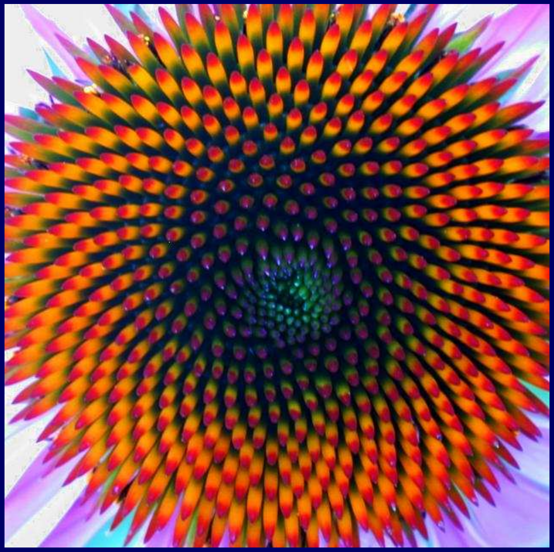
Great Theoretical Ideas In Computer Science

Anupam Gupta CS 15-251 Fall 2005

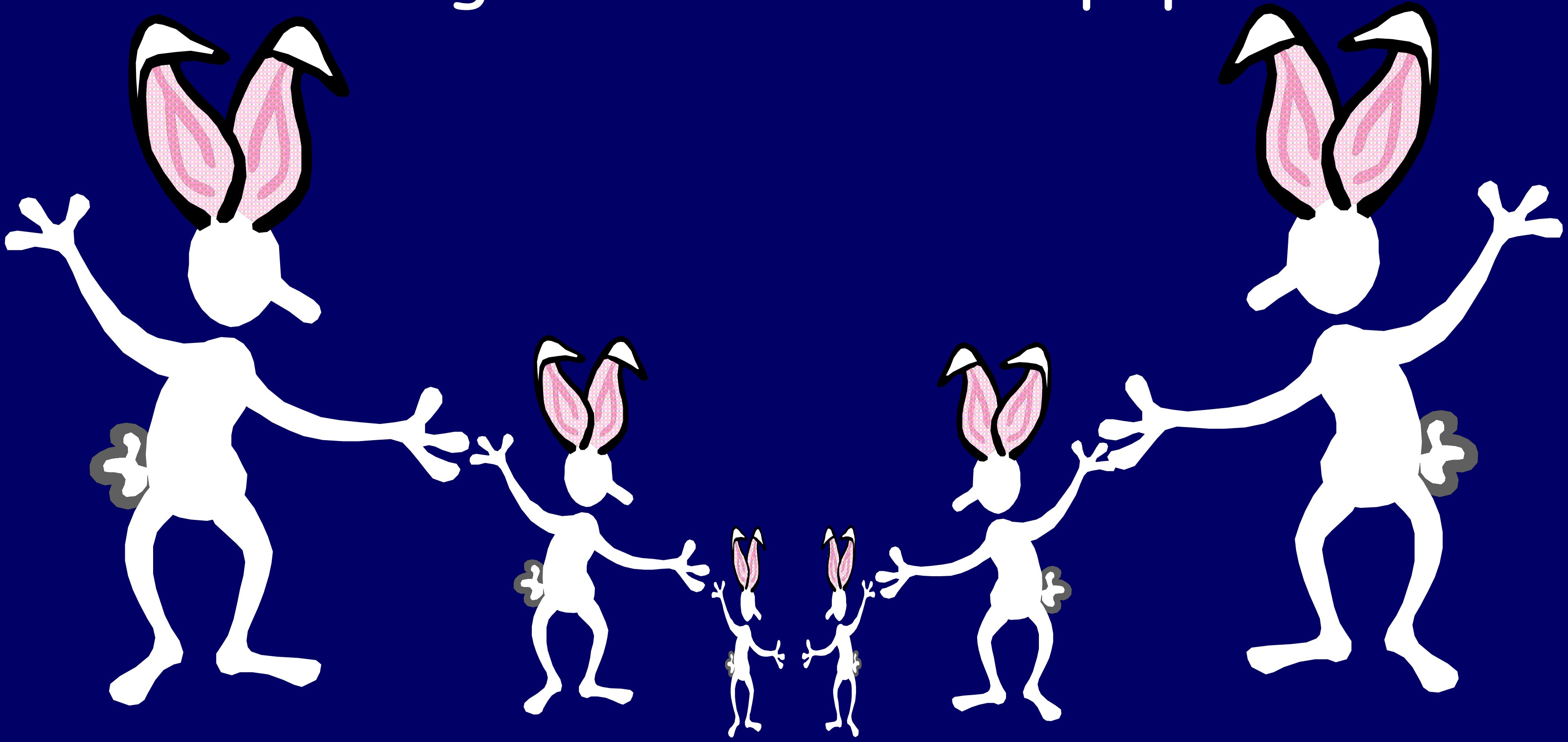
Lecture 13 October 11, 2005 Carnegie Mellon University

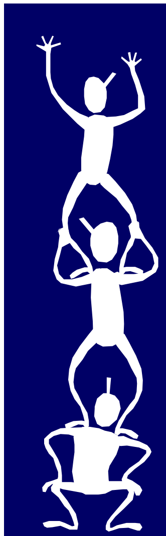
The Fibonacci Numbers And An Unexpected Calculation



Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.





Inductive Definition or Recurrence Relation for the Fibonacci Numbers

Stage 0, Initial Condition, or Base Case:

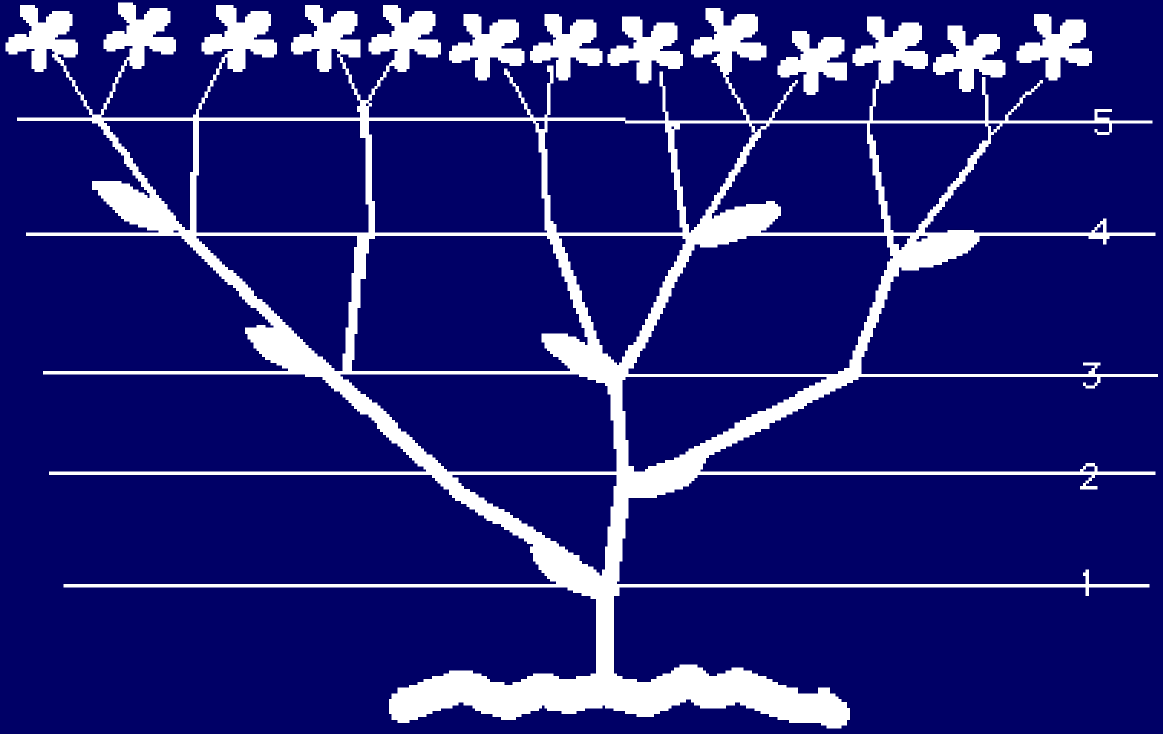
Fib(0) = 0; Fib (1) = 1

Inductive Rule

For n>1, Fib(n) = Fib(n-1) + Fib(n-2)

n	0	1	2	3	4	5	6	7
Fib(n)	0	1	1	2	3	5	8	13

Sneezwort (Achilleaptarmica)



Each time the plant starts a new shoot it takes two months before it is strong enough to support branching.

Counting Petals

5 petals: buttercup, wild rose, larkspur, columbine (aquilegia)

8 petals: delphiniums

13 petals: ragwort, corn marigold, cineraria, some daisies

21 petals: aster, black-eyed susan, chicory

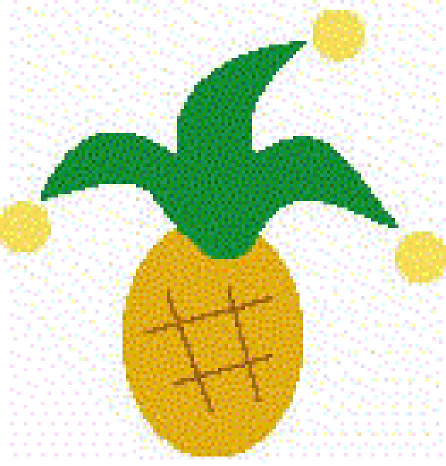
34 petals: plantain, pyrethrum

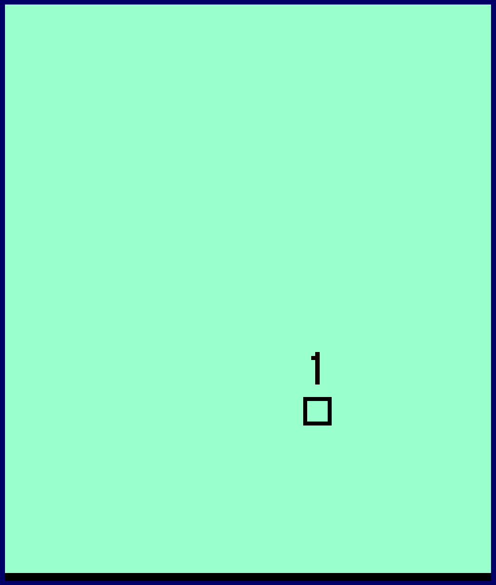
55, 89 petals: michaelmas daisies, the asteraceae family.

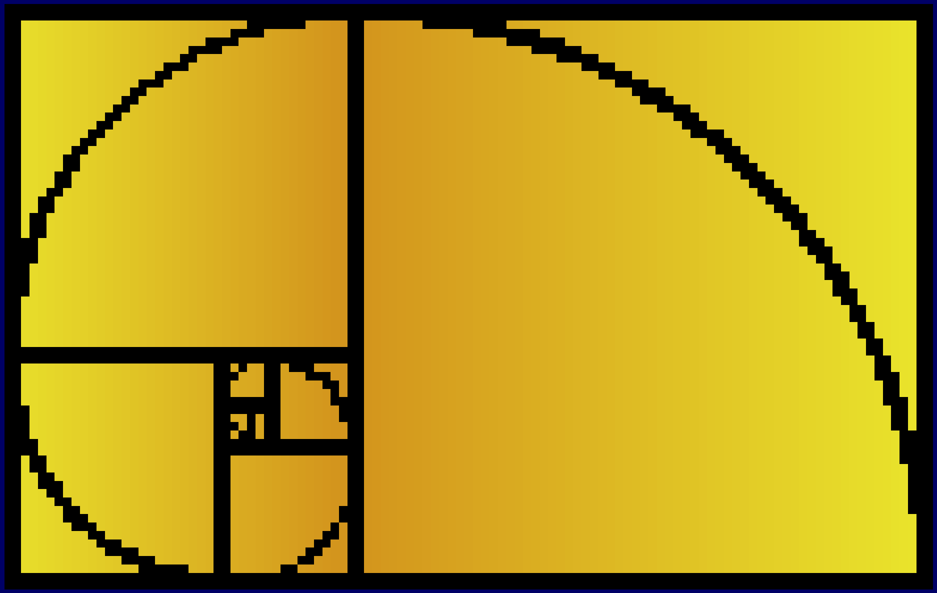
Pineapple whorls

Church and Turing were both interested in the number of whorls in each ring of the spiral.

The ratio of consecutive ring lengths approaches the Golden Ratio.

