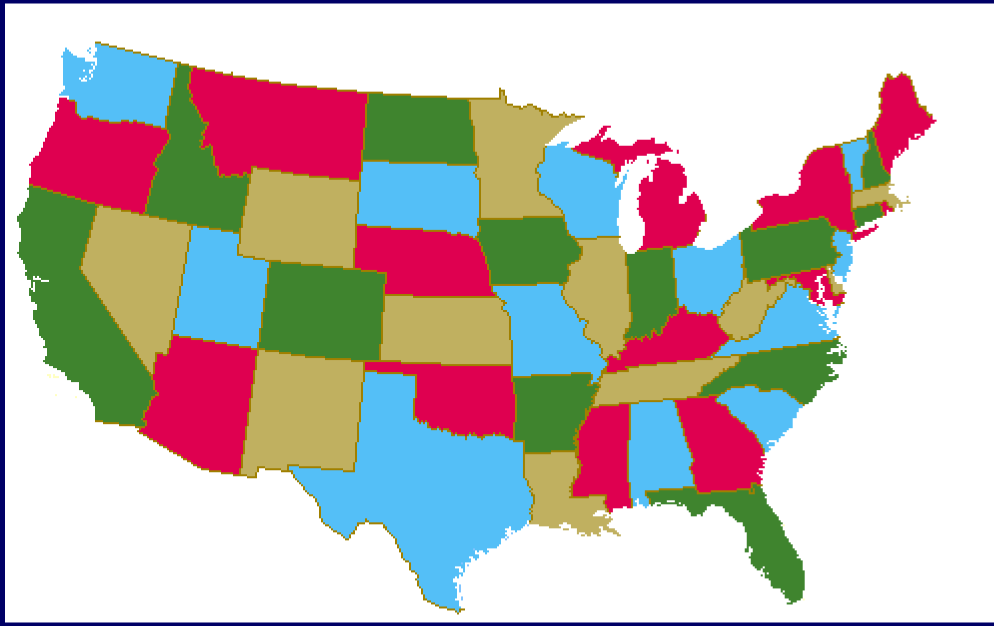
If have a plane graph on n.

There is some node v with deg. ≤ 5.

Remove v and color by I.H.

Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree <= 5.

4-color theorem remains challenging



http://www.math.gatech.edu/~thomas/FC/fourcolor.html

Graph Spectra

We now move to a different *representation* of graphs that is extremely powerful, and useful in many areas of computer science: AI, information retrieval, computer vision, machine learning, CS theory,...

Adjacency matrix

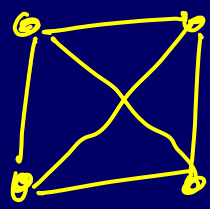
Suppose we have a graph G with n vertices and edge set E. The *adjacency matrix* is the nxn matrix $A=[a_{ij}]$ with

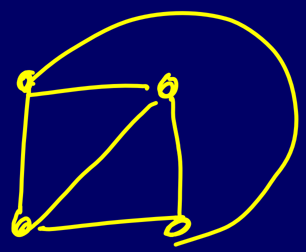
$a_{ij} = 1$ if (i,j) is an edge

$a_{ij} = 0$ if (i,j) is not an edge

Example

K_4





$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Counting Paths

The number of paths of length k from node i to node j is the entry in position (i,j) in the matrix A^k

