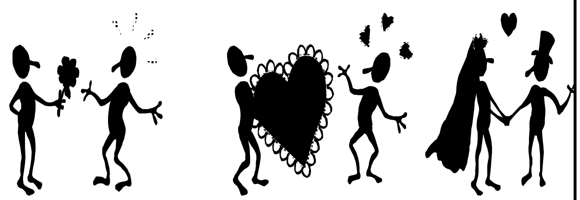
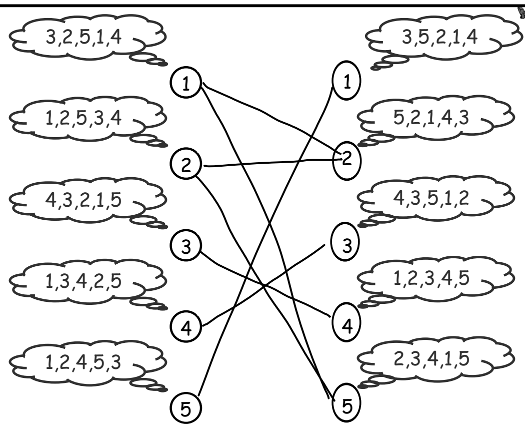
Great Theoretical Ideas In Computer Science
Anupam Gupta CS 15-251 Fall 2005
Lecture 11 Oct 04, 2005 Carnegie Mellon University

The Mathematics Of 1950's Dating: Who wins the battle of the sexes?



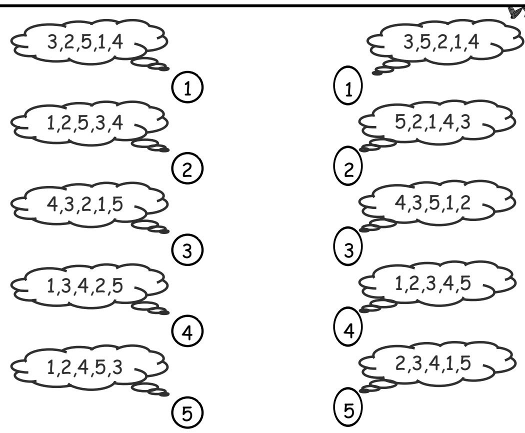
WARNING: This lecture contains mathematical content that may be shocking to some students.



Dating Scenario

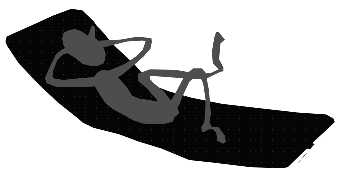
- There are n boys and n girls
- Each girl has her own ranked preference list of all the boys
- Each boy has his own ranked preference list of the girls
- The lists have no ties

Question: How do we pair them off? What criteria come to mind?



There is more than one notion of what constitutes a "good" pairing.

- Maximizing total satisfaction
  - Hong Kong and to an extent the United States
- Maximizing the minimum satisfaction
  - Western Europe
- Minimizing the maximum difference in mate ranks
  - Sweden
- Maximizing number of people who get their first choice
  - Barbie and Ken Land



We will ignore the issue of what is "equitable"!

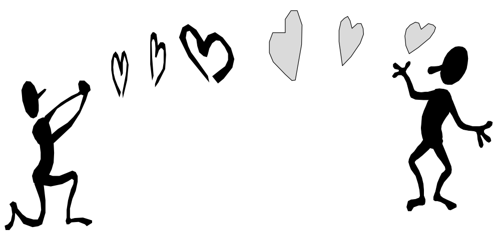
Rogue Couples

Suppose we pair off all the boys and girls.

Now suppose that some boy and some girl prefer each other to the people to whom they are paired.

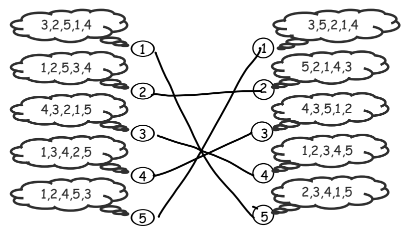
They will be called a <u>rogue couple</u>.

Why be with them when we can be with each other?



Stable Pairings

A pairing of boys and girls is called <u>stable</u> if it contains no rogue couples.



What use is fairness, if it is not stable?

Any list of criteria for a good pairing must include stability. (A pairing is doomed if it contains a rogue couple.)

Any reasonable list of criteria must contain the stability criterion.

Some social and political wisdom:

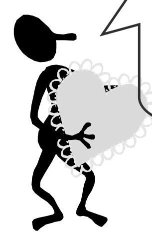
<u>Sustainability</u> is a prerequisite of *fair* policy.