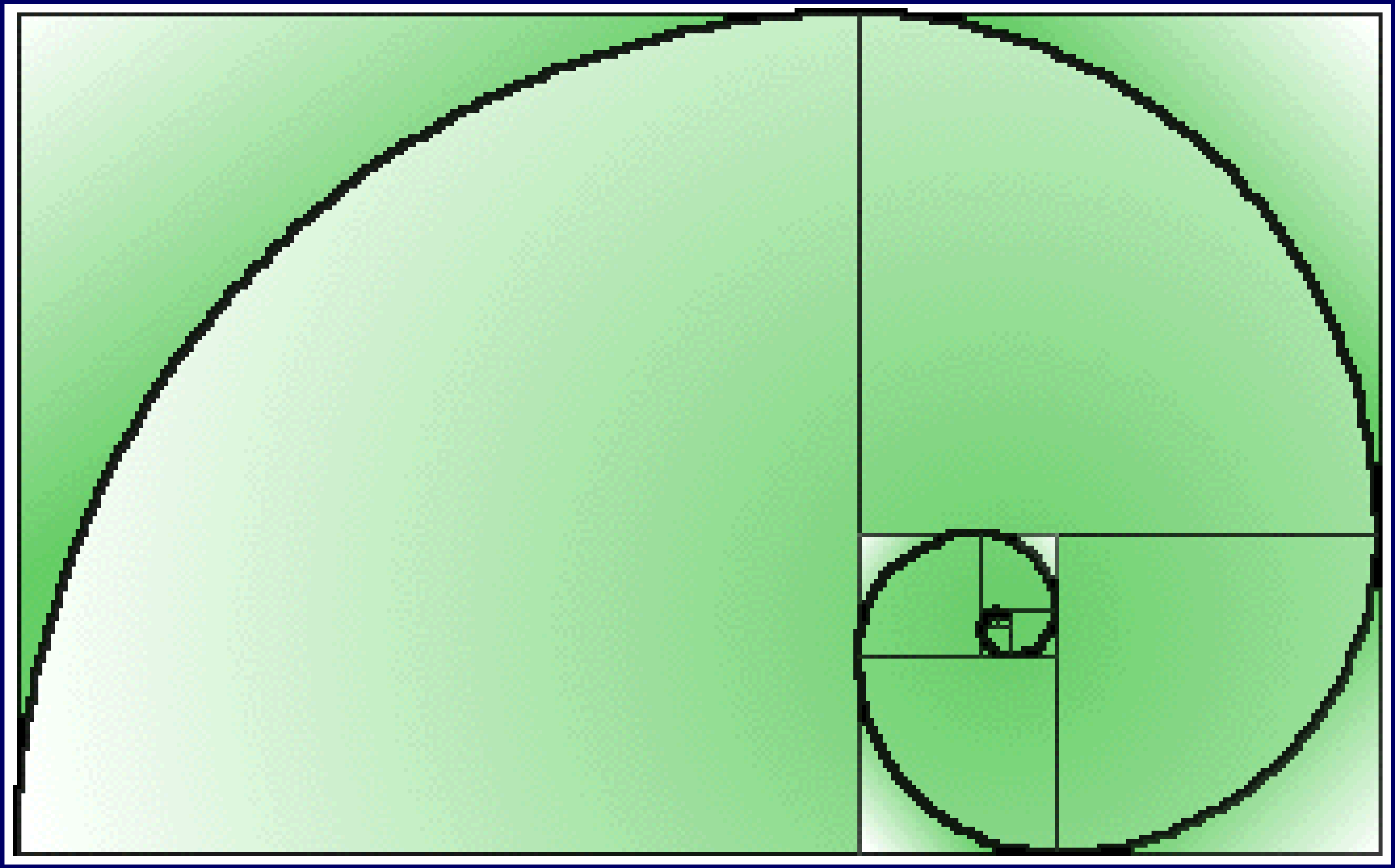



1



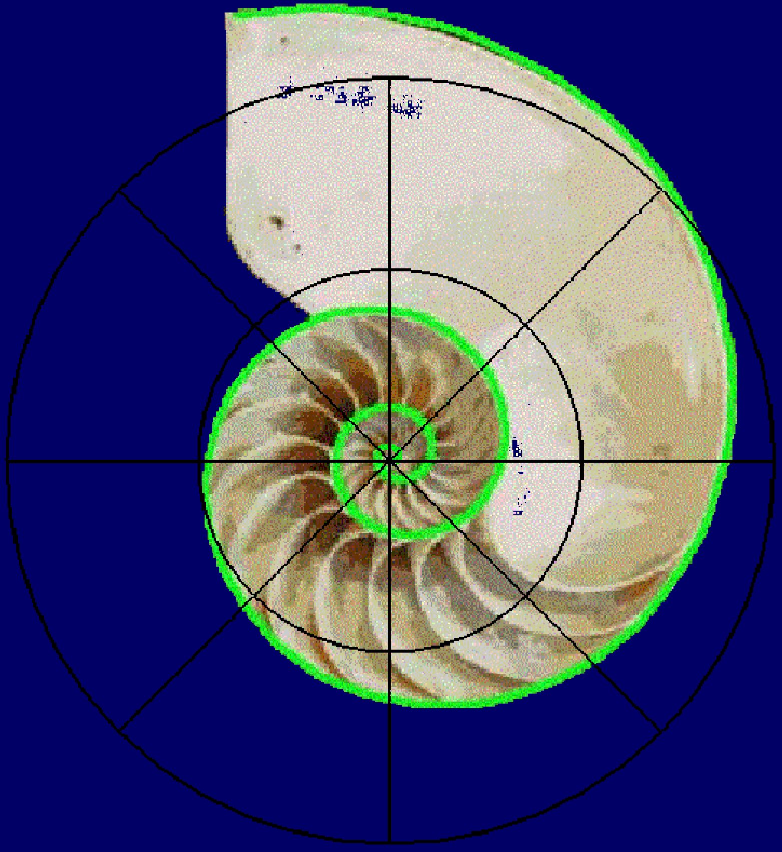
Bernoulli Spiral

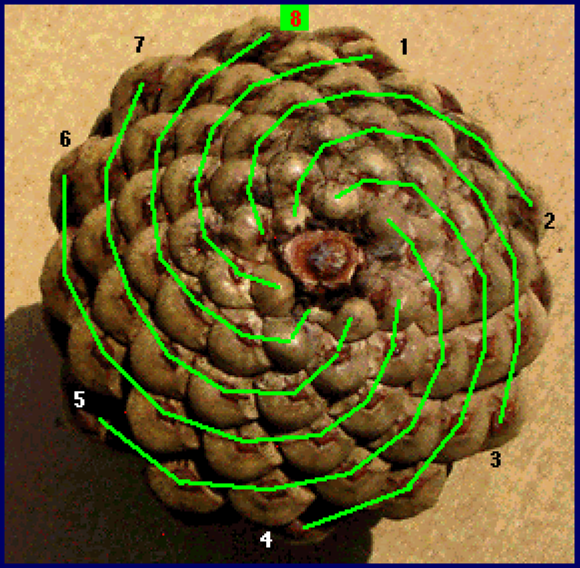
When the growth of the organism is proportional to its size

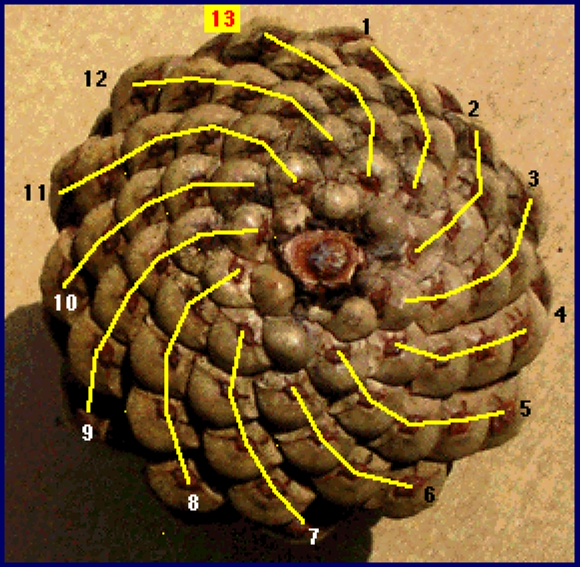


Bernoulli Spiral

When the growth of the organism is proportional to its size




8
7
1
6
2
5
3
4


13
1
12
2
11
3
10
4
9
5
8
6
7

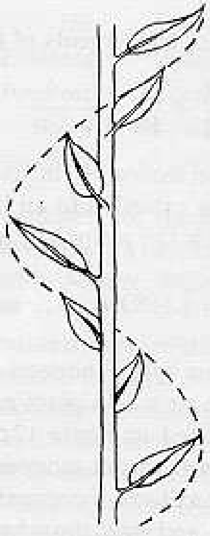
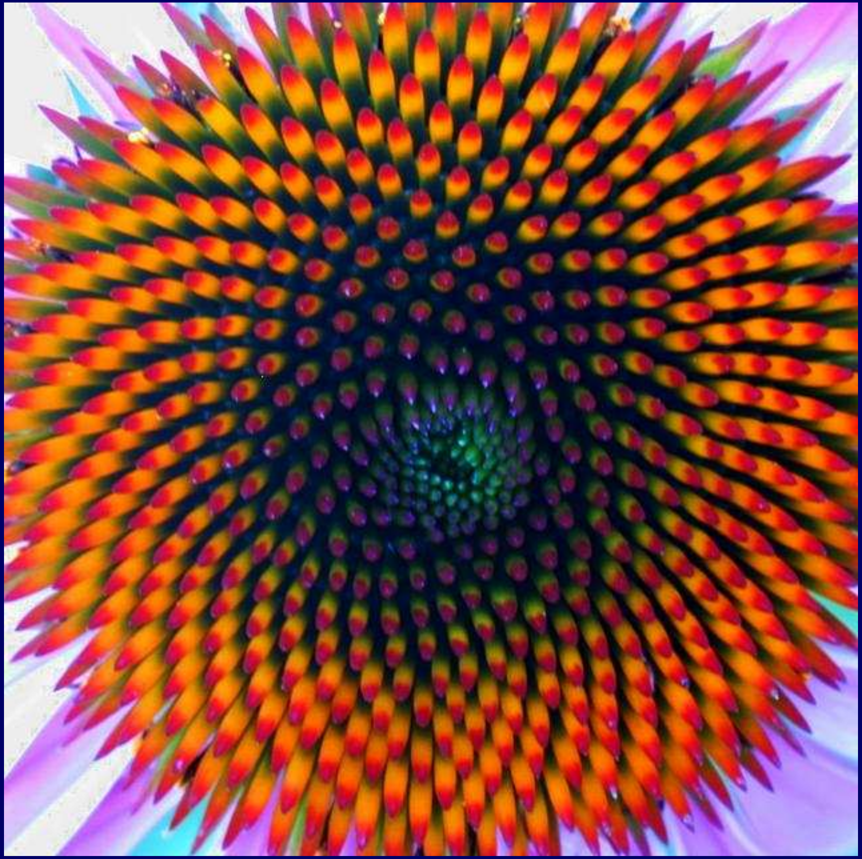


Fig. 12.4. Phyllotaxis



Golden Ratio: the divine proportion

$\phi = 1.6180339887498948482045...$

"Phi" is named after the Greek sculptor <u>Phi</u>dias

Definition of ϕ (Euclid)

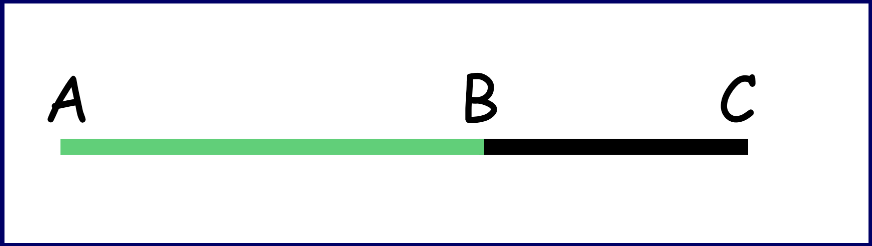
Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

$$\phi = \frac{AC}{AB} = \frac{AB}{BC}$$

$$\phi^2 = \frac{AC}{BC}$$

$$\phi^2 - \phi = \frac{AC}{BC} - \frac{AB}{BC} = \frac{BC}{BC} = 1$$

$$\boxed{\phi^2 - \phi - 1 = 0}$$



Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

$$= 1 + \frac{1}{1 + \frac{1}{\phi}}$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}}$$

Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+\ldots.}}}}}}}}}$$

Continued Fraction Representation

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+....}}}}}}}}}$$

Remember?

We already saw the convergents of this CF

[1,1,1,1,1,1,1,1,1,1,1,…]

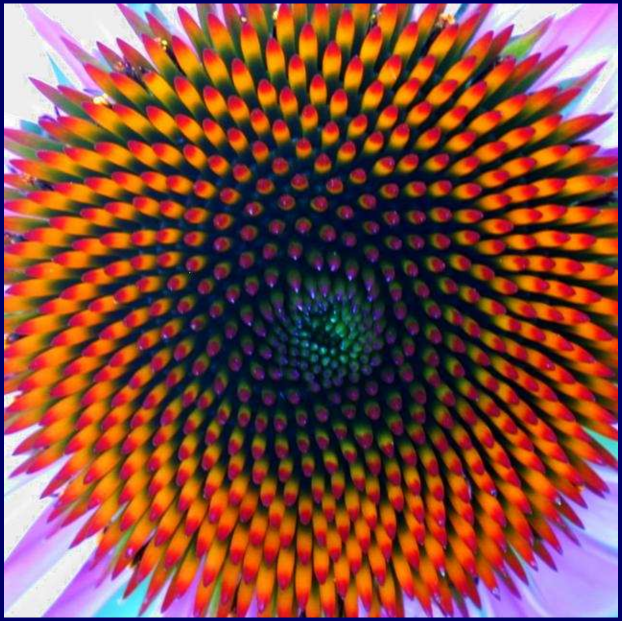
are of the form

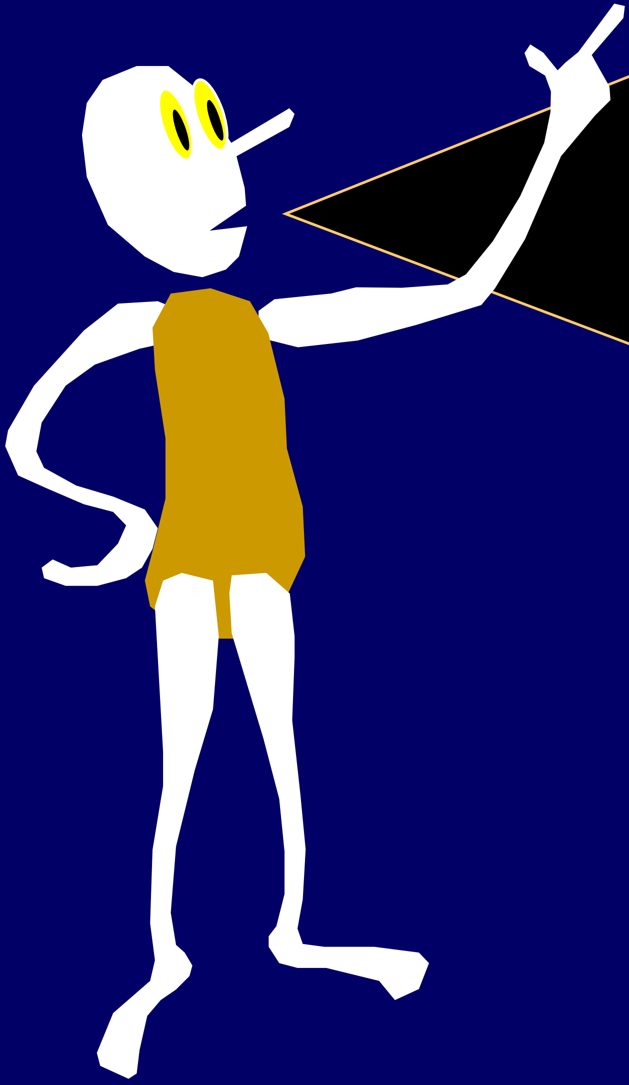
Fib(n+1)/Fib(n)

Hence: $$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi = \frac{1+\sqrt{5}}{2}$$

Continued Fraction Representation

$$\phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+....}}}}}}}}}$$





Let us take a slight detour and look at a different representation.

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

Example: $f_5 = 5$

4 = 2 + 2
2 + 1 + 1
1 + 2 + 1
1 + 1 + 2
1 + 1 + 1 + 1

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$f_1 = 1$

0 = the empty sum

$f_2 = 1$

1 = 1

$f_3 = 2$

2 = 1 + 1

2

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

# of sequences that begins with 1

# of sequences that begin with 2

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

\# of sequences beginning with a 1

\# of sequences beginning with a 2

Fibonacci Numbers Again

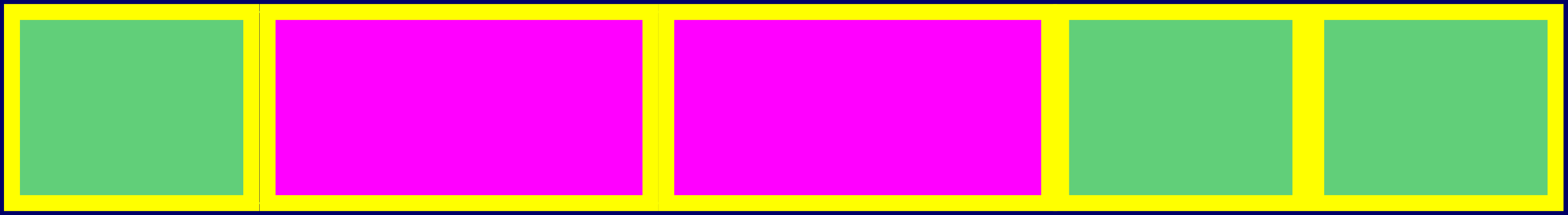
Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

$f_1 = 1$ $f_2 = 1$

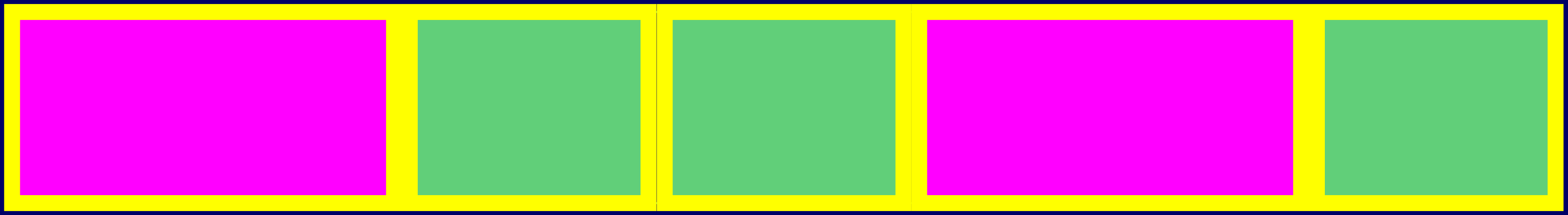
Visual Representation: Tiling

Let f_{n+1} be the number of different ways to tile a 1 × n strip with squares and dominoes.



