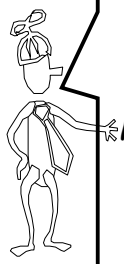
Great Theoretical Ideas In Computer Science		
John Lafferty		CS 15-251 Fall 2005
Lecture 9	Sept 27, 2005	Carnegie Mellon University

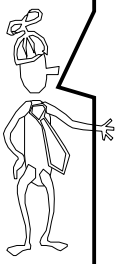
Modular Arithmetic and the RSA Cryptosystem

$\equiv_p 1$



MAX(a,b) + MIN(a,b) = a+b

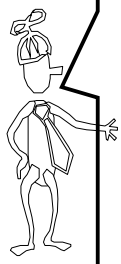
What is multiplicative version?



n|m means that m is an integer multiple of n.

We say that "n divides m".

True: 5|25 2|-66 7|35,
False: 4|5 8|2



Greatest Common Divisor:

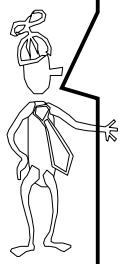
GCD(x,y) = greatest $k \geq 1$ s.t. k|x and k|y.

GCD: Greatest Common Divisor

What is the GCD of 12 and 18?

$12 = 2^2 3$

$18 = 2 3^2$



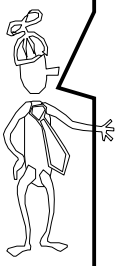
Least Common Multiple:

LCM(x,y) = smallest $k \geq 1$ s.t. x|k and y|k.

Prop: $\left(\frac{x}{\gcd(x,y)}\right) y$

GCD(x,y) = xy/LCM(x,y)

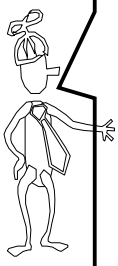
LCM(x,y) = xy/GCD(x,y)



GCD(x,y) = xy/LCM(x,y)
LCM(x,y) = xy/GCD(x,y)

$x = 2^2*3 = 12$; $y = 3^2*5 = 45$

GCD(12,45) = 3
LCM(12,45) = $2^2*3^2*5 = 180$
x*y = 540



GCD(x,y) * LCM(x,y) = xy

MAX(a,b) + MIN(a,b) = a+b

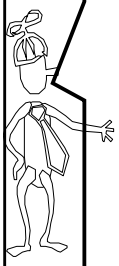
(a mod n) means the remainder when a is divided by n.

If dn + r = a, 0 <= r < n
Then r = (a mod n)
and d = (a div n)

Modular equivalence of integers a and b:

$a \equiv b \pmod{n}$
$a \equiv_n b$
"a and b are equivalent modulo n"

iff (a mod n) = (b mod n)
iff $n|(a-b)$



31 equals 81 modulo 2

$31 \equiv 81 \pmod{2}$

$31 \equiv_2 81$

(31 mod 2) = 1 = (81 mod 2)

$2|(31-81)$

$\equiv_n$ is an equivalence relation

In other words,

Reflexive:
$a \equiv_n a$
Symmetric:
$(a \equiv_n b) \Rightarrow (b \equiv_n a)$
Transitive:
$(a \equiv_n b \text{ and } b \equiv_n c) \Rightarrow (a \equiv_n c)$

$n | a-b$, $n | b-c \Rightarrow n | a-b+b-c = a-c$
$\Rightarrow a \equiv c \mod n$

$a \equiv_n b \leftrightarrow n|(a-b)$

"a and b are equivalent modulo n"

$\equiv_n$ induces a natural partition of the integers into n classes:

a and b are said to be in the same "residue class" or "congruence class" exactly when $a \equiv_n b$.

$a \equiv_n b \leftrightarrow n|(a-b)$

"a and b are equivalent modulo n"

Define the residue class [i] to be the set of all integers that are congruent to i modulo n.

Residue Classes Mod 3:

[0] = { …, -6, -3, 0, 3, 6, ..}
[1] = { …, -5, -2, 1, 4, 7, ..}
[2] = { …, -4, -1, 2, 5, 8, ..}
[-6] = { …, -6, -3, 0, 3, 6, ..}
[7] = { …, -5, -2, 1, 4, 7, ..}
[-1] = { …, -4, -1, 2, 5, 8, ..}

Equivalence mod n implies equivalence mod any divisor of n.

If $(x \equiv_n y)$ and $(k|n)$
Then: $x \equiv_k y$

Example: $10 \equiv_6 16 \Rightarrow 10 \equiv_3 16$

If $(x \equiv_n y)$ and $(k|n)$
Then: $x \equiv_k y$

$n|(x-y)$, $k|n \Rightarrow$
$k|(x-y)$

Fundamental lemma of plus, minus, and times modulo n:

If $(x \equiv_n y)$ and $(a \equiv_n b)$
Then: 1) $x+a \equiv_n y+b$
2) $x-a \equiv_n y-b$
3) $xa \equiv_n yb$

Proof of 3: xa = yb (mod n)

$n \mid x-y \quad , \quad n \mid a-b$

$\Rightarrow n \mid a(x-y) \quad , \quad n \mid y(a-b)$

$\Rightarrow n \mid a(x-y) + y(a-b) = ax - yb$

$\Rightarrow ax = yb \pmod{n}$

Fundamental lemma of plus minus, and times modulo n:

When doing plus, minus, and times modulo n, I can at any time in the calculation replace a number with a number in the same residue class modulo n

Please calculate:

249 * 504 mod 251

-2 * 2 = -4 = 247

A Unique Representation System Modulo n:

We pick exactly one representative from each residue class. We do all our calculations using these representatives.

