Great Theoretical Ideas In Computer Science

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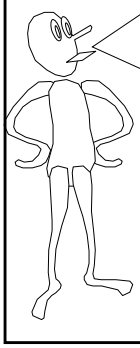
Ancient Wisdom: Primes, Continued Fractions, The Golden Ratio, and Euclid's GCD

$$\frac{3+\sqrt{13}}{2} = 3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cdots}}}}}}}}}$$



Definition: A number > 1 is prime if it has no other factors, besides 1 and itself.

Each number can be factored into primes in a unique way. [Euclid]

Theorem: Each natural has a unique factorization into primes written in non-decreasing order.

Definition: A number > 1 is prime if it has no other factors, besides 1 and itself.

Primes: 2, 3, 5, 7, 11, 13, 17, …

Factorizations:

42 = 2 * 3 * 7

84 = 2 * 2 * 3 * 7

13 = 13

Multiplication might just be a "one-way" function

Multiplication is fast to compute

Reverse multiplication is apparently slow

We have a feasible method to multiply 1000 bit numbers [Egyptian multiplication]

Factoring the product of two random 1000 bit primes has no known feasible approach.

Grade School GCD algorithm

GCD(A,B) is the greatest common divisor, i.e., the largest number that goes evenly into both A and B.

What is the GCD of 12 and 18?

$12 = 2^2 * 3$ $18 = 2*3^2$

Common factors: 2^1 and 3^1

Answer: 6

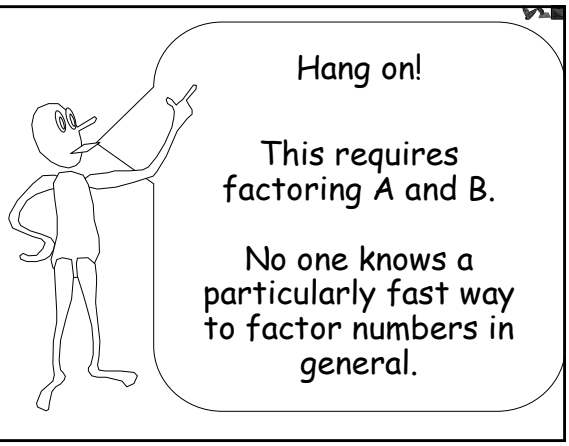
How to find GCD(A,B)?

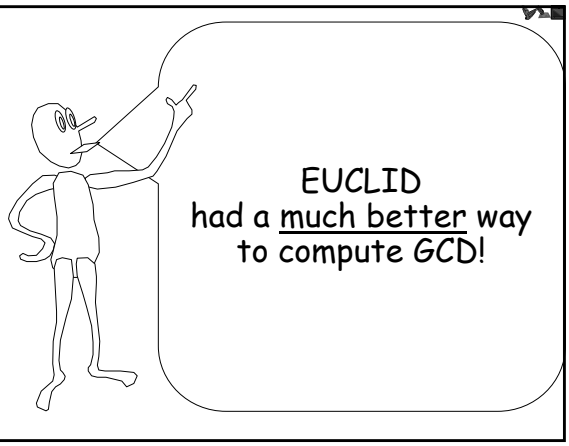
A Naïve method:

Factor A into prime powers.

Factor B into prime powers.

Create GCD by multiplying together each common prime raised to the highest power that goes into both A and B.





Ancient Recursion: Euclid's GCD algorithm

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

A small example

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Note: GCD(67, 29) = 1

= GCD(29, 9)

= GCD(9, 2)

= GCD(2, 1) = GCD(1, 0) = 1

67 mod 29 = 29×2 + 9

A small example

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Note: GCD(67, 29) = 1

Euclid(67,29)	67 mod 29 = 9
Euclid(29,9)	29 mod 9 = 2
Euclid(9,2)	9 mod 2 = 1
Euclid(2,1)	2 mod 1 = 0
Euclid(1,0) outputs 1	

Does the algorithm stop?

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Claim: After first step, A ≥ B ≥ 0

GCD(9, 387) = GCD(387, 9)

Important questions to ask

Is the algorithm correct?

Does the algorithm stop?

How many steps does the algorithm run for?

But is it correct?