Visual Representation: Tiling

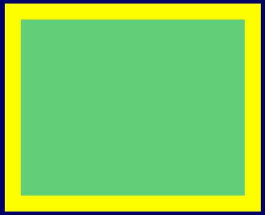
Let f_{n+1} be the number of different ways to tile a 1 × n strip with squares and dominoes.



Visual Representation: Tiling

1 way to tile a strip of length 0

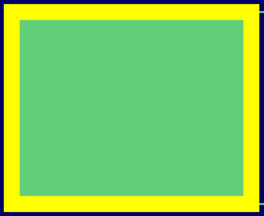
1 way to tile a strip of length 1:



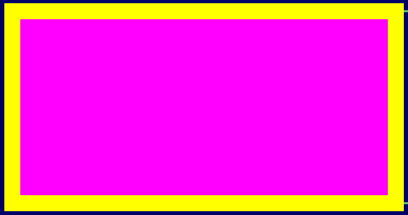
2 ways to tile a strip of length 2:

$$f_{n+1} = f_n + f_{n-1}$$

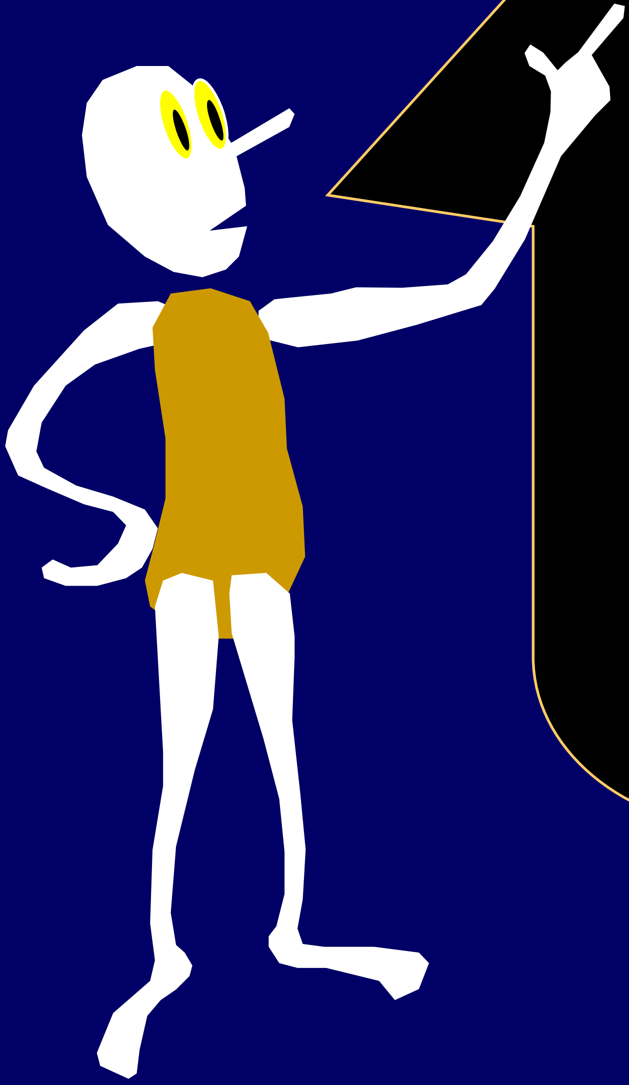
f_{n+1} is number of ways to title length n.



f_n tilings that start with a square.



f_{n-1} tilings that start with a domino.



Let's use this visual representation to prove a couple of Fibonacci identities.

Fibonacci Identities

Some examples:

$$F_{2n} = F_1 + F_3 + F_5 + \ldots + F_{2n-1}$$

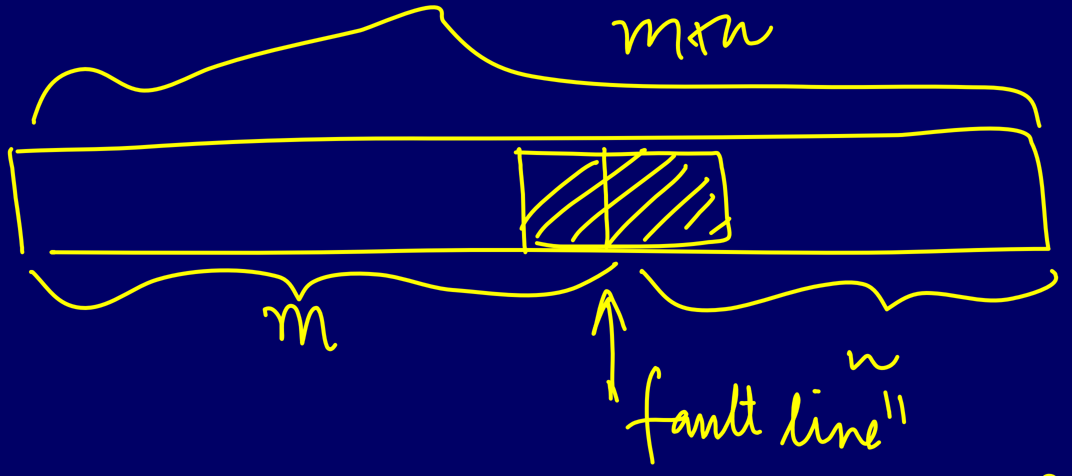
← try this yourselves

$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$

# of ways to tile m+n



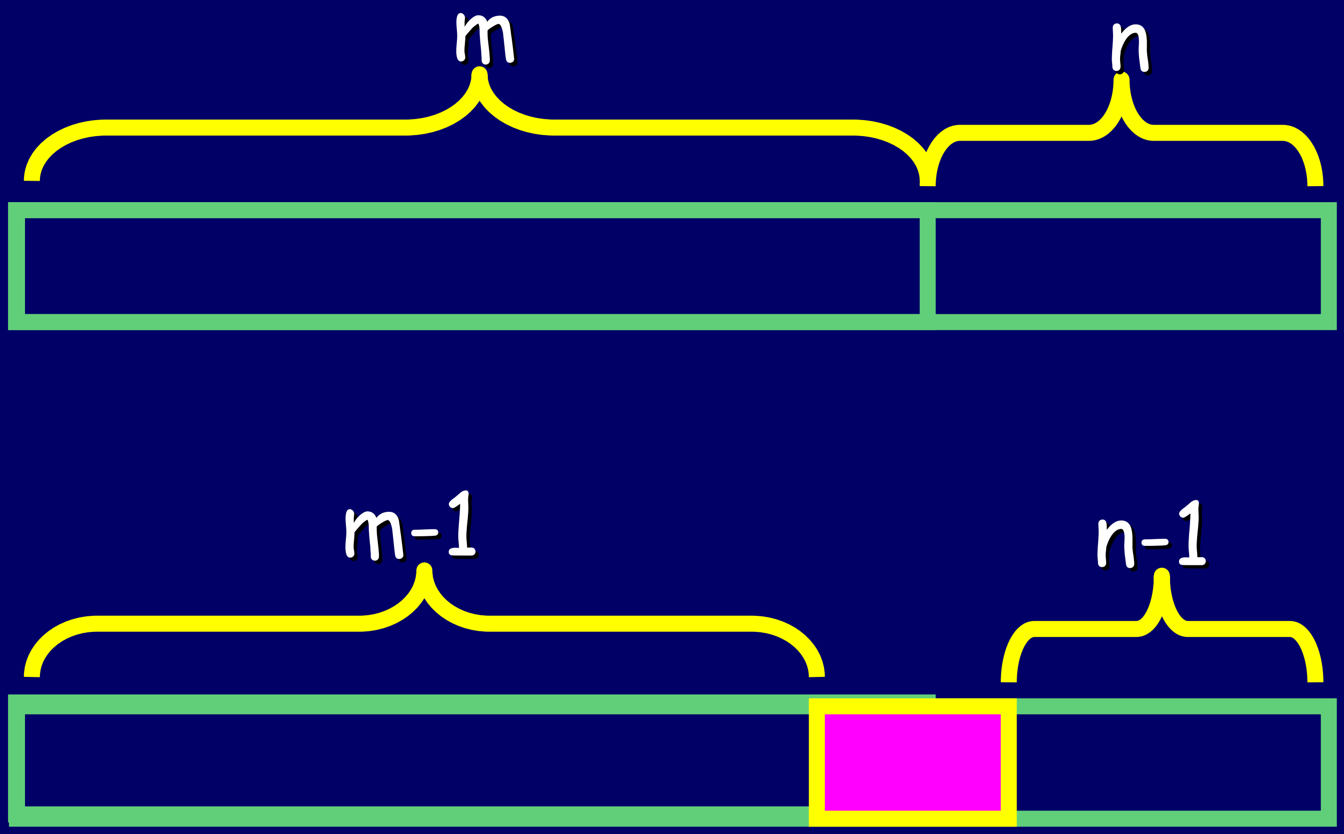
# of tilings where no tile straddles fault line

$= F_{m+1} F_{n+1}$

# of tilings where domino straddles fault line

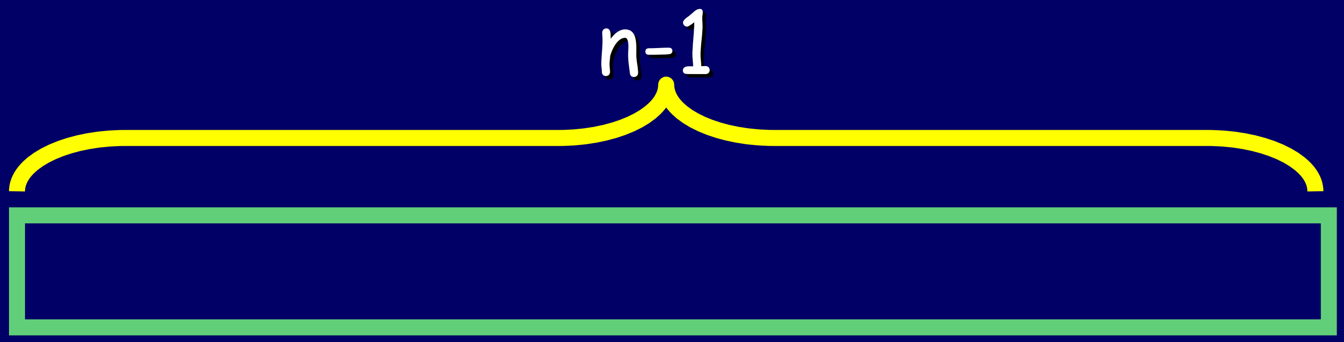
$= F_m F_n$

$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$



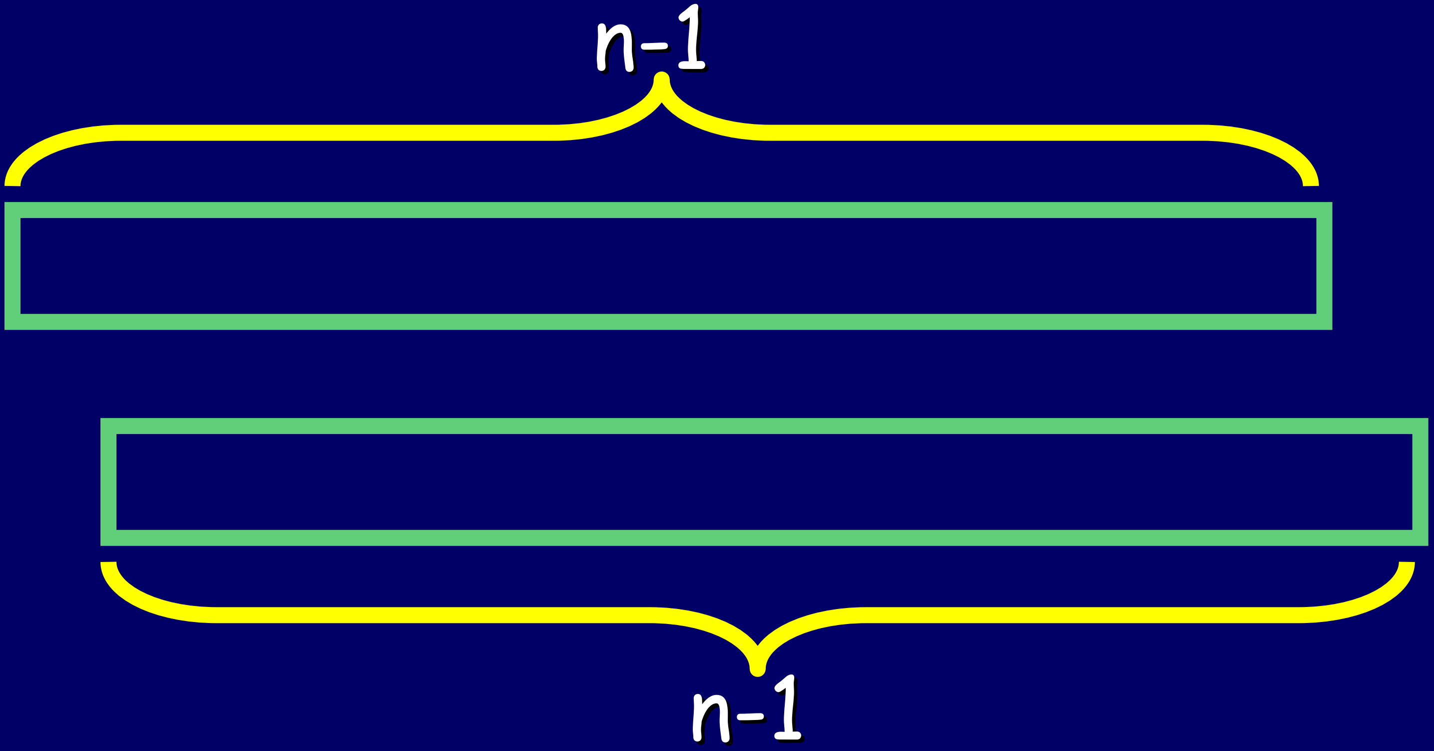
$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