Unique representation system modulo 3

Finite set S = {0, 1, 2}

+ and * defined on S:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Unique representation system modulo 3

Finite set S = {0, 1, -1}

+ and * defined on S:

+	0	1	-1
0	0	1	-1
1	1	-1	0
-1	-1	0	1

*	0	1	-1
0	0	0	0
1	0	1	-1
-1	0	-1	1

The reduced system modulo n:

$Z_n = \{0, 1, 2, \ldots, n-1\}$

Define $+_n$ and $*_n$:

$a +_n b = (a+b \bmod n)$

$a *_n b = (a*b \bmod n)$

$Z_n = \{0, 1, 2, \ldots, n-1\}$

$a +_n b = (a+b \bmod n)$ $\quad$ $a *_n b = (a*b \bmod n)$

$+_n$ and $*_n$ are associative binary operators from $Z_n \times Z_n \rightarrow Z_n$:

When $\heartsuit = +_n$ or $*_n$:

[Closed] $x,y \in Z_n \Rightarrow x \heartsuit y \in Z_n$

[Associative]

$x,y,z \in Z_n \Rightarrow (x \heartsuit y) \heartsuit z = x \heartsuit (y \heartsuit z)$

$Z_n = \{0, 1, 2, \ldots, n-1\}$

$a +_n b = (a+b \bmod n)$ $\quad$ $a *_n b = (a*b \bmod n)$

$+_n$ and $*_n$ are commutative, associative binary operators from $Z_n \times Z_n \rightarrow Z_n$:

[Commutative]

$x,y \in Z_n \Rightarrow x \heartsuit y = y \heartsuit x$

The reduced system modulo 3

$Z_3 = \{0, 1, 2\}$

Two binary, associative operators on Z_3:

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$*_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

The Boolean interpretation of $Z_2 = \{0, 1\}$

0 means FALSE $\quad$ 1 means TRUE

$+_2$ XOR	0	1
0	0	1
1	1	0

$*_2$ AND	0	1
0	0	0
1	0	1

The reduced system $Z_4 = \{0, 1,2,3\}$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

The reduced system
$Z_5 = \{0, 1,2,3,4\}$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The reduced system
$Z_6 = \{0, 1,2,3,4,5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	6	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

The reduced system
$Z_6 = \{0, 1,2,3,4,5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

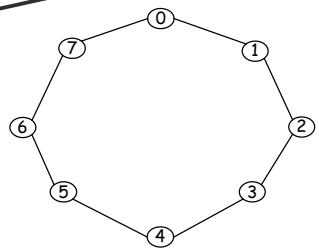
An operator has the permutation property if each row and each column has a permutation of the elements.

For every n, $+_n$ on Z_n has the permutation property

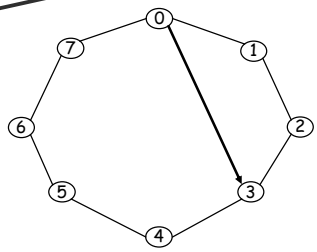
+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

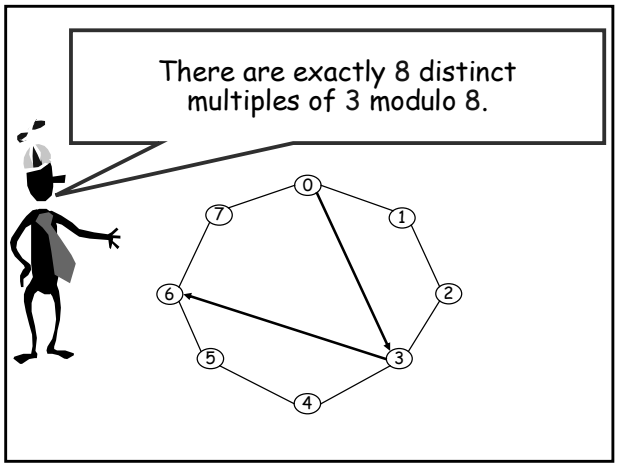
An operator has the permutation property if each row and each column has a permutation of the elements.

There are exactly 8 distinct multiples of 3 modulo 8.



There are exactly 8 distinct multiples of 3 modulo 8.