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Claim: GCD(A,B) = GCD(B, A mod B)

$d|A,\ d|B \Rightarrow d|(A-kB)$ for any k

$\Leftrightarrow d|(A \bmod B),\ d|B.$

$d|A \bmod B,\ d|B \Rightarrow d|(A \bmod B + kB)\ \forall k$

$\Rightarrow d|A$

But is it correct?

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Claim: GCD(A,B) = GCD(B, A mod B) ← value of GCD is an invariant!

But is it correct?

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Claim: GCD(A,B) = GCD(B, A mod B)

d|A and d|B ⇔ d| (A - kB)

The set of common divisors of A, B equals the set of common divisors of B, A-kB.

Does the algorithm stop?

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Claim: A mod B < $\frac{1}{2}$ A

Case ① $B = \frac{1}{2}A \Rightarrow A \bmod B = 0.$

② $B < \frac{1}{2}A \Rightarrow A \bmod B \leq B < \frac{1}{2}A$

③ $B > \frac{1}{2}A \Rightarrow A \bmod B = A - B < \frac{1}{2}A.$ ☺

Does the algorithm stop?

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Claim: A mod B < $\frac{1}{2}$ A

Proof:

If B > $\frac{1}{2}$ A then A mod B = A - B < $\frac{1}{2}$ A

If B < $\frac{1}{2}$ A then <u>any</u> X Mod B < B < $\frac{1}{2}$ A

If B = $\frac{1}{2}$ A then A mod B = 0

Does the algorithm stop?

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

GCD(A,B) calls GCD(B, A mod B)

Less than $\frac{1}{2}$ of A

Euclid's GCD Termination

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

GCD(A,B) calls GCD(B, $<\frac{1}{2}A$)

Euclid's GCD Termination

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

GCD(A,B) calls GCD(B, $<\frac{1}{2}A$)

which calls GCD($<\frac{1}{2}A$, B mod $<\frac{1}{2}A$)

Less than $\frac{1}{2}$ of A

Euclid's GCD Termination

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Every two recursive calls, the input numbers drop by half.

Euclid's GCD Termination

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

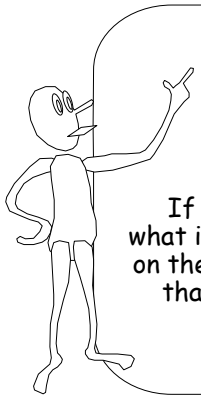
Theorem: If two input numbers have an n bit binary representation, Euclid's Algorithm will not take more than 2n calls to terminate.

Important questions to ask

Is the algorithm correct? ✓

Does the algorithm stop? ✓

How many steps does the algorithm run for? ✓



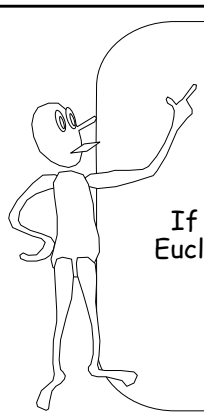
Trick Question:

If X and Y are less than n, what is a reasonable upper bound on the number of recursive calls that Euclid(X,Y) will make?.

$$\pi \cong \left(\frac{2143}{22}\right)^{1/4}$$

is correct to within 0'03%

(3 parts in 10^4)



Answer:

If X and Y are less than n, Euclid(X,Y) will make no more than $2\log_2 n$ calls.

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Euclid(67,29)	67 – 2*29 = 67 mod 29 = 9
Euclid(29,9)	29 – 3*9 = 29 mod 9 = 2
Euclid(9,2)	9 – 4*2 = 9 mod 2 = 1
Euclid(2,1)	2 – 2*1 = 2 mod 1 = 0
Euclid(1,0) outputs 1	

Let <r,s> denote the number r*67 + s*29. Calculate all intermediate values in this representation.

67 = (1,0) 29 = <0,1>

Euclid(67,29)	9 = 67 – 2·29 = <1,0> – 2<0,1> = <1,-2>
Euclid(29,9)	2 = 29 – 3.9 = <0,1> – 3<1,-2> = <3,1-7>
Euclid(9,2)	1 = = <13,-30>
Euclid(2,1)	0

$1 = \gcd(67,29) = 13 \times 67 - 30 \cdot 29$

$\gcd(X,Y) = r\,X + s\,Y$

Let <r,s> denote the number r*67 + s*29. Calculate all intermediate values in this representation.