The study of stability will be the subject of the entire lecture.

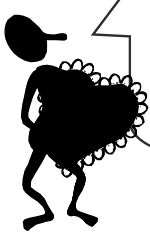
We will:

- Analyze various mathematical properties of an algorithm that looks a lot like 1950's dating
- Discover the **naked mathematical truth** about which sex has the romantic edge
- Learn how the world's largest, most successful dating service operates

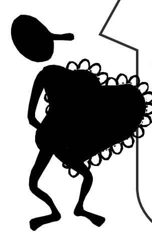
Given a set of preference lists, how do we find a stable pairing?



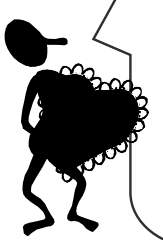
Given a set of preference lists, how do we find a stable pairing?



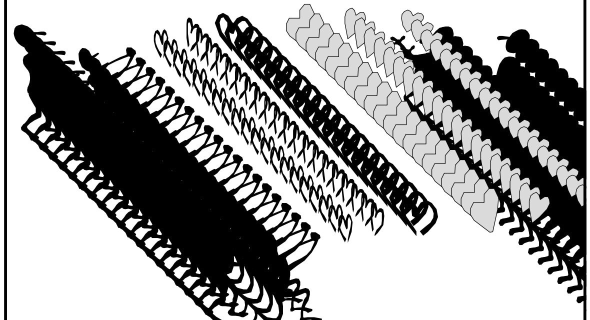
Better Questions:

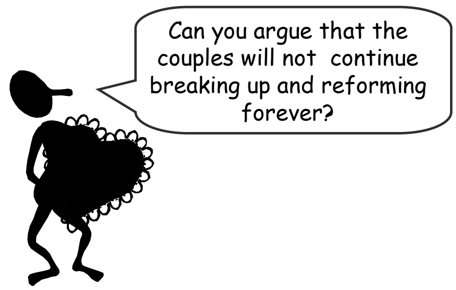


Think about this question:

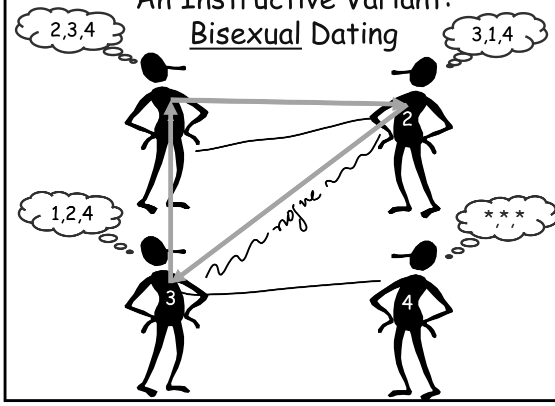


Idea: Allow the pairs to keep breaking up and reforming until they become stable.





An Instructive Variant: Bisexual Dating

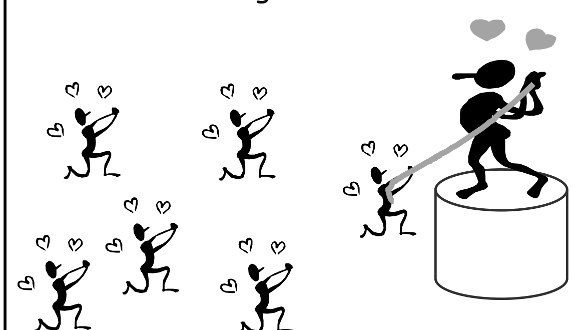


Insight

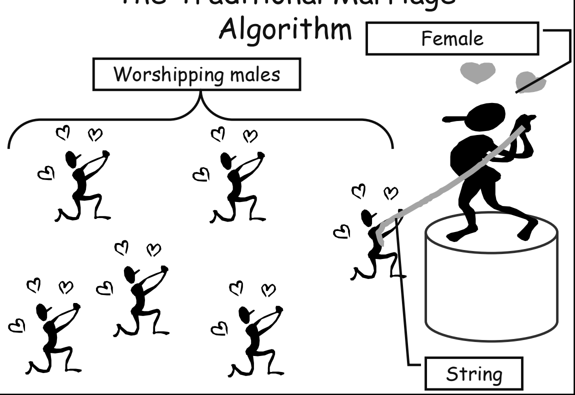
Any proof that heterosexual couples do not break up and re-form forever must contain a step that fails in the bisexual case

If you have a proof idea that works equally well in the hetero and bisexual versions, then your idea is <u>not adequate</u> to show the couples eventually stop.