There are exactly 8 distinct multiples of 3 modulo 8.
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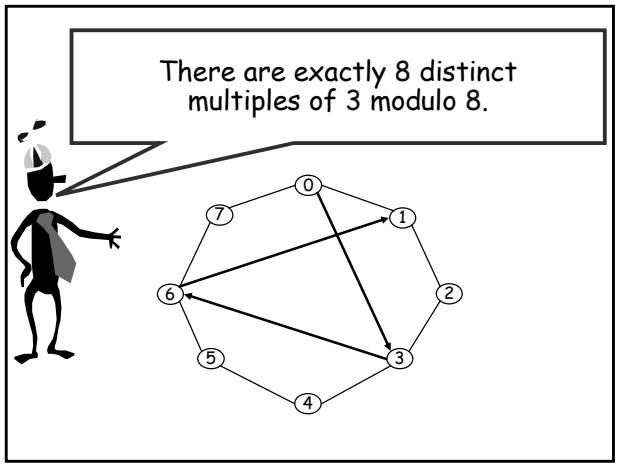
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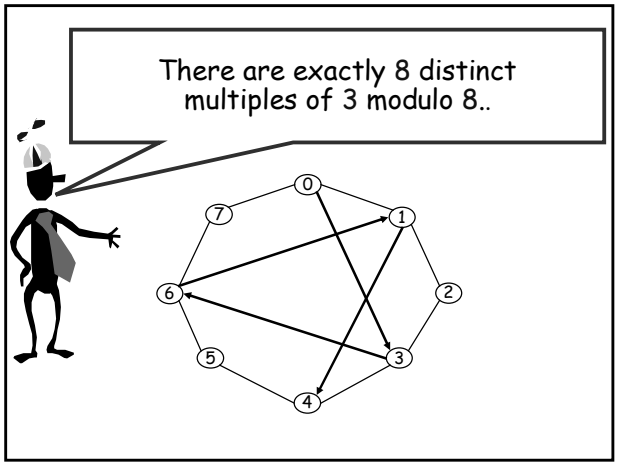
There are exactly 8 distinct multiples of 3 modulo 8..
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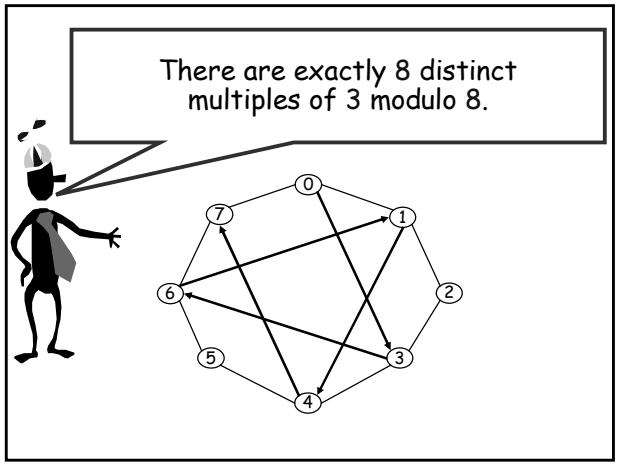
There are exactly 8 distinct multiples of 3 modulo 8.
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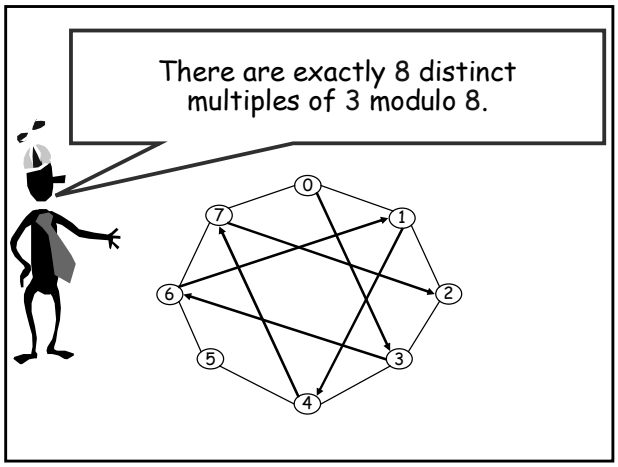
There are exactly 8 distinct multiples of 3 modulo 8.
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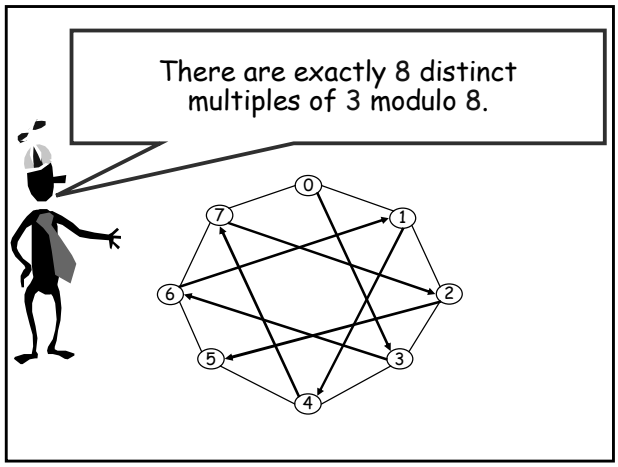
There are exactly 8 distinct multiples of 3 modulo 8.
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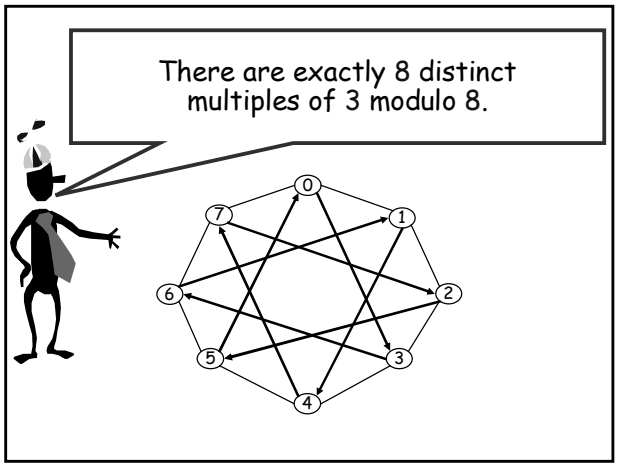
There are exactly 8 distinct multiples of 3 modulo 8.