67=<1,0> 29=<0,1>

Euclid(67,29)	9=<1,0> – 2*<0,1>	9 =<1,-2>
Euclid(29,9)	2=<0,1> – 3*<1,-2>	2=<-3,7>
Euclid(9,2)	1=<1,-2> – 4*<-3,7>	1=<13,-30>
Euclid(2,1)	0=<-3,7> – 2*<13,-30>	0=<-29,67>

Euclid(1,0) outputs 1 = 13*67 – 30*29

Euclid's Extended GCD algorithm

Input: X,Y

Output: r,s,d such that rX+sY = d = GCD(X,Y)

		67=<1,0> 29=<0,1>
Euclid(67,29)	9=67 – 2*29	9 =<1,-2>
Euclid(29,9)	2=29 – 3*9	2=<-3,7>
Euclid(9,2)	1=9 – 4*2	1=<13,-30>
Euclid(2,1)	0=2 – 2*1	0=<-29,67>

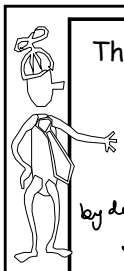
Euclid(1,0) outputs 1 = 13*67 – 30*29

The multiplicative inverse of $y \in Z_n^*$ is the unique $z \in Z_n^*$ such that $y \times_n z \equiv_n 1$.

The unique inverse of a must exist because the y row contains a permutation of the elements and hence contains a unique 1.

Z_5^*

×	1	z	3	4
1	1	2	3	4
2	2	4	1	3
y	3	1	4	2
4	4	3	2	1



The multiplicative inverse of $y \in Z_n^*$ is the unique $z \in Z_n^*$ such that $y \times_n z \equiv_n 1$.

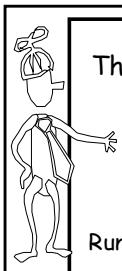
To quickly compute the inverse of y:

by defn $y \in Z_n^* \Rightarrow \gcd(y,n) = 1$ in Z_n^*

$\Rightarrow$ can find $\underbrace{ry + sn = 1}_{\text{mod } n} = \gcd(y,n)$

$\Rightarrow ry \equiv 1 \pmod n$

$r = y^{-1} \pmod n$

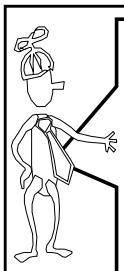


The multiplicative inverse of $y \in Z_n^*$ is the unique $z \in Z_n^*$ such that $y \times_n z \equiv_n 1$.

To quickly compute the inverse of y:

Run ExtendedEuclid(x,n).
returns a, b, and d such that ay+bn = d
But d = GCD(y,n) = 1, so ay + bn = 1

Hence ay = 1 (mod n)
Thus, a is the multiplicative inverse of y.



The RSA story

Pick 2 distinct, random 1000 bit primes, p and q.

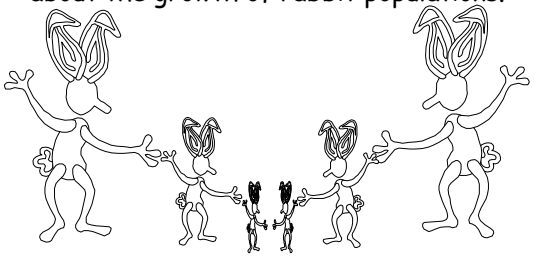
Multiply them to get n = (p*q)
Multiply (p-1) and (q-1) to compute $\phi(n)$
Randomly pick an e s.t. GCD(e,n) = 1.
<u>Publish n and e</u>

Compute multiplicative inverse of e mod $\phi(n)$ to get a <u>secret</u> number d.

$(m^e)^d = m^{ed} = m^1 \pmod n$

Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.



Inductive Definition or Recurrence Relation for the Fibonacci Numbers

Stage 0, Initial Condition, or Base Case:
Fib(0) = 0; Fib (1) = 1

Inductive Rule
For n>1, Fib(n) = Fib(n-1) + Fib(n-2)

n	0	1	2	3	4	5	6	7
Fib(n)	0	1	1	2	3	5	8	13

A (Simple) Continued Fraction Is Any Expression Of The Form:

$$2\cancel{a}+\cfrac{1}{3\cancel{b}+\cfrac{1}{5\cancel{c}+\cfrac{1}{8\cancel{d}+\cfrac{1}{13\cancel{e}+\cfrac{1}{\cancel{f}+\cfrac{1}{g+\cfrac{1}{h+\cfrac{1}{i+\cfrac{1}{j+\ldots}}}}}}}}}$$

where a, b, c, … are whole numbers.