The Traditional Marriage Algorithm



The Traditional Marriage Algorithm



Traditional Marriage Algorithm

For each day that some boy gets a "No" do:
- **Morning**
  - Each girl stands on her balcony
  - Each boy proposes under the balcony of the best girl whom he has not yet crossed off
- **Afternoon (for those girls with at least one suitor)**
  - To today's best suitor: **"Maybe, come back tomorrow"**
  - To any others: **"No, I will never marry you"**
- **Evening**
  - Any rejected boy crosses the girl off his list

If no boys get a "No"
Each girl marries the boy to whom she just said "maybe"

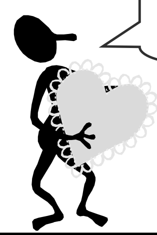
TMA

Girls

	1	2	3	4	5
Day 1:	X X 5		1	3	
Day 2.	5	2	X 4	3	
Day 3:	5	2 X	4	3	
Day 4:	5	2	4	3	1

⇒ get a matching!!

Does the Traditional Marriage Algorithm always produce a <u>stable</u> pairing?



Does TMA always terminate?

- It might encounter a situation where algorithm does not specify what to do next (a.k.a. "core dump error")
- It might keep on going for an infinite number of days

Traditional Marriage Algorithm

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If no boys get a "No"

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<u>Improvement Lemma</u>:
If a girl has a boy on a string, then she will always have someone at least as good on a string, (or for a husband).

- She would only let go of him in order to "maybe" someone better
- She would only let go of that guy for someone even better
- She would only let go of that guy for someone even better
- AND SO ON . . . .

Informal Induction

<u>Improvement Lemma</u>:
If a girl has a boy on a string, then she will always have someone at least as good on a string, (or for a husband).

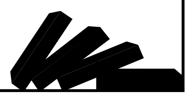
Formal Induction

Improvement Lemma:
If a girl has a boy on a string, then she will always have someone at least as good on a string, (or for a husband).

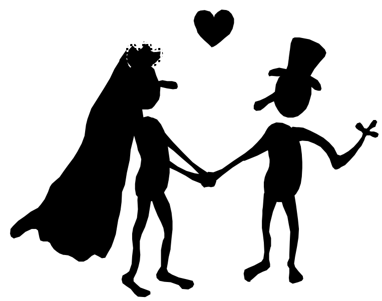
PROOF: Let q be the day she first gets b on a string. If the lemma is false, there must be a smallest k such that the girl has some b* inferior to b on day q+k.

One day earlier, she has someone as good as b. Hence, a better suitor than b* returns the next day. She will choose the better suitor contradicting the assumption that her prospects went below b on day q+k.

Formal Induction



Corollary: Each girl will marry her absolute favorite of the boys who visit her during the TMA



Lemma: No boy can be rejected by all the girls

Proof: ~~boy a~~ Suppose boy b is rejected by all n girls

But each girl rejected b ⇒ each girl has someone on a string

⇒ # boys = n boys on strings
+ boy b
≥ n+1 boys   Contradiction!!

Lemma: No boy can be rejected by all the girls

Proof by contradiction.

Suppose boy b is rejected by all the girls.
At that point:

- Each girl must have a suitor other than b (By Improvement Lemma, once a girl has a suitor she will always have at least one)
- The n girls have n suitors, b not among them. Thus, there are at least n+1 boys

Contradiction

Theorem: The TMA always terminates in at most n^2 days

M = Master list of all n boys preferences.
there are n^2 names to begin with.
one each day, at least one name is struck off.
⇒ # of days names are struck off $\leq n^2$.
(Since each boy's list must have at least one name not crossed off
$n(n-1)$ names

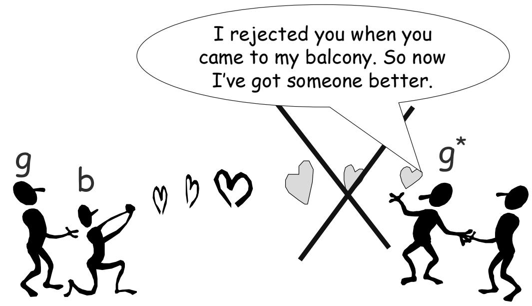
Theorem: The TMA always terminates in at most n^2 days

- A "master list" of all n of the boys lists starts with a total of $n \times n = n^2$ girls on it.
- Each day that at least one boy gets a "No", at least one girl gets crossed off the master list
- Therefore, the number of days is bounded by the original size of the master list. In fact, since each list never drops below 1, the number of days is bounded by $n(n-1) = n^2$.