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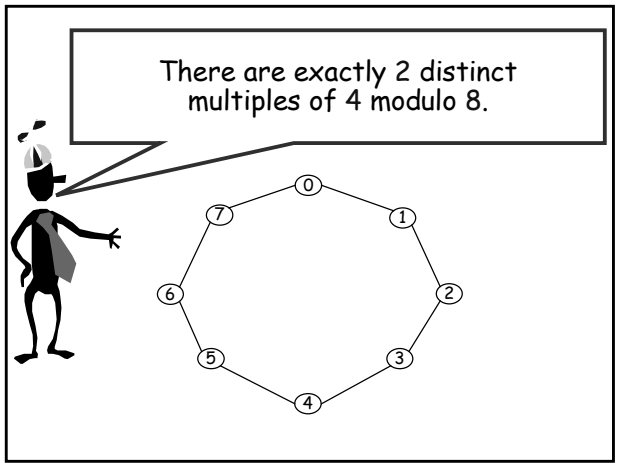
There are exactly 2 distinct multiples of 4 modulo 8.
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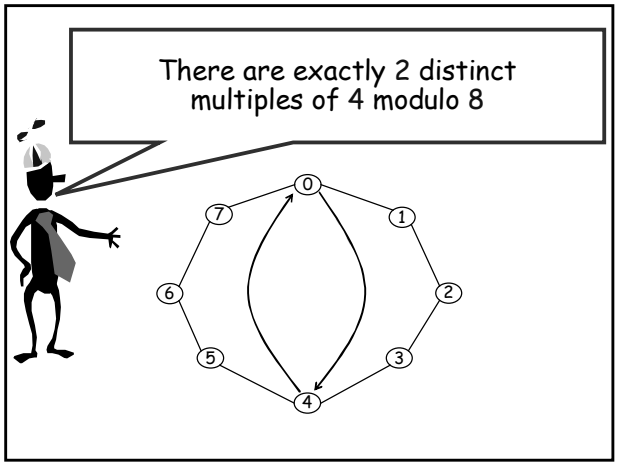
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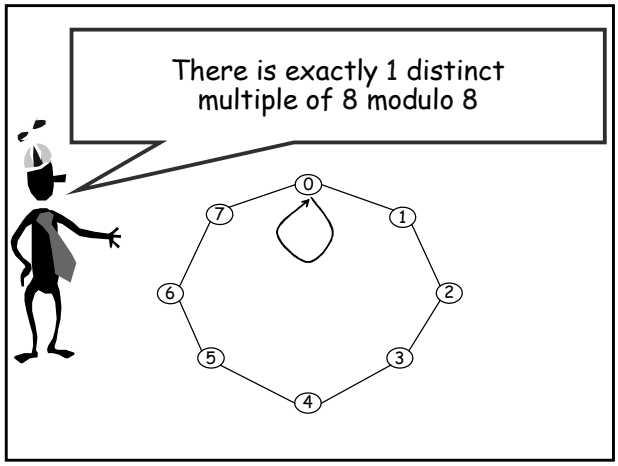
There is exactly 1 distinct multiple of 8 modulo 8
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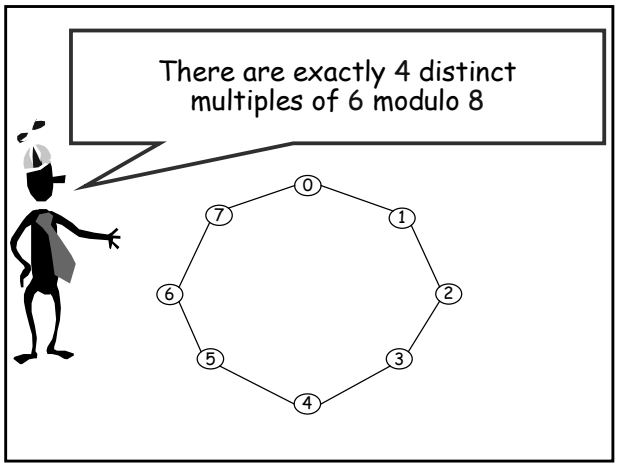
There are exactly 4 distinct multiples of 6 modulo 8
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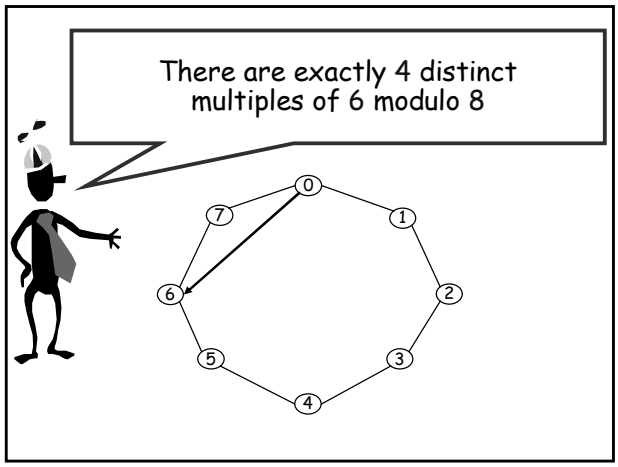
There are exactly 4 distinct multiples of 6 modulo 8
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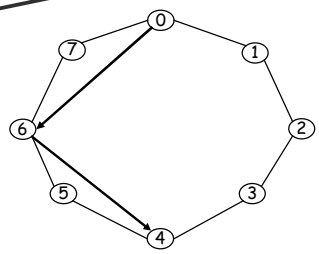
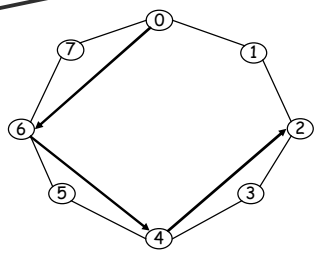
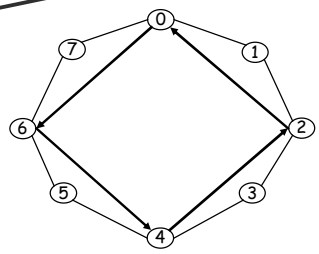
There are exactly 4 distinct multiples of 6 modulo 8