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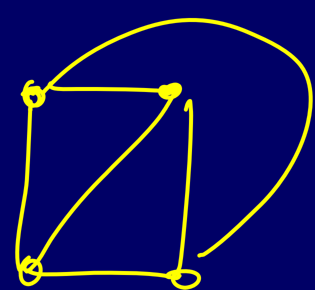
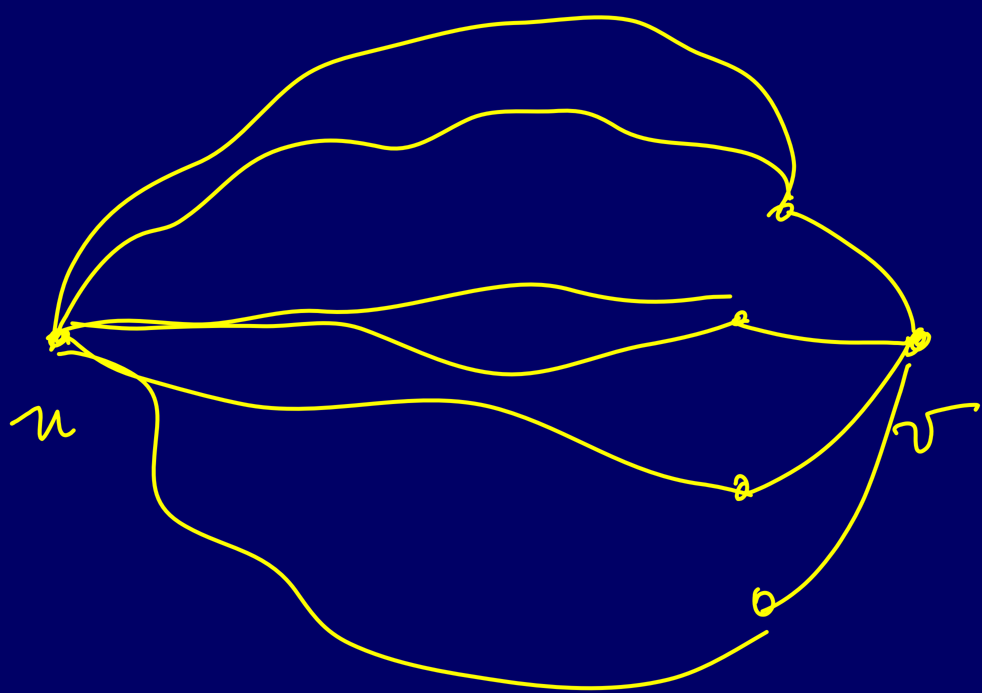
The number of paths of length k from node i to node j is the entry in position (i,j) in the matrix A^k

$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$



By induction on K

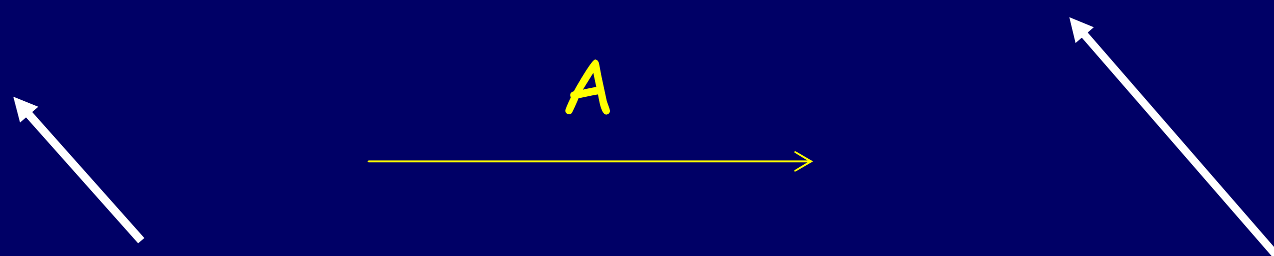


$$(A^K)_{uv} = \sum_j (A^{K-1})_{uj} A_{jv} = \sum_{j \in N(v)} (A^{K-1})_{uj}$$

Eigenvalues

An nxn matrix *A* is a linear transformation from n-vectors to n-vectors

An *eigenvector* is a vector *fixed* (up to length) by the transformation. The associated *eigenvalue* is the scaling of the vector.



Eigenvalues

Vector **x** is an eigenvector of **A** with eigenvalue λ if

$$Ax = \lambda x$$

A symmetric nxn matrix has at most n distinct real eigenvalues

Characteristic Polynomial

The *determinant* of A is the product of its eigenvalues:

$$\det A = \lambda_1 \lambda_2 \ldots \lambda_n$$

The *characteristic polynomial* of A is the polynomial

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda-\lambda_1)(\lambda- \lambda_2) \ldots (\lambda-\lambda_n)$$

Example: K_4

Example: K_4

$$A\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\3\\3\end{pmatrix} \qquad \lambda = 3$$