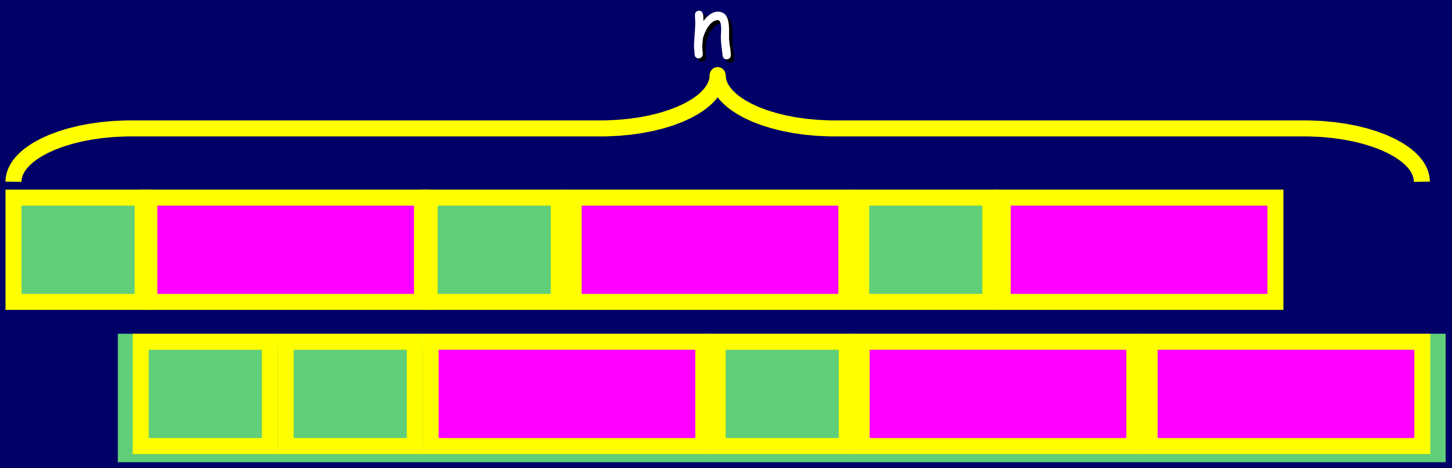


F_n tilings of a strip of length n-1

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

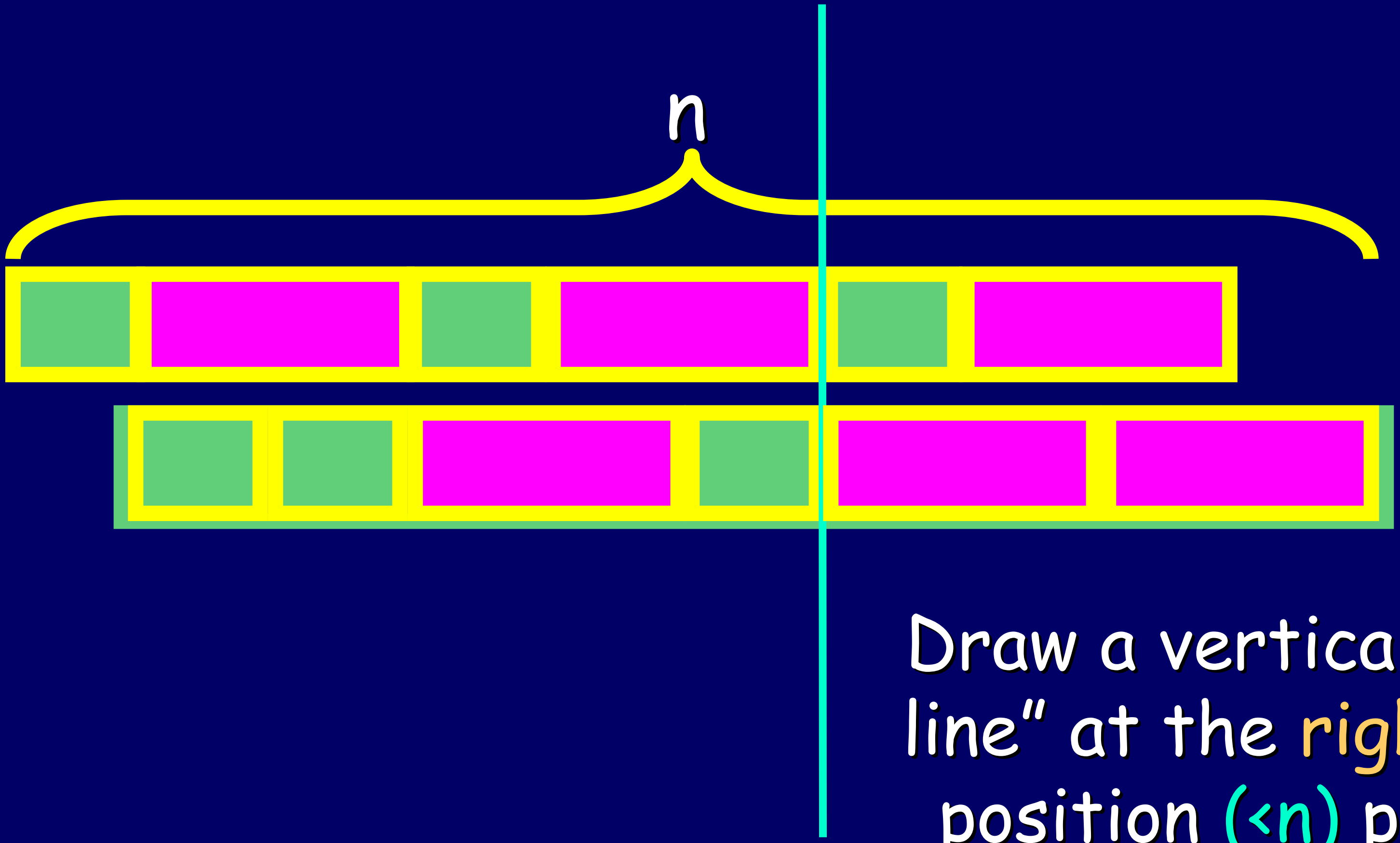


$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



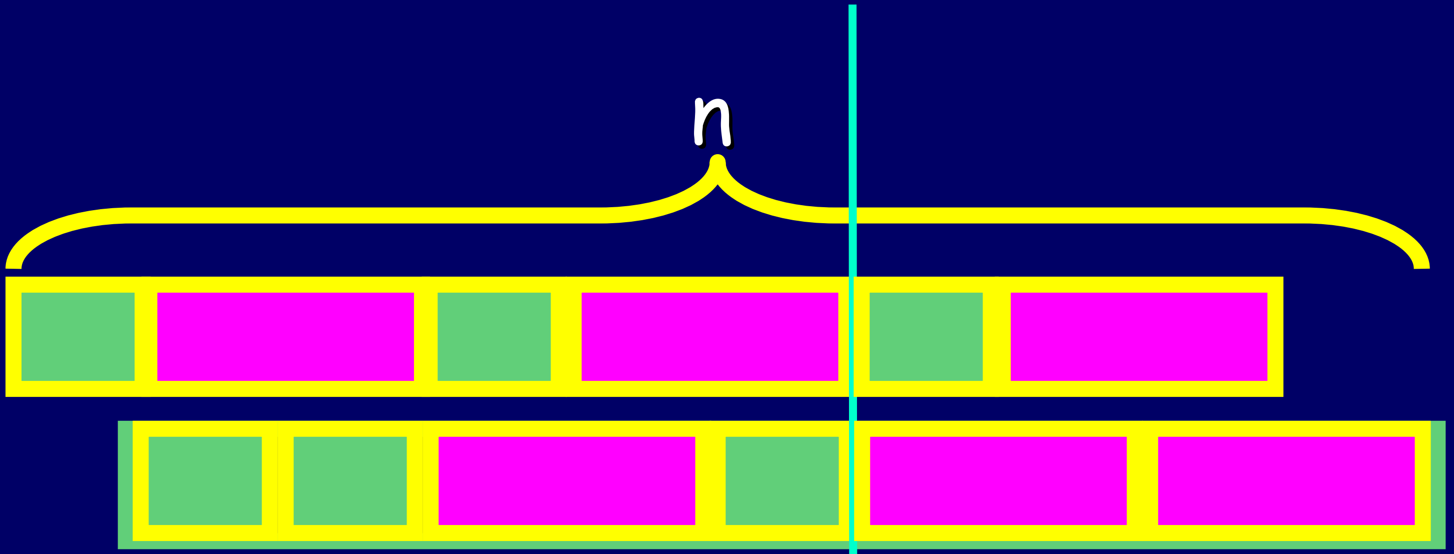
$(F_n)^2$ tilings of two strips of size n-1

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



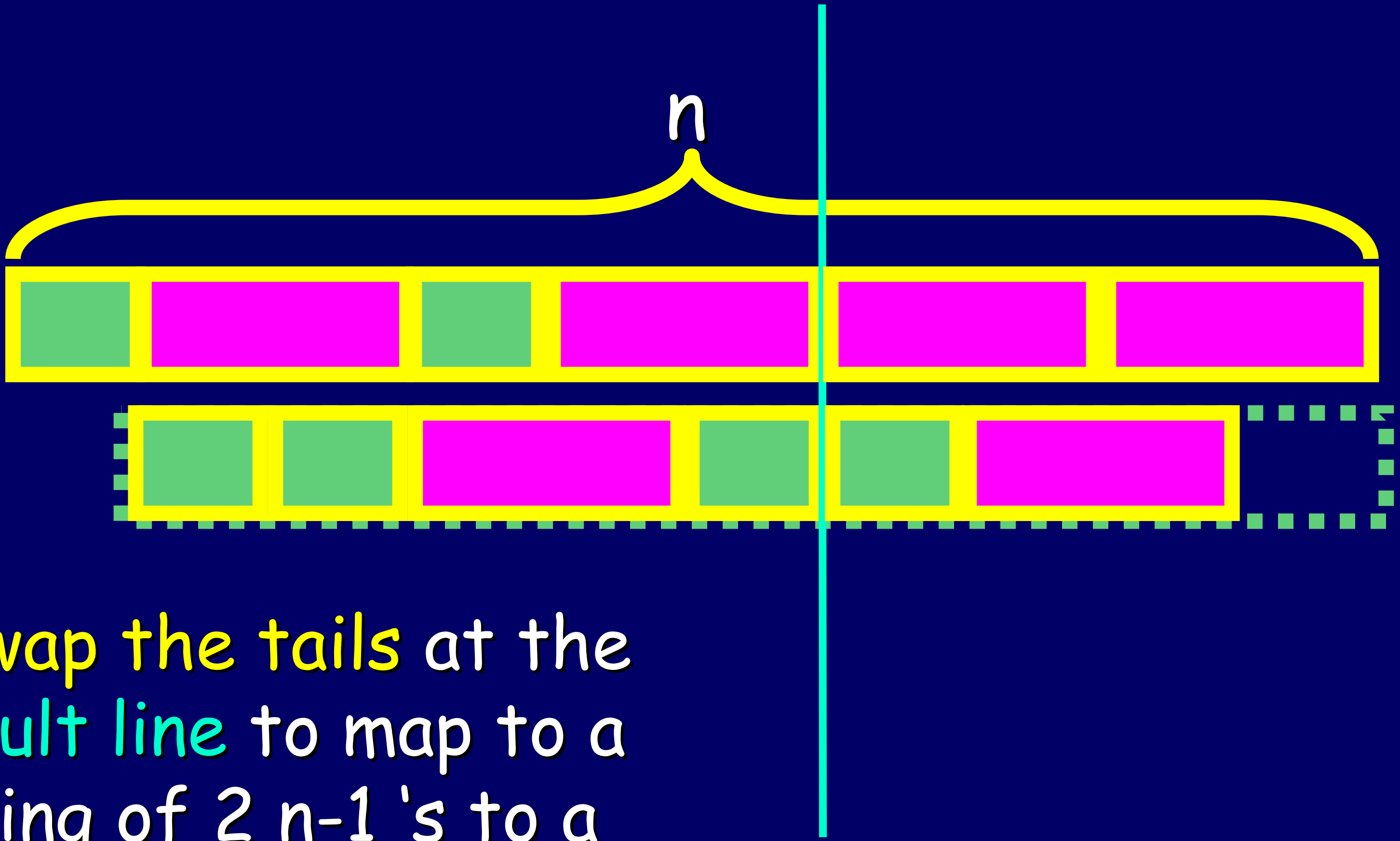
Draw a vertical "fault line" at the rightmost position (<n) possible without cutting any dominoes

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



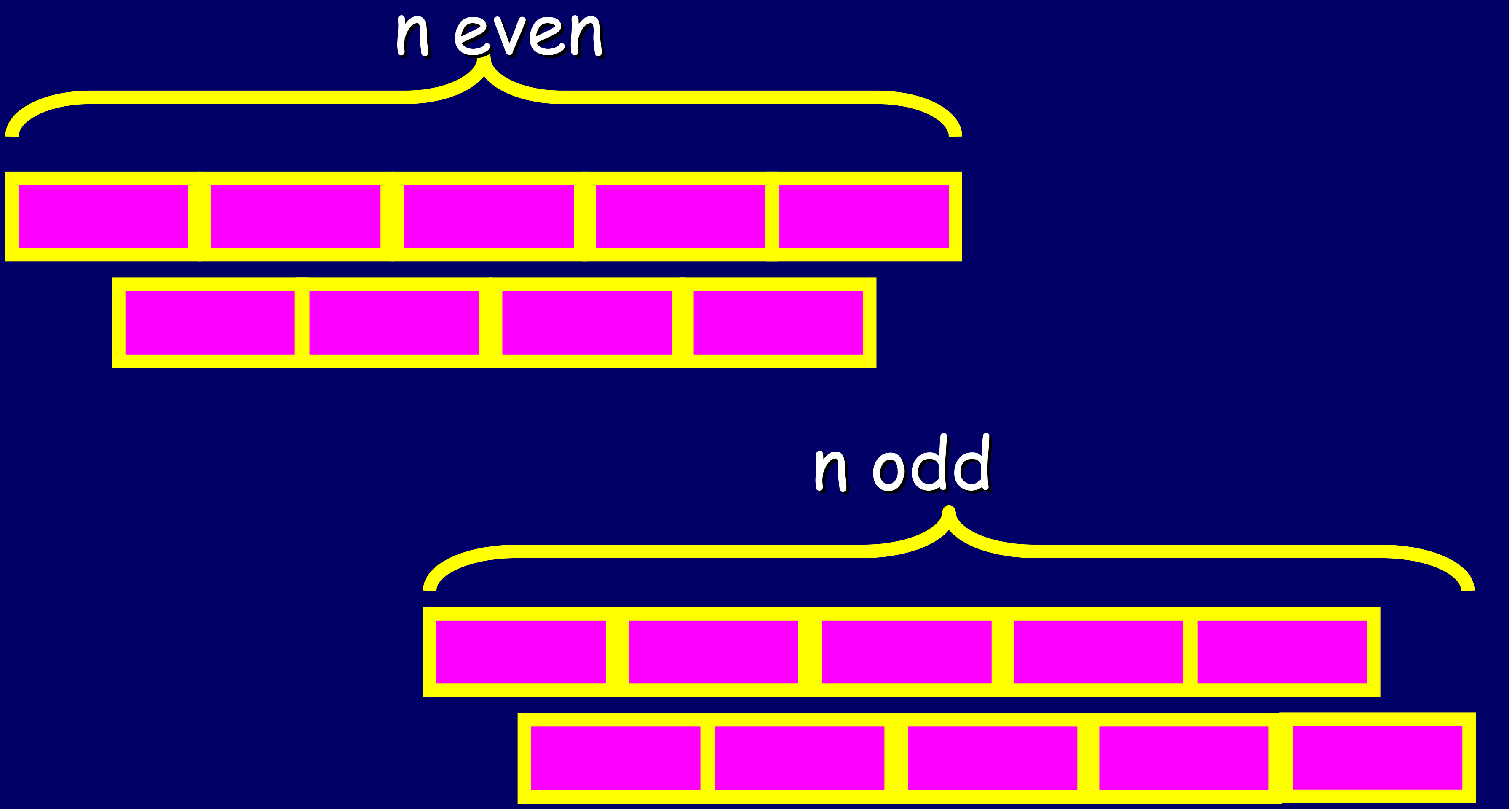
Swap the tails at the fault line to map to a tiling of 2 n-1 's to a tiling of an n-2 and an n.