A Continued Fraction can have a finite or infinite number of terms.

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\cfrac{1}{g+\cfrac{1}{h+\cfrac{1}{i+\cfrac{1}{j+\ldots}}}}}}}}}$$

We also denote this fraction by [a,b,c,d,e,f,…]

A Finite Continued Fraction

$$2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2}}}$$

Denoted by [2,3,4,2,0,0,0,…]

An Infinite Continued Fraction

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}}}}}}}$$

Denoted by [1,2,2,2,…]

Recursively Defined Form For CF

$$\text{CF} = \text{whole number, or}$$

$$= \text{whole number} + \frac{1}{\text{CF}}$$

Ancient Greek Representation: Continued Fraction Representation

$$\frac{5}{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2}}$$

Ancient Greek Representation: Continued Fraction Representation

$$\frac{5}{3} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$$

= [1,1,1,1,0,0,0,…]

Ancient Greek Representation: Continued Fraction Representation

$$? = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

Ancient Greek Representation: Continued Fraction Representation

$$\frac{8}{5} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

= [1,1,1,1,1,0,0,0,…]

Ancient Greek Representation: Continued Fraction Representation

$$\frac{13}{8} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}$$

= [1,1,1,1,1,1,0,0,0,…]

A Pattern?

Let r_1 = [1,0,0,0,…] = 1
r_2 = [1,1,0,0,0,…] = 2/1
r_3 = [1,1,1,0,0,0…] = 3/2
r_4 = [1,1,1,1,0,0,0…] = 5/3
and so on.

Theorem:

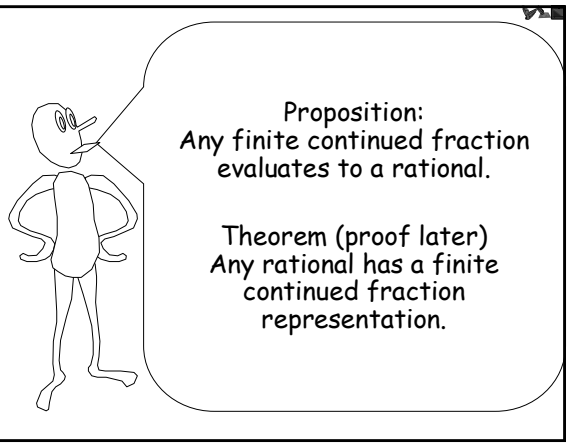
r_n = Fib(n+1)/Fib(n)

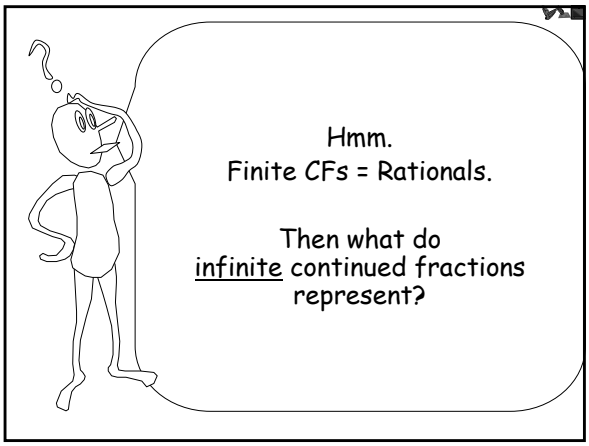
$r_n = 1 + \frac{1}{r_{n-1}}$

$= 1 + \frac{F_{n-1}}{F_n}$ by IH.