There are exactly 4 distinct multiples of 6 modulo 8



There are exactly 4 distinct multiples of 6 modulo 8



There are exactly ? distinct multiples of ? modulo ?

Can you see the general rule?

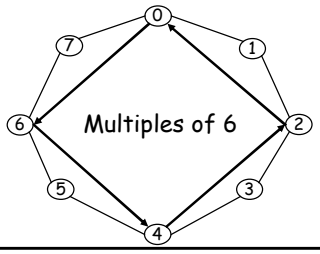
There are exactly LCM(n,c)/c distinct multiples of c modulo n

There are exactly LCM(n,c)/c distinct multiples of c modulo n

There are exactly n/(nc/LCM(n,c)) distinct multiples of c modulo n

There are exactly n/GCD(c,n) distinct multiples of c modulo n

The multiples of c modulo n is the set:
$\{0, c, c +_n c, c +_n c +_n c, \ldots\}$
$=\{kc \bmod n \mid 0 <= k <= n-1\}$



Theorem: There are exactly
$k = n/GCD(c,n) = LCM(c,n)/c$
distinct multiples of c modulo n:
$\{ c*i \bmod n \mid 0 <= i < k \}$

Clearly, $c/GCD(c,n) \geq 1$ is a whole number
$ck = n\,[c/GCD(c,n)] \equiv_n 0$
There are <= k distinct multiples of c mod n:
$c*0, c*1, c*2, \ldots, c*(k-1)$
k is all the factors of n missing from c
$cx \equiv_n cy \leftrightarrow n|c(x-y) \Rightarrow k|(x-y) \Rightarrow x-y \geq k$
There are $\geq$ k multiples of c

Is there a fundamental lemma of division modulo n?

$cx \equiv_n cy \Rightarrow x \equiv_n y$?

Is there a fundamental lemma of division modulo n?

$cx \equiv_n cy \Rightarrow x \equiv_n y$? NO!

If c=0 [mod n], $cx \equiv_n cy$ for any x and y. Canceling the c is like dividing by zero.

Repaired fundamental lemma of division modulo n?

$c \neq 0 \pmod n$, $cx \equiv_n cy \Rightarrow x \equiv_n y$?

$2*2 \equiv_6 2*5$, but not $2 \equiv_6 5$.
$6*3 \equiv_{10} 6*8$, but not $3 \equiv_{10} 8$.

When can I divide by c?

Theorem: There are exactly n/GCD(c.n) distinct multiples of c modulo n.

Corollary: If GCD(c,n) > 1, then the number of multiples of c is less than n.

Corollary: If GCD(c,n)>1 then you can't always divide by c.