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



Swap the tails at the fault line to map to a tiling of 2 n-1 's to a tiling of an n-2 and an n.

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^{n-1}$$



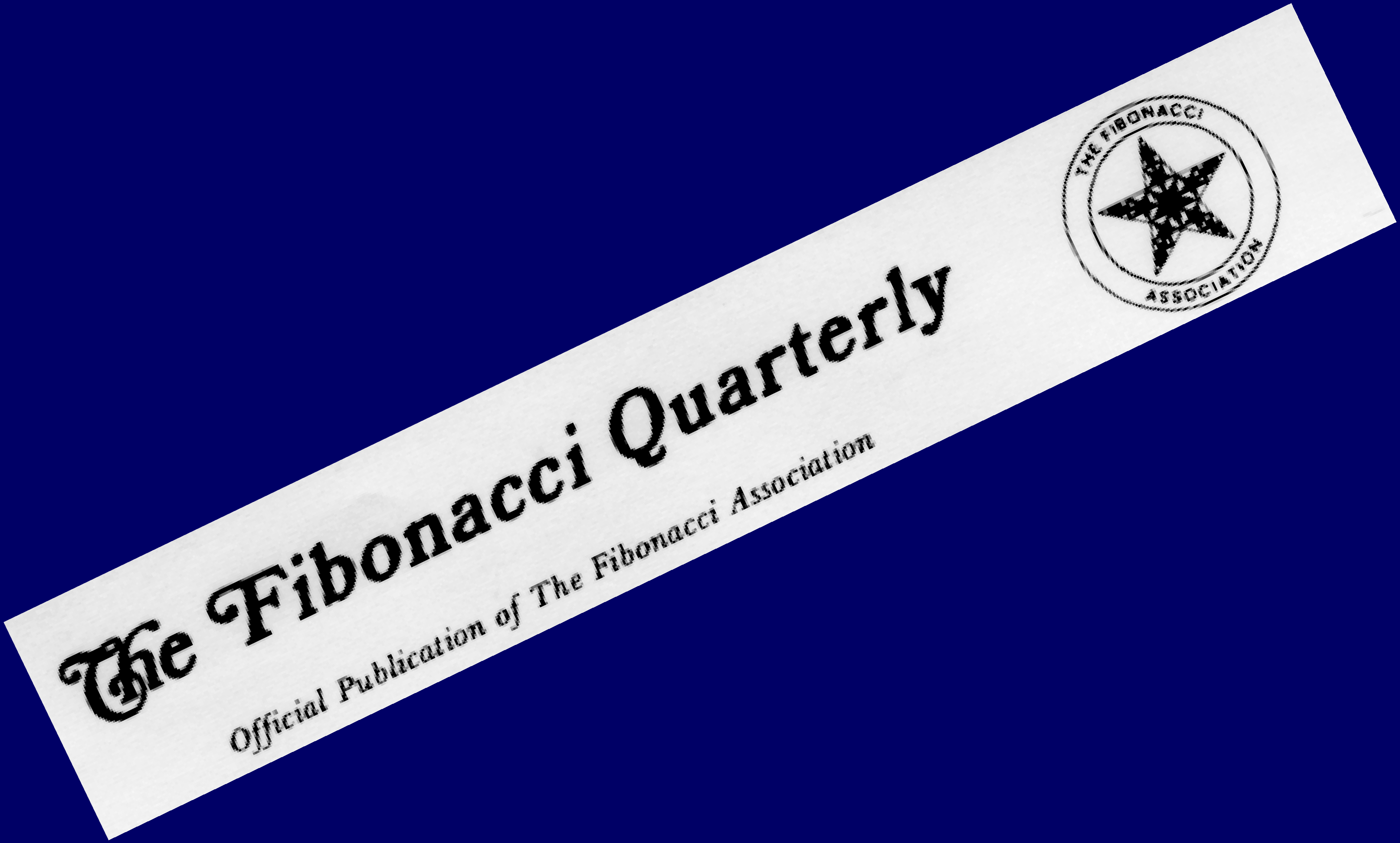
More random facts

The product of any four consecutive Fibonacci numbers is the area of a Pythagorean triangle.

The sequence of final digits in Fibonacci numbers repeats in cycles of 60. The last two digits repeat in 300, the last three in 1500, the last four in 15,000, etc.

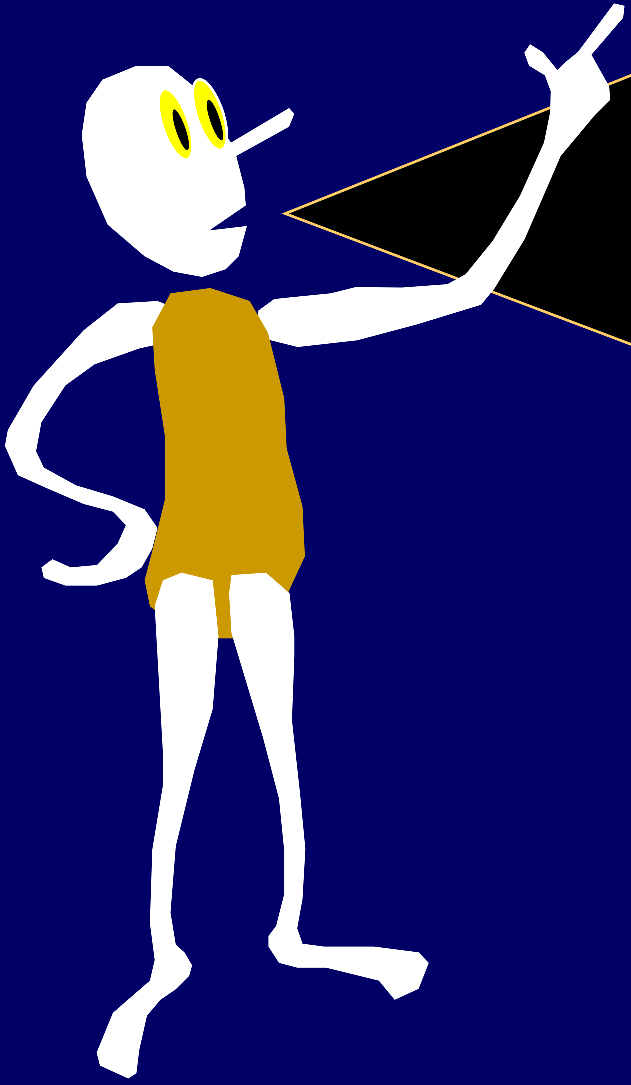
Useful to convert miles to kilometers.

The Fibonacci Quarterly



The Fibonacci Quarterly

Official Publication of The Fibonacci Association



Let's take a break from the Fibonacci Numbers in order to talk about polynomial division.

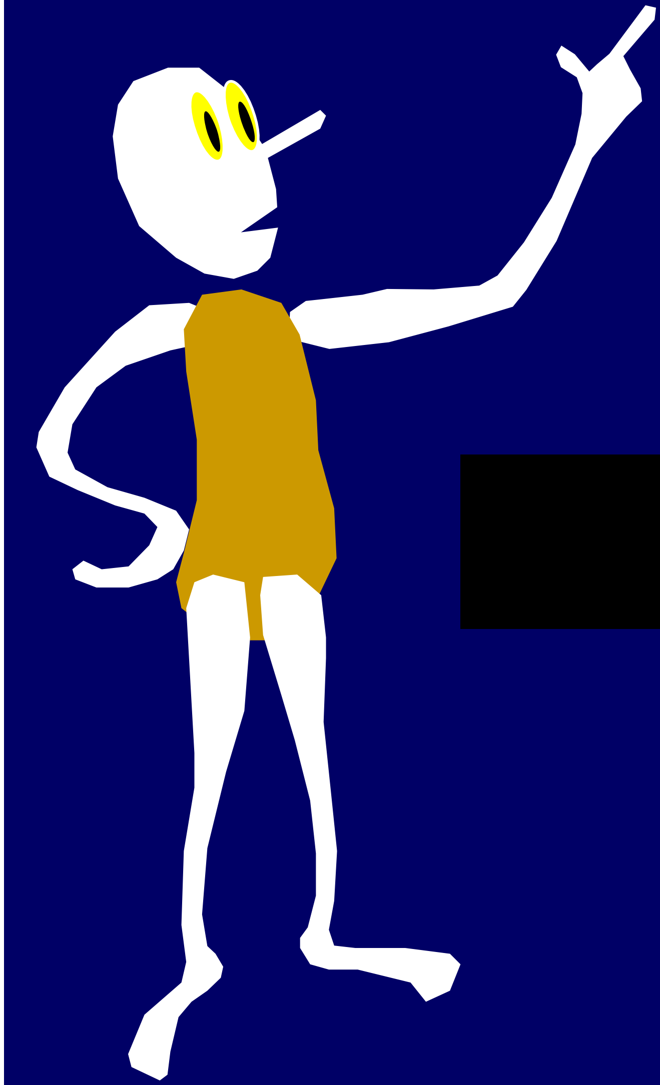
How to divide polynomials?

$$\frac{1}{1 - X} \; ?$$

$$\begin{array}{r|l}
 & 1 + X \;\; + X^2 \\
\hline
1 - X & 1 \\
 & -(1 - X) \\
\cline{2-2}
 & \qquad X \\
 & \qquad -(X - X^2) \\
\cline{2-2}
 & \qquad\qquad X^2 \\
 & \qquad\qquad -(X^2 - X^3) \\
\cline{2-2}
 & \qquad\qquad\qquad X^3 \quad \ldots
\end{array}$$

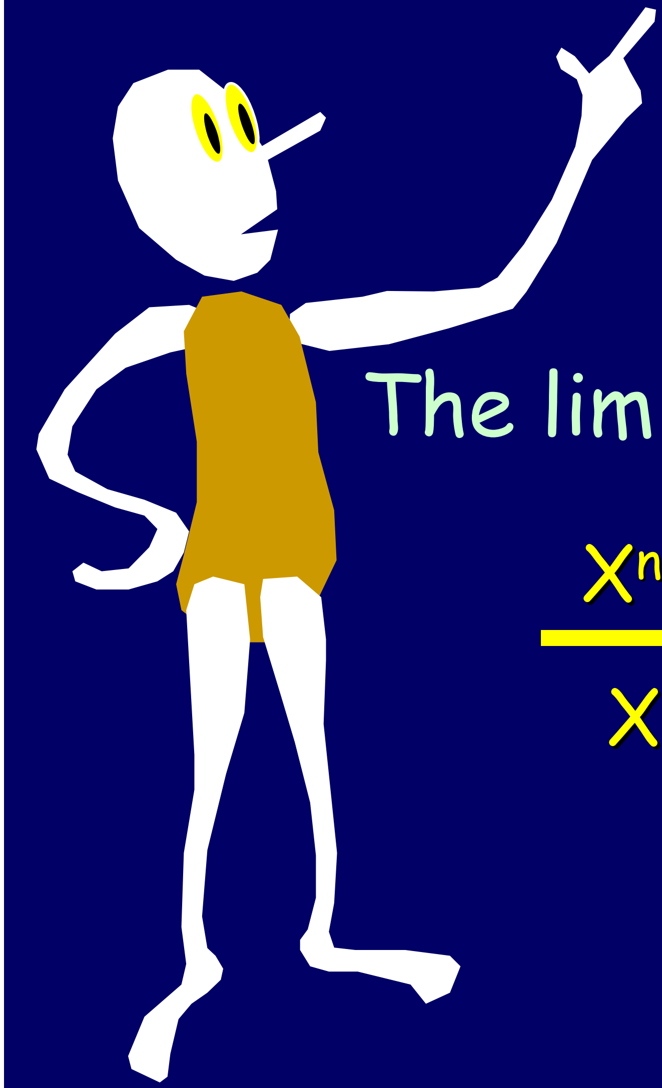
$$= \quad 1 + X + X^2 + X^3 + X^4 + X^5 + X^6 + X^7 + \ldots$$

$$1 + X^1 + X^2 + X^3 + \dots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$



The Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$

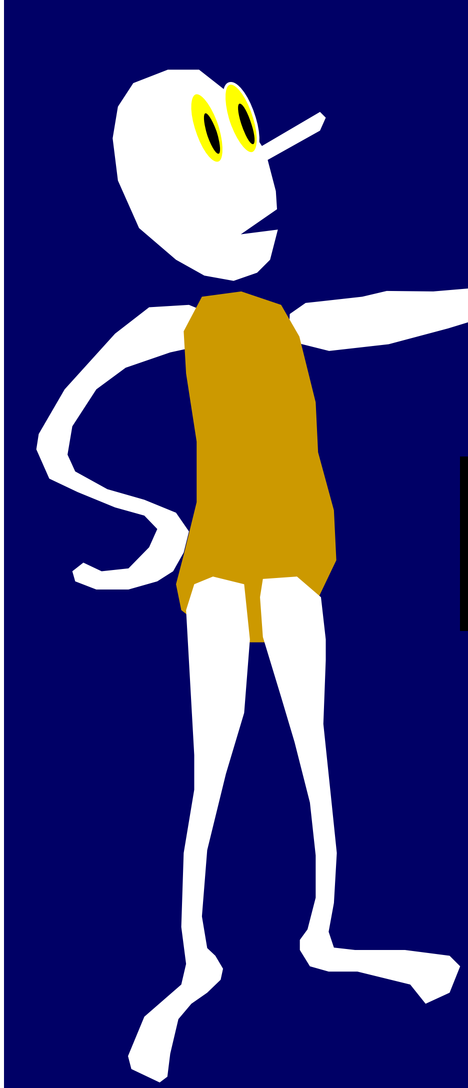


The limit as n goes to infinity of

$$\frac{X^{n+1} - 1}{X - 1} = \frac{-1}{X - 1}$$

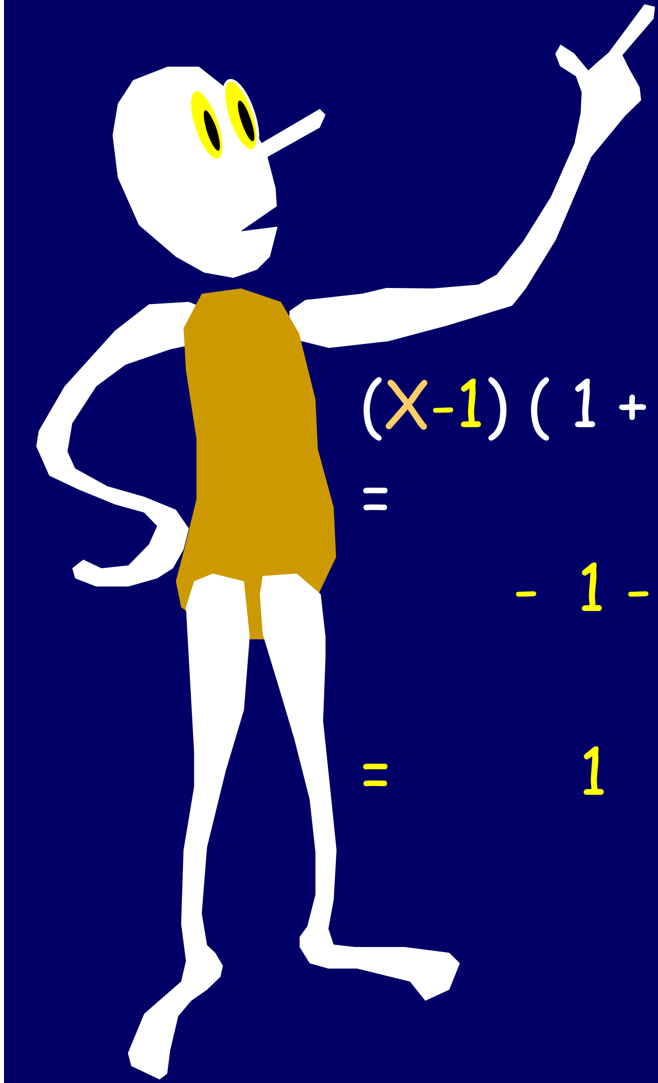
$$= \frac{1}{1 - X}$$

$$1 + X^1 + X^2 + X^3 + \dots + X^n + \dots\dots = \frac{1}{1 - X}$$



The Infinite Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



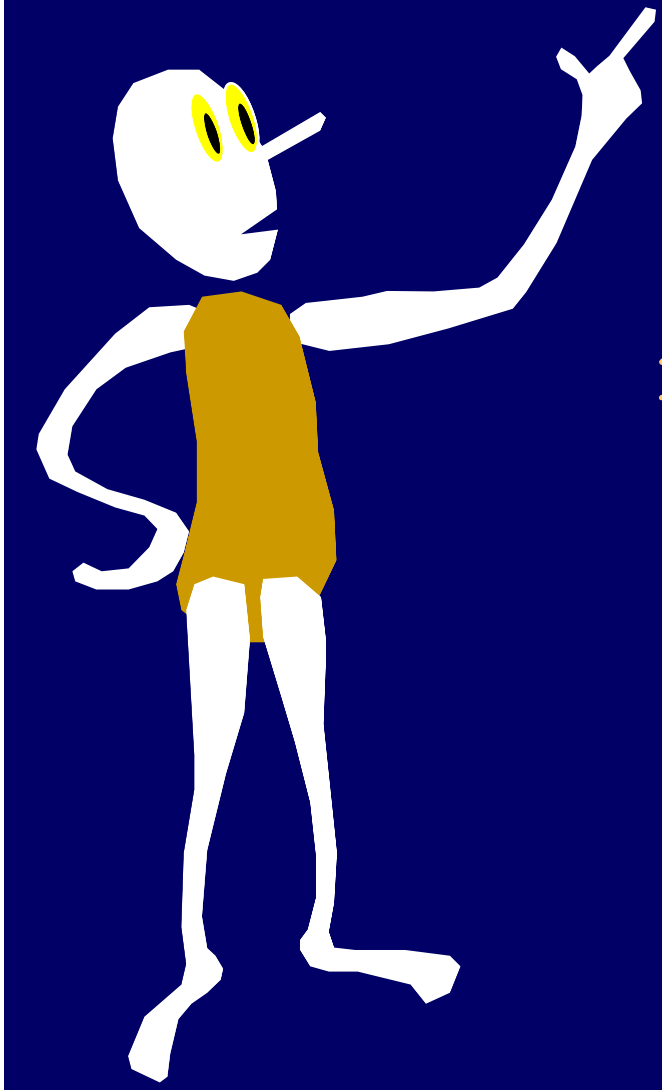
$$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots)$$

$$= X^1 + X^2 + X^3 + \ldots + X^n + X^{n+1} + \ldots.$$

$$- 1 - X^1 - X^2 - X^3 - \ldots - X^{n-1} - X^n - X^{n+1} - \ldots$$

$$= 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots.. = \frac{1}{1 - X}$$



$$1 + X + X^2 + \ldots$$

$$1 - X \,\big)\, 1$$

$$-(1 - X)$$

$$X$$

$$-(X - X^2)$$

$$X^2$$

$$-(X^2 - X^3)$$

$$X^3$$

$$\ldots$$

Something a bit more complicated

$$\boxed{\frac{X}{1 - X - X^2}}$$

$X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6$

$1 - X - X^2 \;\overline{)\; X}$

$-(X - X^2 - X^3)$

$X^2 + X^3$

$-(X^2 - X^3 - X^4)$

$2X^3 + X^4$

$-(2X^3 - 2X^4 - 2X^5)$

$3X^4 + 2X^5$

$-(3X^4 - 3X^5 - 3X^6)$

$5X^5 + 3X^6$

$-(5X^5 - 5X^6 - 5X^7)$

$8X^6 + 5X^7$

$-(8X^6 - 8X^7 - 8X^8)$

Hence

$$\frac{X}{1 - X - X^2}$$

$$= 0\times1 + 1\,X^1 + 1\,X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \ldots$$

$0 \times X^0$

$$= F_0\,1 + F_1\,X^1 + F_2\,X^2 + F_3\,X^3 + F_4\,X^4 +$$
$$F_5\,X^5 + F_6\,X^6 + \ldots$$

Going the Other Way

$F_0 = 0, F_1 = 1$

$(1 - X - X^2) \times$

$(F_0 1 + F_1 X^1 + F_2 X^2 + \ldots + F_{n-2} X^{n-2} + F_{n-1} X^{n-1} + F_n X^n + \ldots$

Going the Other Way

$$(1 - X - X^2) \times$$

$$(F_0\, 1 + F_1 X^1 + F_2 X^2 + \dots + F_{n-2} X^{n-2} + F_{n-1} X^{n-1} + F_n X^n + \dots$$

$$= (F_0\, 1 + F_1 X^1 + F_2 X^2 + \dots + F_{n-2} X^{n-2} + F_{n-1} X^{n-1} + F_n X^n + \dots$$

$$- F_0 X^1 - F_1 X^2 - \dots - F_{n-3} X^{n-2} - F_{n-2} X^{n-1} - F_{n-1} X^n - \dots$$

$$- F_0 X^2 - \dots - F_{n-4} X^{n-2} - F_{n-3} X^{n-1} - F_{n-2} X^n - \dots$$

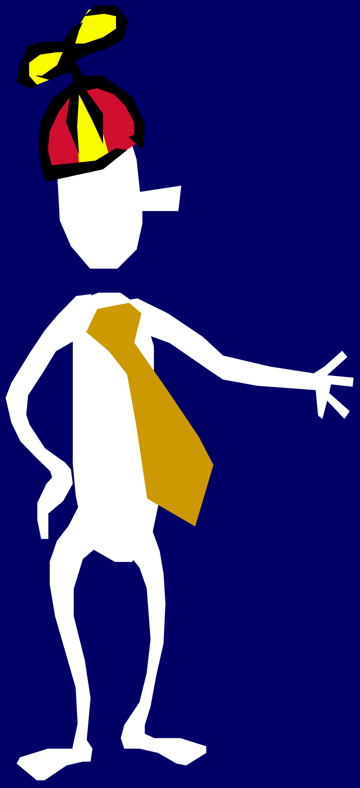
$$= F_0\, 1 + (F_1 - F_0) X^1$$

$$\boxed{F_0 = 0, F_1 = 1}$$

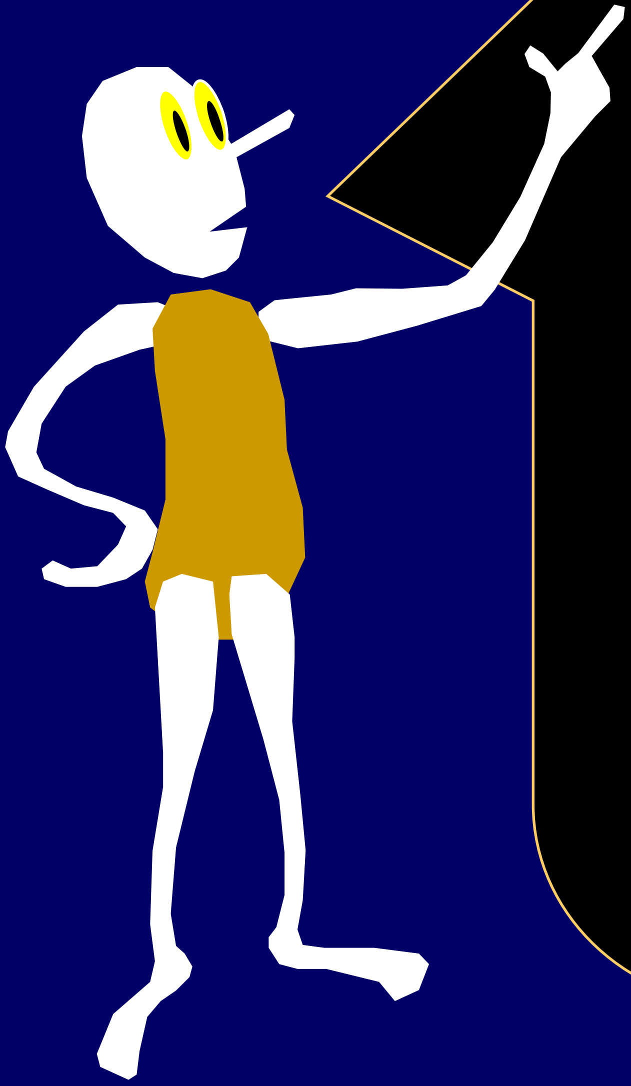
$$= X$$

Thus

$$F_0\,1 + F_1\,X^1 + F_2\,X^2 + \ldots + F_{n-1}\,X^{n-1} + F_n\,X^n + \ldots$$



$$= \frac{X}{1 - X - X^2}$$



So much for trying to take a break from the Fibonacci numbers...

Formal Power Series

Infinite polynomials a.k.a. formal power series:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

Addition and Multiplication

$$P_1 = \sum_{i \geq 0} a_i x^i \qquad P_2 = \sum_{i \geq 0} b_i x^i$$

$$P_1 + P_2 = \sum_{i \geq 0} (a_i + b_i) x^i$$

$$x P_i = \sum_{i \geq 0} a_i x^{i+1}$$

Multiplying two power series

$$P_1 * P_2 = \left(\sum_{i \geq 0} a_i X^i\right)\left(\sum_{j \geq 0} b_j X^j\right)$$

$$= \sum_{k \geq 0} \left(a_k b_0 + a_{k-1} b_1 + a_{k-2} b_2 + \cdots + a_1 b_{k-1} + a_0 b_k\right) X^k$$

$$= \sum_{k \geq 0} \underbrace{\left(\sum_{j=0}^{k} a_j b_{k-j}\right)}_{c_k} X^k$$

$$= \sum_{k \geq 0} c_k X^k .$$

$$\left(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..\right)$$

$$\times \left(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..\right) =$$

$$= \sum_{k \geq 0} \underbrace{\left(\sum_{j=0}^{k} a^j b^{k-j} \right)}_{} X^k$$

$$= a^0 b^k + a^1 b^{k-1} + a^2 b^{k-2} + \ldots + a^k b^0$$

$$= b^k \left[\frac{a^0}{b^0} + \frac{a^1}{b^1} + \frac{a^2}{b^2} + \ldots + \frac{a^k}{b^k} \right]$$

$$= b^k \left[\frac{(a/b)^{k+1} - 1}{(a/b) - 1} \right] = \frac{a^{k+1} - b^{k+1}}{a - b} \quad ☺$$

Geometric Series (Quadratic Form)

Fibonacci Numbers

Recurrence Relation Definition:

$$F_0 = 0, \quad F_1 = 1,$$

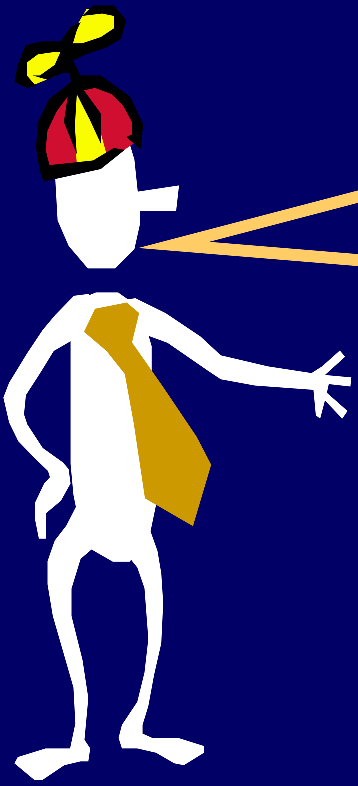
$$F_n = F_{n-1} + F_{n-2}, n > 1$$

Getting the Fibonacci Power Series

$$(1-x-x^2)\,P_{fib} = x$$

(by previous slides)

$$\Rightarrow \quad P_{fib} = \frac{x}{(1-x-x^2)}$$



Solve for P.

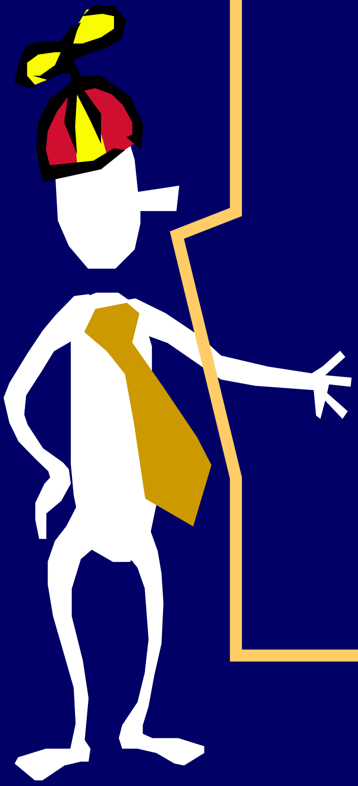
$$P - PX - PX^2 = X$$

$$P(1-X-X^2) = X$$

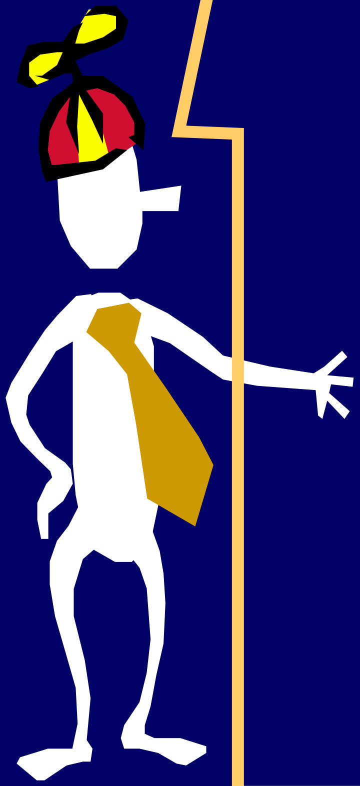
$$P = X/(1-X-X^2)$$

What is the Power Series Expansion of $x/(1-x-x^2)$?