$= \frac{F_n + F_{n-1}}{F_n}$

$= \frac{F_{n+1}}{F_n}$ ☺





An infinite continued fraction

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}}}}}}}$$

Quadratic Equations

$X^2 - 3x - 1 = 0$

$$X = \frac{3+\sqrt{13}}{2}$$

$X^2 = 3x + 1$

$X = 3 + \frac{1}{X}$

$= 3 + \frac{1}{3 + \frac{1}{X}} = 3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{X}}}$

Quadratic Equations

$X^2 - 3x - 1 = 0$

$$X = \frac{3+\sqrt{13}}{2}$$

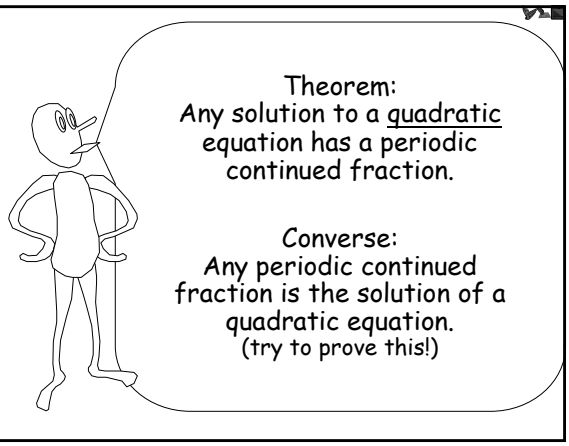
$X^2 = 3X + 1$

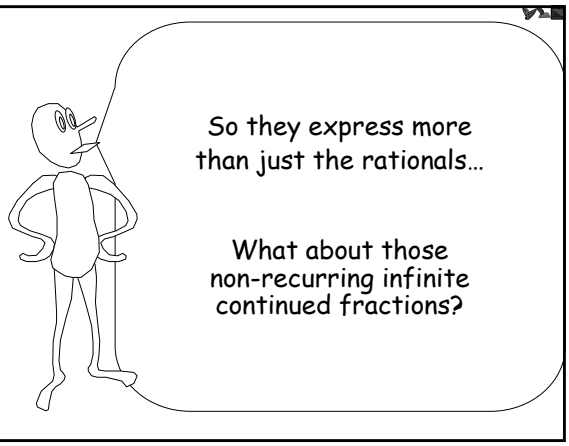
$X = 3 + 1/X$

$X = 3 + 1/X = 3 + 1/[3 + 1/X] = \ldots$

A Periodic CF

$$\frac{3+\sqrt{13}}{2} = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \ldots}}}}}}}}$$





Non-periodic CFs

$$e-1=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\ldots}}}}}}}}}$$

What is the pattern?

$$\pi=3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ldots}}}}}}}}}$$

No one knows!

$$\pi = 3 + \cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+}}}}$$

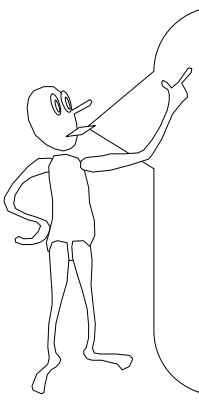
Ramanujan

$$\pi \approx \left(\frac{2143}{22}\right)^{1/4}$$

$$\pi^4 \approx \frac{2143}{22}$$

$$\pi^4 = [97, 2, 2, 3, 1, 16539, 1, \ldots]$$

$$= 97 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1539+}}}}} = \frac{2143}{22}$$



What a cool representation!

Finite CF: Rationals

Periodic CF: Quadratic roots

And some numbers reveal hidden regularity.

More good news: Convergents

Let $\alpha = [a_1, a_2, a_3, \ldots]$ be a CF.

Define: $C_1 = [a_1,0,0,0,0..]$
$C_2 = [a_1,a_2,0,0,0,\ldots]$
$C_3 = [a_1,a_2,a_3,0,0,\ldots]$ and so on.

C_k is called the k-th convergent of α

α is the limit of the sequence $C_1, C_2, C_3, \ldots$

Best Approximator Theorem

A rational p/q is the <u>best approximator</u> to a real α if no rational number of denominator smaller than q comes closer to α.

p/q approx $\sqrt{53}$ p'/q', p'/q'

BEST APPROXIMATOR THEOREM:

Given any CF representation of α, each convergent of the CF is a best approximator for α !