$$A\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} \qquad \lambda = -1$$

$$p_A(\lambda) = (\lambda-3)(\lambda+1)^3$$

$$= \lambda^4 - 6\lambda^2 - 8\lambda - 3$$

If graph G has adjacency matrix A with characteristic polynomial

$$p_A(\lambda) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_n$$

then

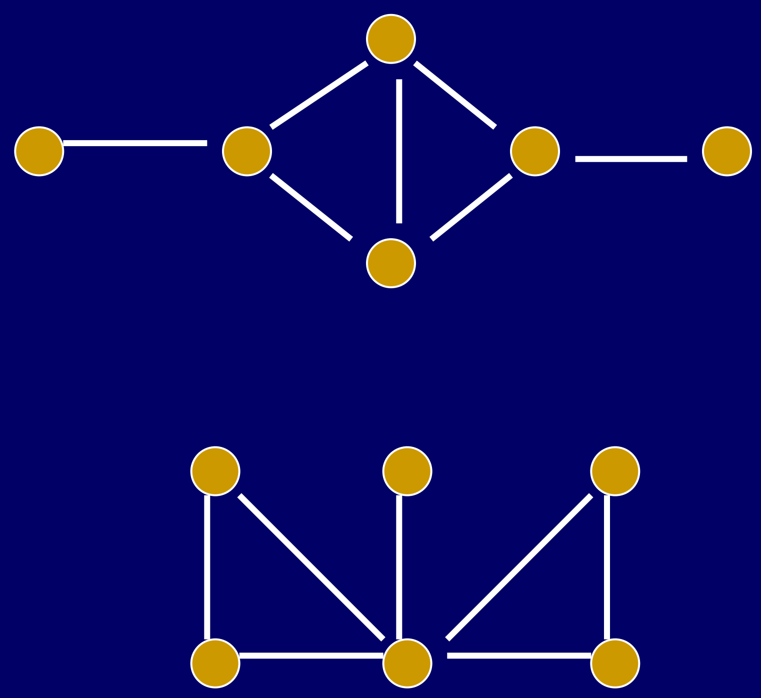
$c_1 = 0$

$-c_2$ = # of edges in G

$-c_3$ = twice # of triangles in G

Two different graphs with the same spectrum

$$p_A(\lambda) = \lambda^6 - 7\lambda^4 - 4\lambda^3 + 7\lambda^2 + 4\lambda - 1$$



Let your spectrum do the counting...

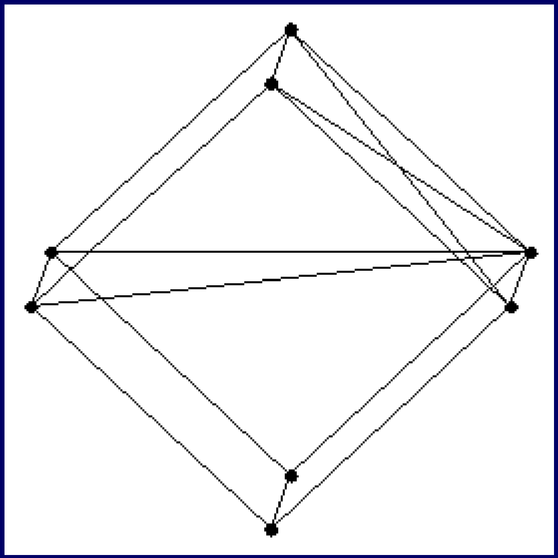
A *closed walk* or *loop* in a graph is a path whose initial and final vertices are the same. We easily get

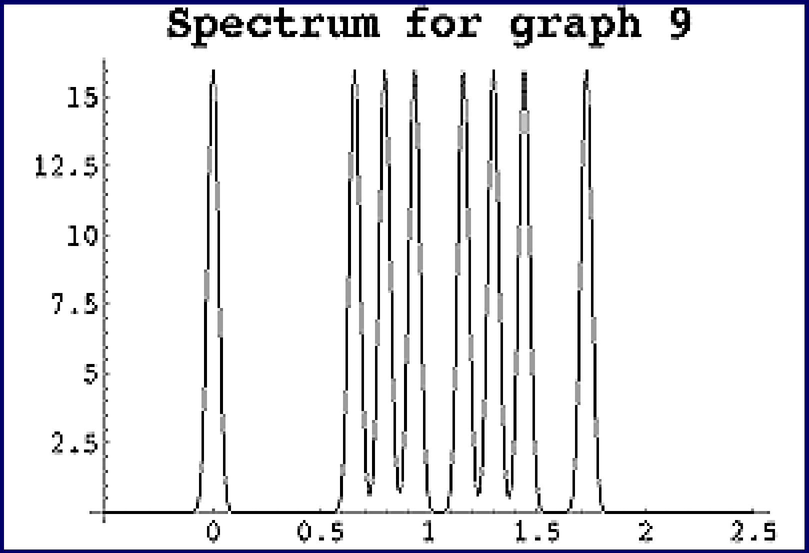
$$\mathrm{trace}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n = 0$$

$$\mathrm{trace}(A^2) = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2$$
$$= \text{twice \# of edges}$$

$$\mathrm{trace}(A^3) = \lambda_1^3 + \lambda_2^3 + \ldots + \lambda_n^3$$
$$= \text{six times \# of triangles}$$

Graph Muzak





http://math.ucsd.edu/~fan/hear/