Proof: There must exist distinct x,y<n such that c*x=c*y (but $x \neq y$)

Fundamental lemma of division modulo n.

$GCD(c,n)=1,\ ca \equiv_n cb \Rightarrow a \equiv_n b$

$n \mid c(a-b)$, but $\gcd(n,c)=1$

So $n \mid a-b \Rightarrow a \equiv b \pmod{n}$

Corollary for general c:

$cx \equiv_n cy \Rightarrow x \equiv_{n/GCD(c,n)} y$

$n \mid cx - cy = c(x-y)$

$\frac{n}{\gcd(c,n)} \mid \frac{c}{\gcd(c,n)} \cdot (x-y)$

$\Rightarrow x \equiv y \left(\bmod \frac{n}{\gcd(c,n)}\right)$

Fundamental lemma of division modulo n.

$GCD(c,n)=1,\ ca \equiv_n cb \Rightarrow a \equiv_n b$

$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$

Multiplication over Z_n^* will have the cancellation property.

$Z_6 = \{0, 1,2,3,4,5\}$

$Z_6^* = \{1,5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Suppose GCD(x,n) =1 and GCD(y,n) =1
Let z = xy and z′ = (xy mod n)

It is clear that GCD(z,n) =1. A moment's thought convinces us that GCD(z′,n)=1

$xy = an + z'$, Suppose $b \mid z'$, $b \mid n$

Then $b \mid an + z' \Rightarrow b \mid xy$

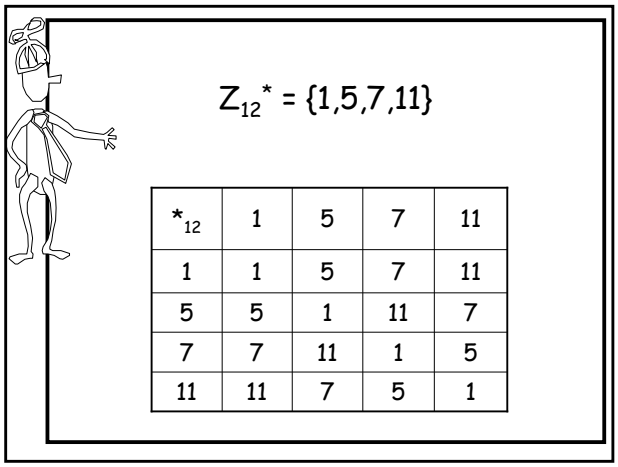
Thus, $b=1$, since $\gcd(xy,n)=1$

$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$

$*_n$ is an associative, binary operator. In particular, Z_n^* is closed under $*_n$:

$x,y \in Z_n^* \Rightarrow x *_n y \in Z_n^*$.

Proof: Let z = xy. Let z′ = z mod n. z = z′ + kn.
Suppose there exists a prime p>1 p|z′ and p|n.
z is the sum of two multiples of p, so p|z.
p|z ⇒ that p|x or p|y. Contradiction of $x,y \in Z_n^*$