Best Approximators of π

$C_1 = 3$

$C_2 = 22/7$

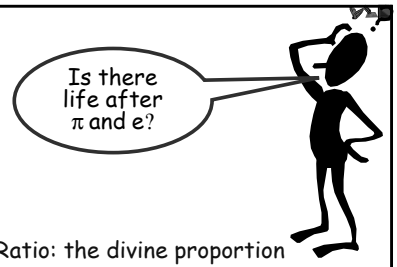
$C_3 = 333/106$

$C_4 = 355/113$

$C_5 = 103993/33102$

$C_6 = 104348/33215$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ldots}}}}}}}}$$

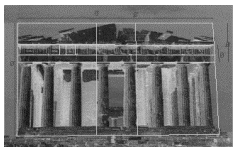


Golden Ratio: the divine proportion

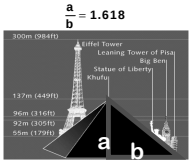
$\phi = 1.6180339887498948482045\ldots$

"Phi" is named after the Greek sculptor <u>Phi</u>dias

Golden ratio supposed to arise in…



Parthenon, Athens (400 B.C.)

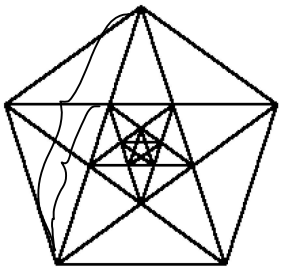


The great pyramid at Gizeh

Ratio of a person's height to the height of his/her navel

Mostly circumstantial evidence…

Pentagon



Definition of ϕ (Euclid)

Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

$$\phi = \frac{AC}{AB} = \frac{AB}{BC}$$

$$\phi^2 = \frac{AC}{BC}$$

$$\phi^2 - \phi = \frac{AC}{BC} - \frac{AB}{BC} = \frac{BC}{BC} = 1$$

$$\boxed{\phi^2 - \phi - 1 = 0}$$

Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

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$$= 1 + \frac{1}{1 + \frac{1}{\phi}}$$

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Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Continued Fraction Representation

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}}}}$$

Remember?

We already saw the convergents of this CF

[1,1,1,1,1,1,1,1,1,1,1,...]

are of the form

Fib(n+1)/Fib(n)

Hence: $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi = \frac{1+\sqrt{5}}{2}$

1,1,2,3,5,8,13,21,34,55,....

2/1	=	2
3/2	=	1.5
5/3	=	1.666...
8/5	=	1.6
13/8	=	1.625
21/13	=	1.6153846...
34/21	=	1.61904...

ϕ = 1.6180339887498948482045

Continued fraction representation of a standard fraction

$$\frac{67}{29} = 2 + \cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2}}}$$

$$\frac{67}{29} = 2 + \frac{1}{\frac{29}{9}} = 2 + \cfrac{1}{3+\cfrac{2}{9}} 2 + \cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2}}}$$

e.g., 67/29 = 2 with remainder 9/29
= 2 + 1/ (29/9)

A Representational Correspondence

$$\frac{67}{29} = 2 + \frac{1}{\frac{29}{9}} = 2 + \cfrac{1}{3+\cfrac{2}{9}} 2 + \cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2}}}$$

Euclid(67,29)	67 div 29 = 2
Euclid(29,9)	29 div 9 = 3
Euclid(9,2)	9 div 2 = 4
Euclid(2,1)	2 div 1 = 2
Euclid(1,0)	

Euclid's GCD = Continued Fractions

$$\frac{A}{B} = \left\lfloor \frac{A}{B} \right\rfloor + \cfrac{1}{\cfrac{B}{A \bmod B}}$$

Euclid(A,B) = Euclid(B, A mod B)
Stop when B=0

Theorem: All fractions have finite continuous fraction expansions

Fibonacci Magic Trick

REFERENCES

Continued Fractions, C. D. Olds

The Art Of Computer Programming, Vol 2, by Donald Knuth

"*Misconceptions About the Golden Ratio*", George Markowsky, *College Mathematics Journal*, Jan 92.

The Golden Ratio: The Story of PHI, the World's Most Astonishing Number, by Mario Livio

Fibonacci Numbers and the Golden Section, Ron Knott's excellent website

Study Bee

GCD
- Euclid's algorithm
- Extended Euclid's algorithm
  - Given X,Y, outputs r,s and GCD(X,Y) such that rX + sY = GCD(X,Y)
- Use it to find X^{-1} (for X in Z_n^*)

Continued Fractions
- Finite CFs = rationals
- Periodic CFs = roots of quadratics
- Convergents
  - e.g. convergents of [1,1,1,...] = F_n/F_{n-1}

Golden Ratio ϕ
- Solution to quadratic $x^2 - x - 1 = 0$.
- ϕ = [1,1,1,1,...]