What does this look like when we expand it as an infinite sum?



ϕ and $-\frac{1}{\phi}$ are roots of $1-x-x^2$.



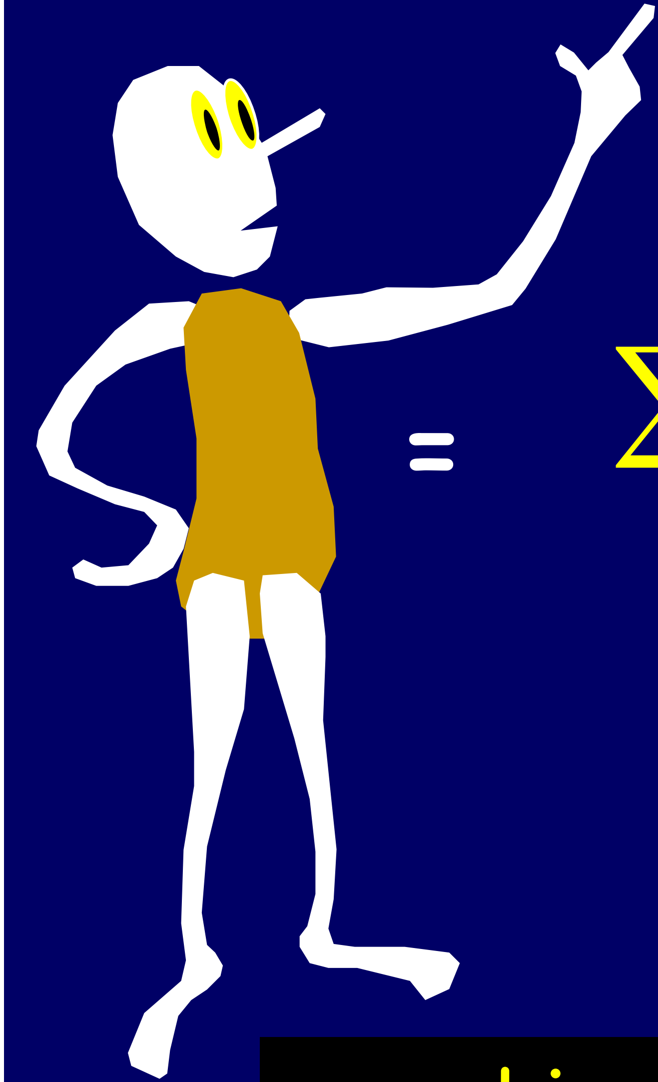
Since the bottom is quadratic we can factor it.

$$X / (1-X-X^2) =$$

$$X/(1- \phi X)(1 - (-\phi)^{-1}X)$$

$$\text{where } \phi = \frac{1+\sqrt{5}}{2}$$

“The Golden Ratio”



$$\frac{X}{(1 - \phi X)(1- (-\phi)^{-1}X)}$$

$$= \sum_{n=0..\infty} \quad ? \quad X^n$$

Linear factors on the bottom

$$\frac{X}{(1-\phi X)\left(1-\left(\frac{1}{-\phi}\right)X\right)}$$

[Forget about X in numerator for now]

$$\frac{1}{(1-aX)(1-bX)} = (1+aX+a^2X^2+\cdots)$$

$$(1+bX+b^2X^2+\cdots$$

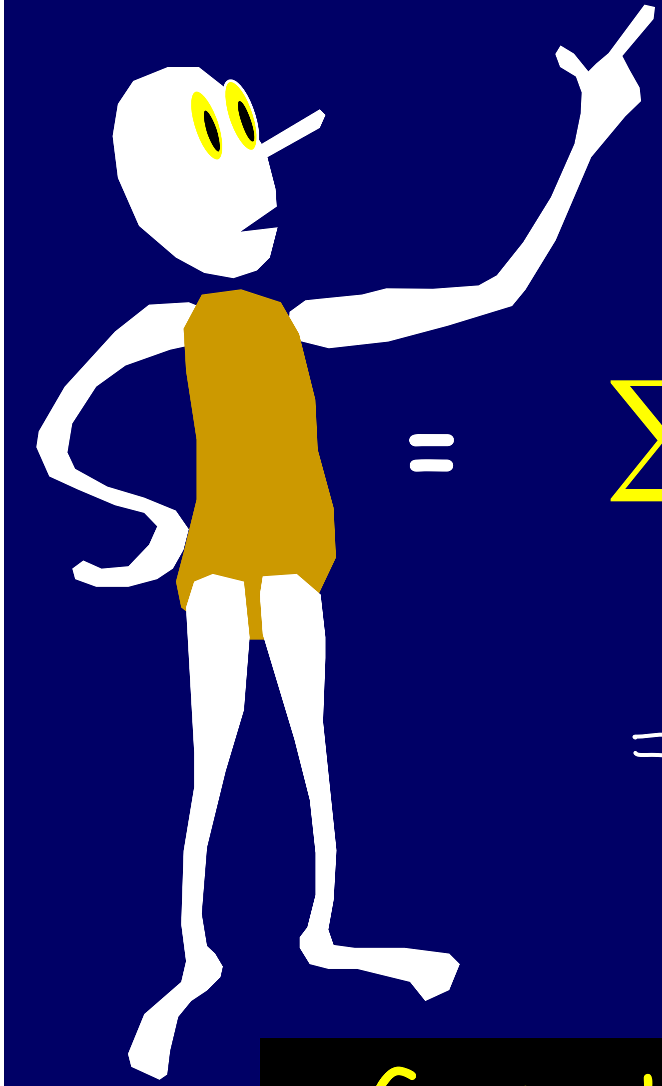
$$= \sum_{i\geq 0} \frac{(b^{i+1}-a^{i+1})}{(b-a)} X^i$$

$$\left(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..\right)\left(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..\right) =$$

$$= \frac{1}{(1 - aX)(1-bX)}$$

$$= \sum_{n=0..\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

Geometric Series (Quadratic Form)

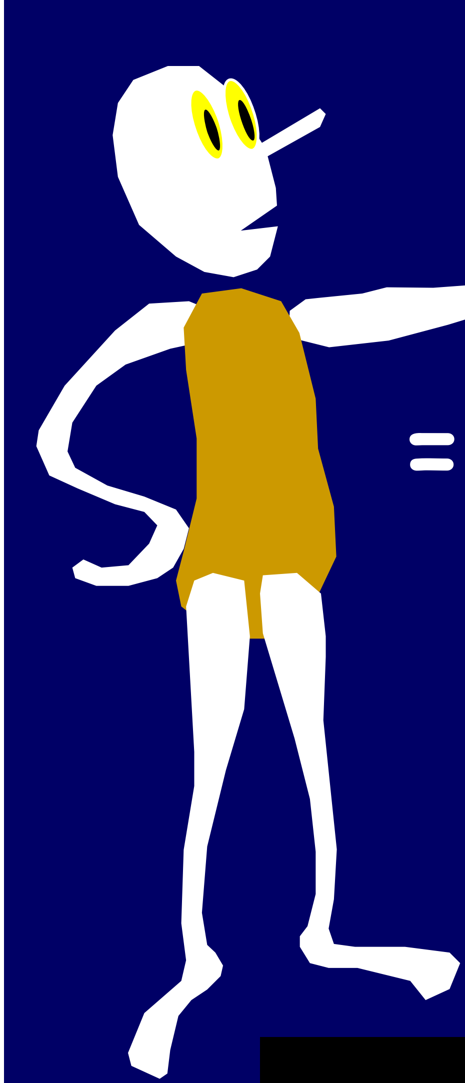


$$\frac{1 \; * \; X}{(1 - \phi X)(1- (-\phi^{-1}X)}$$

$$= \sum_{n=0..\infty} \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\sqrt{5}} X^n * X$$

$$= \sum_{n\geq 0} \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\sqrt{5}} \cdot X^{n+1}$$

Geometric Series (Quadratic Form)



$$\frac{X}{(1-\phi X)(1-(-\phi^{-1}X)} = \frac{X}{1-X-X^2}$$

$$= \underbrace{\sum_{n=0..\infty} \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\sqrt{5}} X^{n+1}}$$

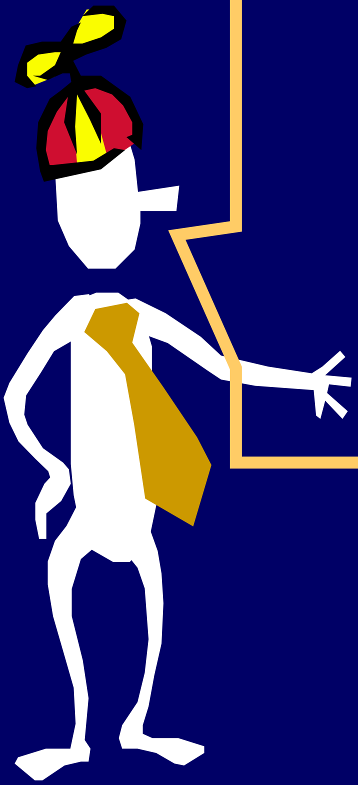
$$= \sum F_{n+1} X^{n+1}$$

Power Series Expansion of F

$$\Rightarrow \quad F_n = \frac{\phi^n - \left(\frac{1}{-\phi}\right)^n}{\sqrt{5}}$$

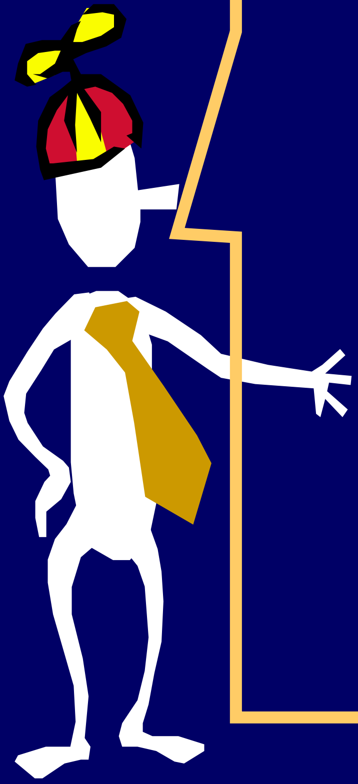
closed form expression for the Fibonacci numbers!

$$\underline{\frac{x}{1-x-x^2}} = F_0x^0 + F_1x^1 + F_2x^2 + F_3x^3 + \cdots = \sum_{i=0}^{\infty} F_i x^i$$



$$\frac{x}{1-x-x^2} = \sum_{i=0}^{\infty} \underbrace{\frac{1}{\sqrt{5}}\left(\phi^i - \left(-\frac{1}{\phi}\right)^i\right)}_{=F_i} x^i$$

Leonhard Euler (1765)
J. P. M. Binet (1843)
A de Moivre (1730)



The i^{th} Fibonacci number is:

$$\frac{1}{\sqrt{5}}\left(\phi^i - \left(-\frac{1}{\phi}\right)^i\right)$$

Remember:

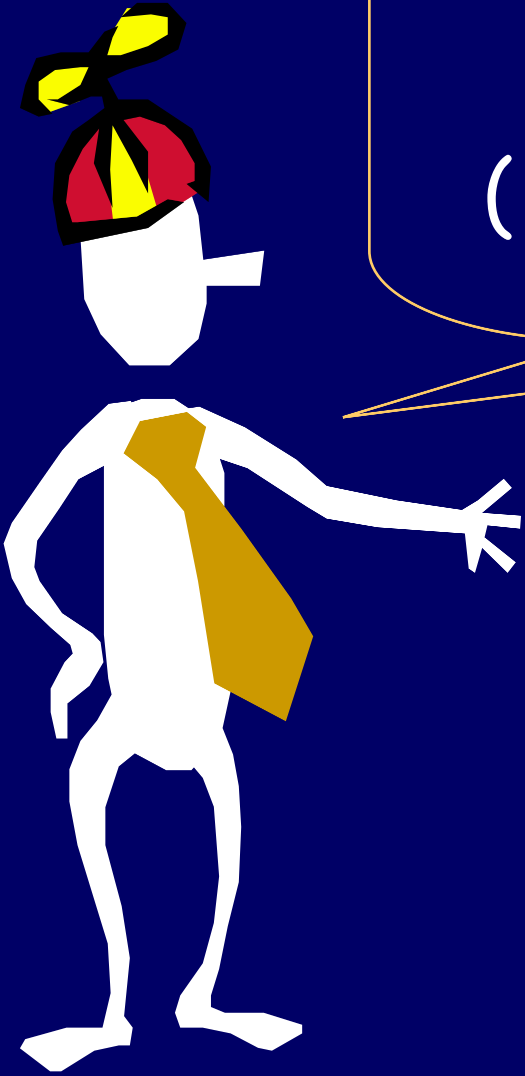
$$\mathbf{F_n} = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}} = \frac{\phi^n}{\sqrt{5}} \boxed{- \frac{\left(\frac{-1}{\phi}\right)^n}{\sqrt{5}}}$$

Less than .277

$$\mathbf{F_n} = \text{ closest integer to } \frac{\phi^n}{\sqrt{5}} = \left[\frac{\phi^n}{\sqrt{5}}\right]$$

$$\frac{\mathbf{F_n}}{\mathbf{F_{n-1}}} = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} = \frac{\phi^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} + \frac{-\left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}}$$

$$\boxed{\lim_{n\to\infty} \frac{\mathbf{F_n}}{\mathbf{F_{n-1}}} = \phi}$$



What is the coefficient of X^k in the expansion of:

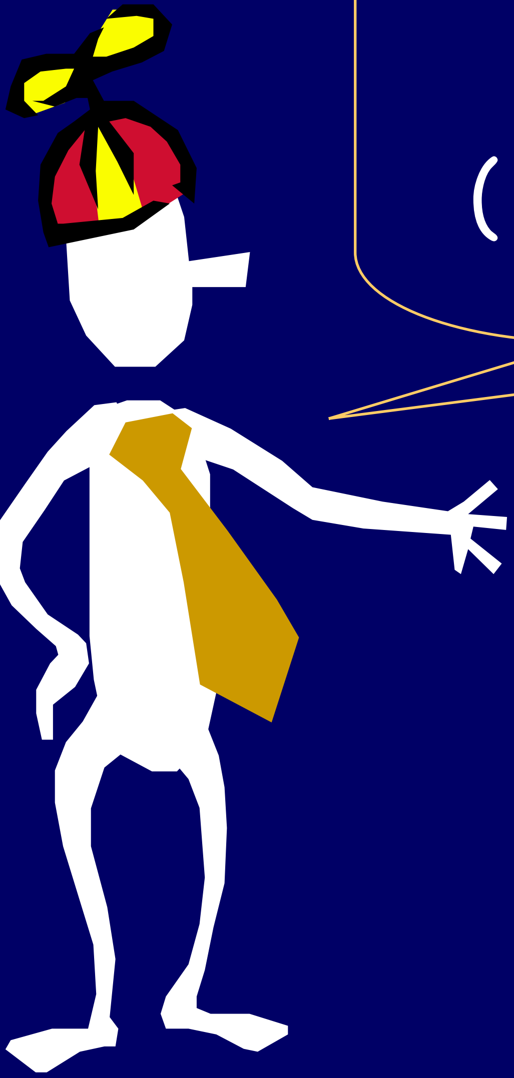
$$(1 + X + X^2 + X^3 + X^4 + \dots)^n \text{ ?}$$

Each path in the choice tree for the cross terms has n choices of exponent $e_1, e_2, \dots, e_n \geq 0$. Each exponent can be any natural number.