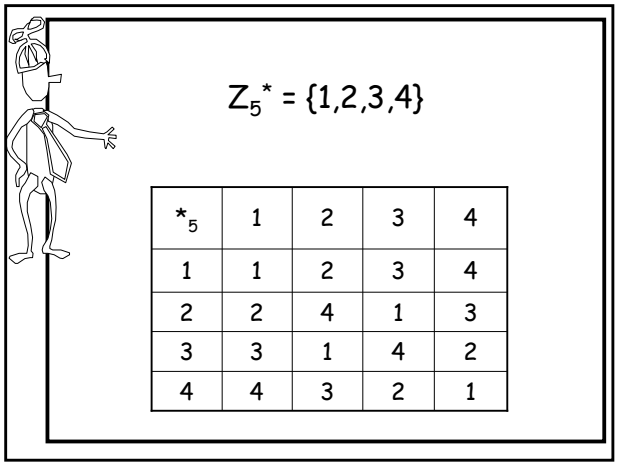


$Z_{12}^* = \{1,5,7,11\}$

$*_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

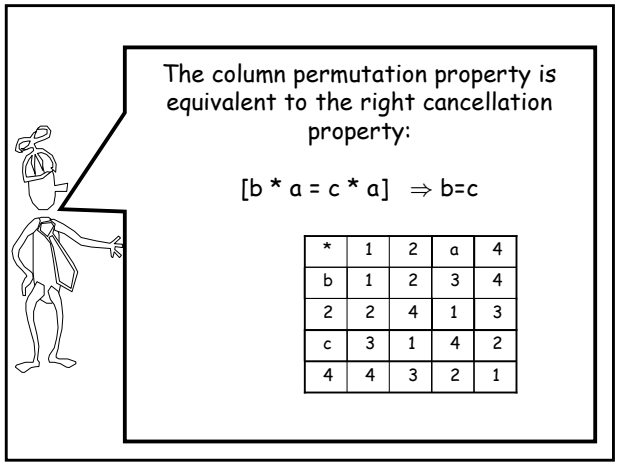
Z_{15}^*

*	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1



$Z_5^* = \{1,2,3,4\}$

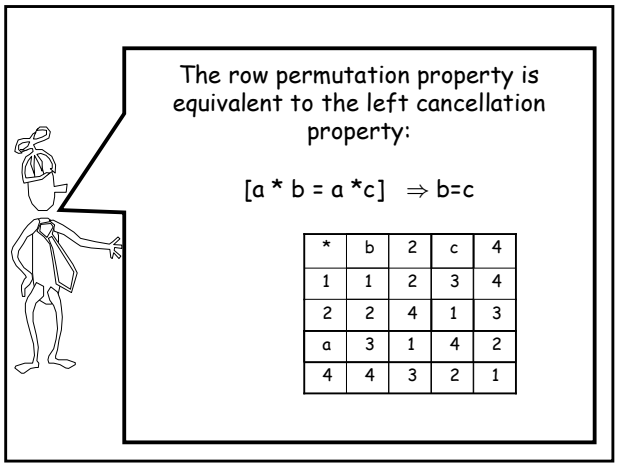
$*_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1



The column permutation property is equivalent to the right cancellation property:

[b * a = c * a] $\Rightarrow$ b=c

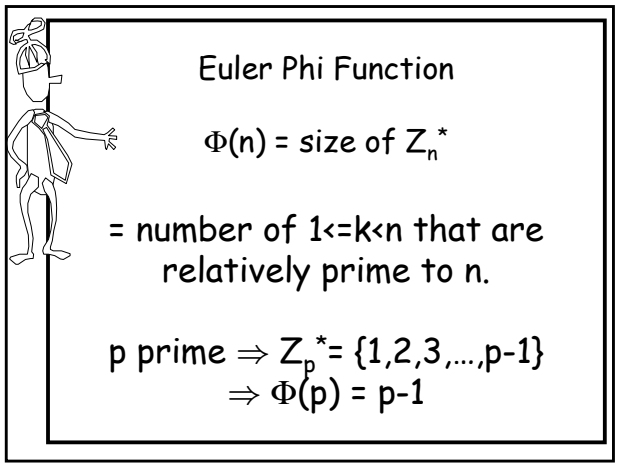
*	1	2	a	4
b	1	2	3	4
2	2	4	1	3
c	3	1	4	2
4	4	3	2	1



The row permutation property is equivalent to the left cancellation property:

[a * b = a *c] $\Rightarrow$ b=c

*	b	2	c	4
1	1	2	3	4
2	2	4	1	3
a	3	1	4	2
4	4	3	2	1



Euler Phi Function

$\Phi(n)$ = size of Z_n^*

= number of 1<=k<n that are relatively prime to n.

p prime $\Rightarrow Z_p^* = \{1,2,3,\ldots,p-1\}$
$\Rightarrow \Phi(p) = p-1$

$Z_{12}^* = \{1,5,7,11\}$
$\phi(12) = 4$

$*_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

$\phi(pq) = (p-1)(q-1)$
if p,q distinct primes

pq = # of numbers from 1 to pq
p = # of multiples of q up to pq
q = # of multiples of p up to pq
1 = # of multiple of both p and q up to pq

$\phi(pq) = pq - p - q + 1 = (p-1)(q-1)$

Let's consider how
we do arithmetic in Z_n and in Z_n^*

The additive inverse of $a \in Z_n$ is the unique $b \in Z_n$ such that $a +_n b \equiv_n 0$. We denote this inverse by "–a".

It is trivial to calculate:
"-a" = (n-a).

The multiplicative inverse of $a \in Z_n^*$ is the unique $b \in Z_n^*$ such that $a *_n b \equiv_n 1$. We denote this inverse by "a^{-1}" or "1/a".

The unique inverse of a must exist because the a row contains a permutation of the elements and hence contains a unique 1.

*	1	b	3	4
1	1	2	3	4
2	2	4	1	3
a	3	1	4	2
4	4	3	2	1

The multiplicative inverse of $a \in Z_n^*$ is the unique $b \in Z_n^*$ such that
$a *_n b \equiv_n 1$. We denote this inverse by "a^{-1}" or "1/a".

<u>Efficient algorithm to compute a^{-1} from a and n.</u>

Execute the Extended Euclidean Algorithm on a and n. It will give two integers r and s such that:
$ra + sn = (a,n) = 1$

Taking both sides mod n, we obtain:
$rn \equiv_n 1$
Output r, which is the inverse of a

$Z_n = \{0, 1, 2, \ldots, n-1\}$
$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$

Define $+_n$ and $*_n$:

$a +_n b = (a+b \bmod n)$ $\quad$ $a *_n b = (a*b \bmod n)$

$c *_n (a +_n b) \equiv_n (c *_n a) +_n (c *_n b)$

$\langle Z_n, +_n \rangle$	$\langle Z_n^*, *_n \rangle$
1. Closed	1. Closed
2. Associative	2. Associative
3. 0 is identity	3. 1 is identity
4. Additive Inverses	4. Multiplicative Inverses
5. Cancellation	5. Cancellation
6. Commutative	6. Commutative

Fundamental lemma of powers?