Great! We know that TMA will terminate and produce a pairing.

But is it stable?

<u>Theorem</u>: Let T be the pairing produced by TMA. Then T is stable.



<u>Theorem</u>: Let T be the pairing produced by TMA. T is stable.

- Let b and g be any couple in T.
- Suppose b prefers g^* to g. We will argue that g^* prefers her husband to b.
- During TMA, b proposed to g^* before he proposed to g. Hence, at some point g^* rejected b for someone she preferred. By the Improvement lemma, the person she married was also preferable to b.
- Thus, every boy will be rejected by any girl he prefers to his wife. T is stable.

<u>Opinion Poll</u>

Forget TMA for a moment

How should we define what we mean when we say "the optimal girl for boy b"?

Flawed Attempt:
"The girl at the top of b's list"

The Optimal Girl

A boy's optimal girl is the highest ranked girl for whom there is <u>some</u> stable pairing in which the boy gets her.

She is the best girl he can conceivably get in a stable world. Presumably, she might be better than the girl he gets in the stable pairing output by TMA.

The Pessimal Girl

A boy's pessimal girl is the lowest ranked girl for whom there is <u>some</u> stable pairing in which the boy gets her.

> She is the worst girl he can conceivably get in a stable world.

Dating Heaven and Hell

A pairing is male-optimal if every boy gets his optimal mate. This is the best of all possible stable worlds for every boy simultaneously.

A pairing is male-pessimal if every boy gets his pessimal mate. This is the worst of all possible stable worlds for every boy simultaneously.

Dating Heaven and Hell

A pairing is male-optimal if every boy gets his optimal mate. Thus, the pairing is simultaneously giving each boy his optimal.

Is a male-optimal pairing always stable?

Dating Heaven and Hell

A pairing is female-optimal if every girl gets her optimal mate. This is the best of all possible stable worlds for every girl simultaneously.

A pairing is female-pessimal if every girl gets her pessimal mate. This is the worst of all possible stable worlds for every girl simultaneously.

The Naked Mathematical Truth!

> The Traditional Marriage Algorithm always produces a male-optimal, female-pessimal pairing.

<u>Theorem</u>: TMA produces a male-optimal pairing

<u>Proof</u>: Suppose not. Let T be the TMA's pairing

Suppose b is the <u>first</u> boy to be refused by his optimal girl g

then g must be matched to b^* (in T) whom she likes more.

Since b's optimal girl is g

$\exists$ stable pairing S matching b & g

g likes b^* more than she likes b.

b^* likes g at least as much as his opt girl

Theorem: TMA produces a male-optimal pairing

in S g&b matched

g likes b* more than her mate in S

b* likes g more than his mate in S

⇒ rogue couple!

⇒ S is not stable!!

⇒ contradiction !!! ☺

Theorem: TMA produces a male-optimal pairing

- Suppose, for a contradiction, that some boy gets rejected by his optimal girl during TMA. Let t be the earliest time at which this happened.
- In particular, at time t, some boy b got rejected by his optimal girl g because she said "maybe" to a preferred b*. By the definition of t, b* had not yet been rejected by his optimal girl.
- Therefore, b* likes g at least as much as his optimal.

Some boy b got rejected by his optimal girl g because she said "maybe" to a preferred b*. b* likes g at least as much as his optimal girl.

There must exist a stable pairing S in which b and g are married.

- b* wants g more than his wife in S
  - g is as at least as good as his best and he does not have her in stable pairing S
- g wants b* more than her husband in S
  - b is her husband in S and she rejects him for b* in TMA

Some boy b got rejected by his optimal girl g because she said "maybe" to a preferred b*. b* likes g at least as much as his optimal girl.

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  - b is her husband in S and she rejects him for b* in TMA

Contradiction of the stability of S.

What proof technique did we just use?

Theorem: **The TMA pairing, T, is female-pessimal.**

We know it is male-optimal. Suppose there is a stable pairing S where some girl g does worse than in T. ← pairing produced by TMA

Let b be her mate in T.

Let b* be her mate in S.

- By assumption, g likes b better than her mate in S
- b likes g better than his mate in S
  - We already know that g is his optimal girl
- **Therefore, S is not stable**

Contradiction

The largest, most successful dating service in the world uses a computer to run TMA!

REFERENCES

D. Gale and L. S. Shapley, *College admissions and the stability of marriage*, American Mathematical Monthly 69 (1962), 9-15

Dan Gusfield and Robert W. Irving, *The Stable Marriage Problem: Structures and Algorithms*, MIT Press, 1989