Coefficient of X^k is the number of non-negative solutions to:

$$e_1 + e_2 + \dots + e_n = k$$

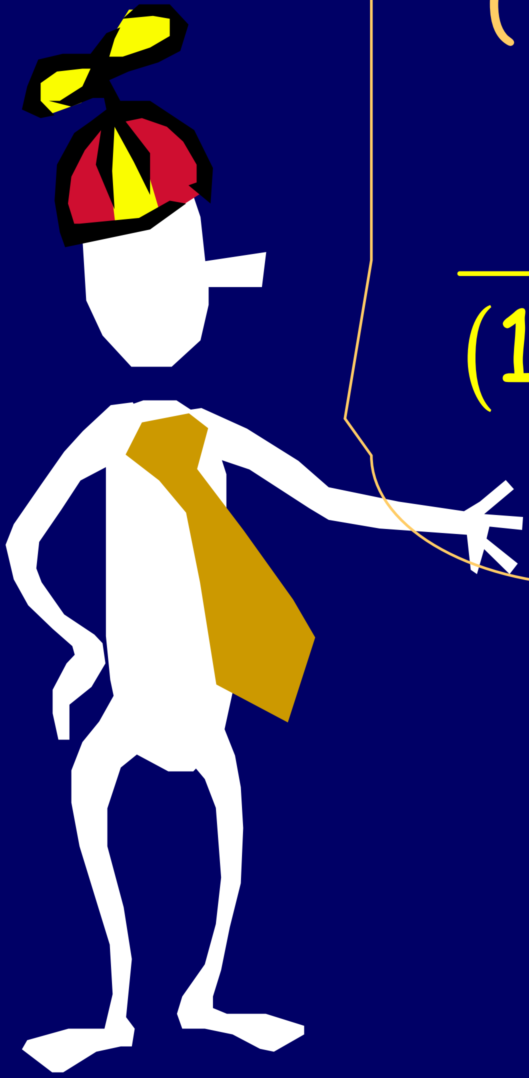
$e_1 + e_2 + \dots + e_n = k$?



What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \dots)^n \,?$$

$$\binom{n+k-1}{n-1}$$



$$(1 + X + X^2 + X^3 + X^4 + \ldots\ldots)^n =$$

$$\frac{1}{(1-X)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} X^k$$

Study Bee

Fibonacci Numbers

- Arise everywhere
- Visual Representations
- Fibonacci Identities

Polynomials

- The infinite geometric series
- Division of polynomials
- Representation of Fibonacci numbers as coefficients of polynomials.

Generating Functions and Power Series

- Simple operations (add, multiply)
- Quadratic form of the Geometric Series
- Deriving the closed form for F_n

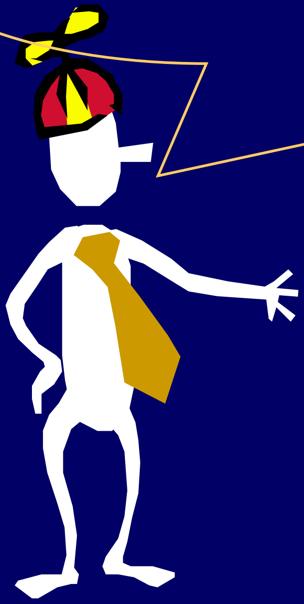
Some Extra Material on Generating Functions

Pirates & Gold, and getting a formula for the sum of squares

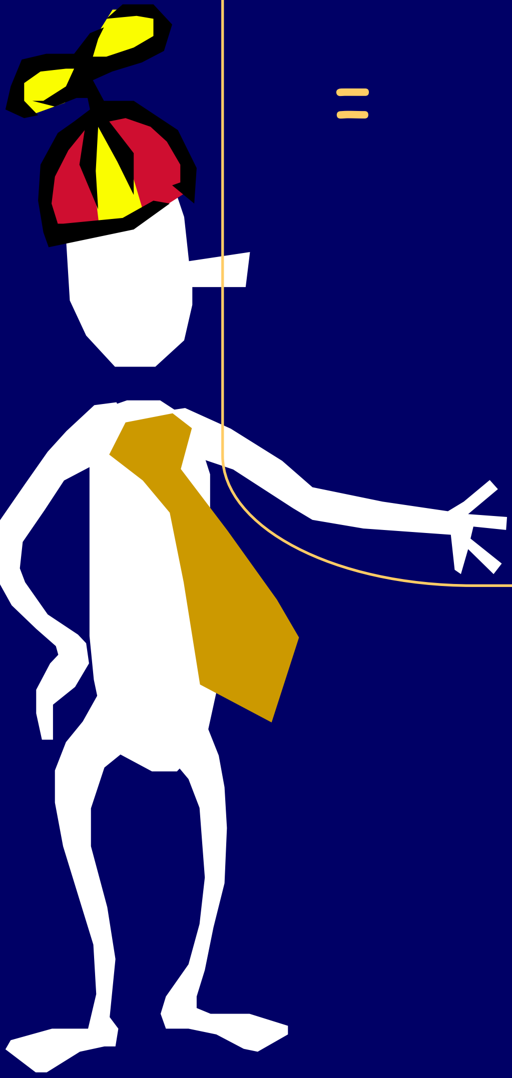
What is the coefficient of X^k in the expansion of:

$$(a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots)\ (\ 1 + X + X^2 + X^3 + \ .\ .\ .\ )$$

$$= (a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots) \,/\, (1 - X) \qquad ?$$



$$a_0 + a_1 + a_2 + \ldots + a_k$$



$$(a_0 + a_1X + a_2X^2 + a_3X^3 + \dots) / (1 - X)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{i=k} a_i \right) X^k$$

Some simple power series

$$1 = 1.x^0 + 0.x^1 + \cdot \cdot \cdot \cdot \cdot$$

$$\Rightarrow \frac{1}{1-x} = 1x^0 + 1.x^1 + 1.x^2 + 1.x^3 \cdot \cdot \cdot$$

$$\Rightarrow \frac{1}{(1-x)^2} = 1.x^0 + (1+1)x^1 + (1+1+1)x^2 + \cdot \cdot \cdot \cdot$$

$$= 1x^0 + 2x^1 + 3.x^2 + \cdot \cdot \cdot \cdot$$

$$\Rightarrow \frac{1}{(1-x)^3} = 1.x^0 + 3.x^1 + 6.x^2 + \cdot \cdot \cdot \cdot$$

$$= \sum \Delta_{n+1} x^n$$

Al-Karaji's Identities

Zero_Ave = $1/(1-X)$;

First_Ave = $1/(1-X)^2$;

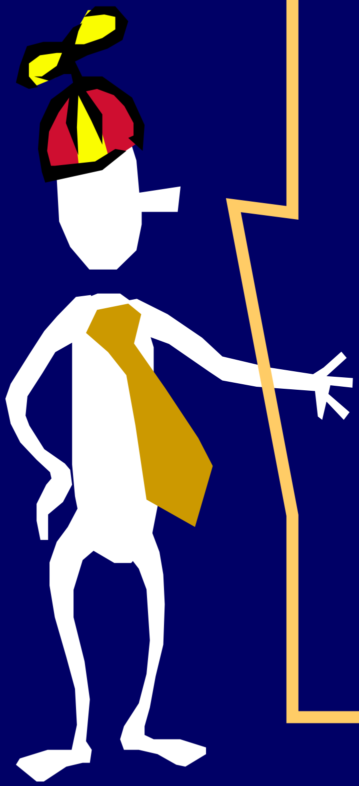
Second_Ave = $1/(1-X)^3$;

Output =

$$1/(1-X)^2 + 2X/(1-X)^3$$

$$(1-X)/(1-X)^3 + 2X/(1-X)^3$$

$$= (1+X)/(1-X)^3$$



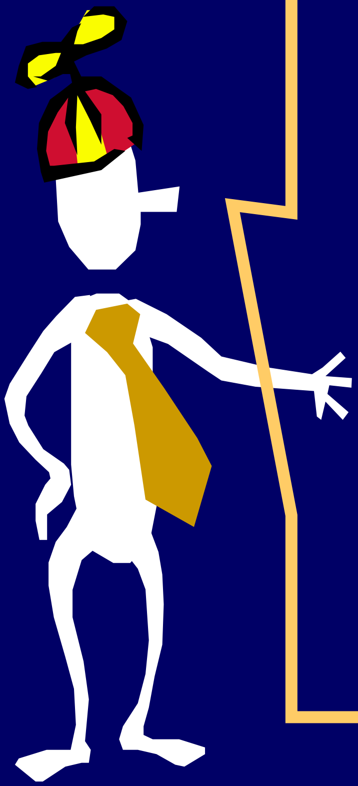
$(1+X)/(1-X)^3$

outputs <1, 4, 9, ..>

$X(1+X)/(1-X)^3$

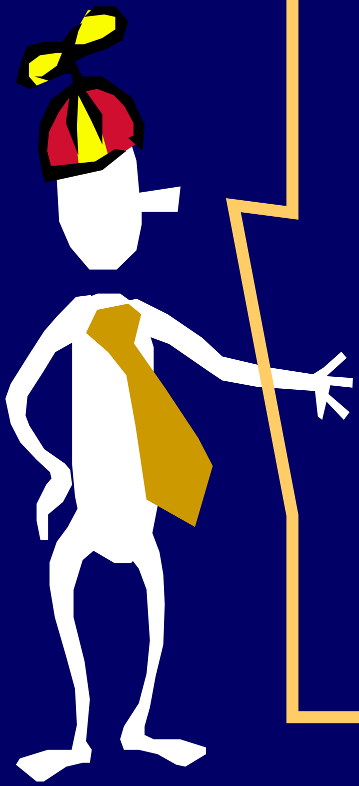
outputs <0, 1, 4, 9, ..>

The k^{th} entry is k^2



$X(1+X)/(1-X)^3 = \sum\ k^2X^k$

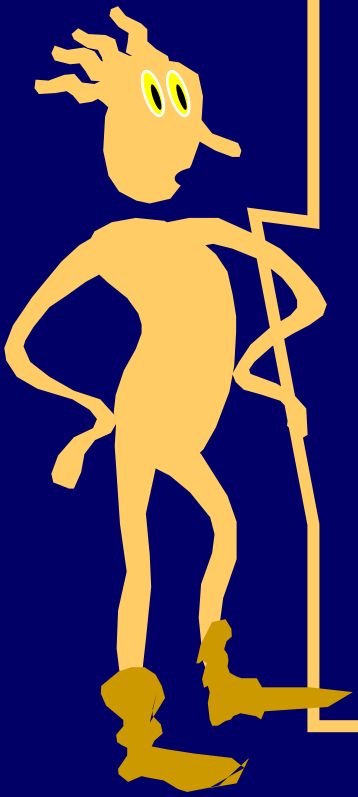
What does $X(1+X)/(1-X)^4$ do?



$X(1+X)/(1-X)^4$ expands to :

$\sum S_k X^k$

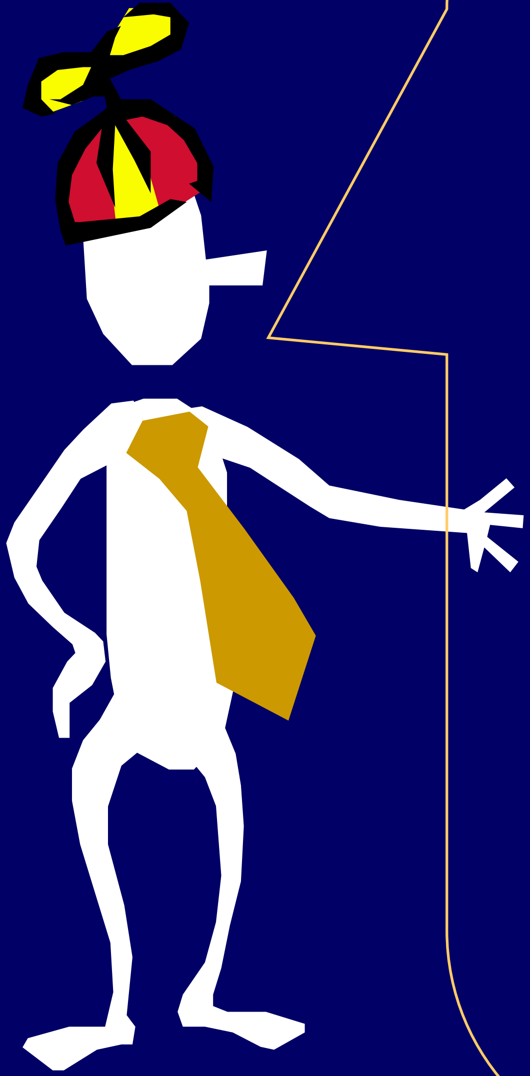
where S_k is the sum of the first k squares



Aha! Thus, if there is an alternative interpretation of the kth coefficient of

$$X(1+X)/(1-X)^4$$

we would have a new way to get a formula for the sum of the first k squares.



Using pirates and gold we found that:

$$\frac{1}{(1-X)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} X^k$$

THUS:

$$\frac{1}{(1-X)^4} = \sum_{k=0}^{\infty} \binom{k+3}{3} X^k$$

Coefficient of X^k in $P_V = (X^2+X)(1-X)^{-4}$ is the sum of the first k squares:

$$\frac{X^2+X}{(1-X)^4} = (X^2+X)\sum_{k=0}^{\infty}\binom{k+3}{3}X^k$$

$$= \sum_{k=0}^{\infty}\left(\binom{k+2}{3}+\binom{k+1}{3}\right)X^k$$

$$\frac{1}{(1-X)^4} = \sum_{k=0}^{\infty}\binom{k+3}{3}X^k$$

Polynomials give us closed form expressions

$$\frac{X^2+X}{(1-X)^4} = \sum_{k=0}^{\infty}\left(\binom{k+2}{3}+\binom{k+1}{3}\right)X^k$$

$$\sum_{i=0}^{i=n} i^2 = \binom{n+2}{3}+\binom{n+1}{3}$$

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