If $(a \equiv_n b)$
Then $x^a \equiv_n x^b$?

$3 = 0 \pmod 3$
$2^0 = 1 \overset{?}{\equiv} 2^3 \pmod 3$

If $(a \equiv_n b)$ Then $x^a \equiv_n x^b$?

NO!

$(16 \equiv_{15} 1)$, but it is not the case that: $2^1 \equiv_{15} 2^{16}$

Calculate a^b mod n:

Except for b, work in a reduced mod system to keep all intermediate results less than $\lfloor \log_2(n) \rfloor + 1$ bits long.

Phase I (Repeated Multiplication)

For $\lfloor \log b \rfloor$ steps

multiply largest so far by a

$(a, a^2, a^4, \ldots)$

Phase II (Make a^b from bits and pieces)

Expand n in binary to see how n is the sum powers of 2. Assemble a^b by multiplying together appropriate powers of a.

By the permutation property, two names for the same set:

$Z_n^* = Z_n^a$

$Z_n^a = \{a *_n x \mid x \in Z_n^*\}, a \in Z_n^*$

*	b	2	c	4
1	1	2	3	4
2	2	4	1	3
a	3	1	4	2
4	4	3	2	1

Two products on the same set:

$Z_n^* = Z_n^a$

$Z_n^a = \{a *_n x \mid x \in Z_n^*\}, a \in Z_n^*$

$\prod x \equiv_n \prod ax$ [as x ranges over Z_n^*]

$\prod x \equiv_n \prod x \; (a^{\text{size of } Zn^*})$ [Commutativity]

$1 = a^{\text{size of } Zn^*}$ [Cancellation]

$a^{\Phi(n)} = 1$

Euler's Theorem

$a \in Z_n^*$, $a^{\Phi(n)} \equiv_n 1$

Fermat's Little Theorem

p prime, $a \in Z_p^* \Rightarrow a^{p-1} \equiv_p 1$

Fundamental lemma of powers.

Suppose $x \in Z_n^*$, and a,b,n are naturals.

If $a \equiv_{\Phi(n)} b$ Then $x^a \equiv_n x^b$

Equivalently,

$x^{a \bmod \Phi(n)} \equiv_n x^{b \bmod \Phi(n)}$

Calculate a^b mod n:

Use the repeated squaring method to compute a, a^2, a^4, …

But take exponents modulo $\Phi(n)$

Defining negative powers.

Suppose $x \in Z_n^*$, and a,n are naturals.

x^{-a} is defined to be the multiplicative inverse of X^a

$X^{-a} = (X^a)^{-1} = (X^{-1})^a$

Rule of integer exponents

Suppose $x,y \in Z_n^*$, and a,b are integers.

$(xy)^{-1} \equiv_n x^{-1} y^{-1}$

$X^a X^b \equiv_n X^{a+b}$

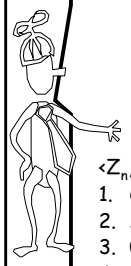
Lemma of integer powers.

Suppose $x \in Z_n^*$, and a,b are integers.

If $a \equiv_{\Phi(n)} b$ Then $x^a \equiv_n x^b$

Equivalently,

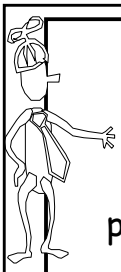
$x^{a \bmod \Phi(n)} \equiv_n x^{b \bmod \Phi(n)}$



$Z_n = \{0, 1, 2, \ldots, n-1\}$
$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$

Quick raising to power.

$\langle Z_n, +_n \rangle$	$\langle Z_n^*, *_n \rangle$
1. Closed	1. Closed
2. Associative	2. Associative
3. 0 is identity	3. 1 is identity
4. Additive Inverses Fast + and -	4. Multiplicative Inverses Fast * and /
5. Cancellation	5. Cancellation
6. Commutative	6. Commutative



Euler Phi Function

$\Phi(n)$ = size of z_n^*

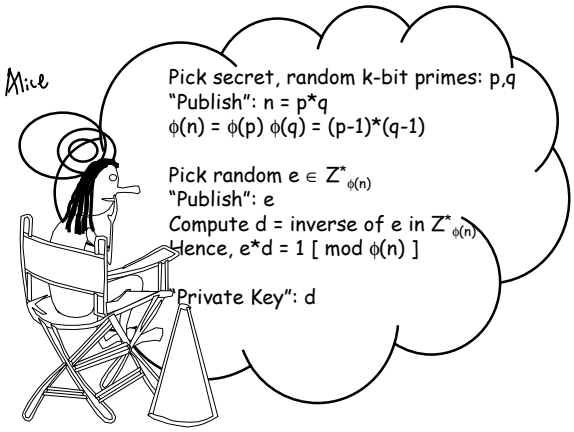
p prime $\Rightarrow Z_p^* = \{1,2,3,\ldots,p-1\}$
$\Rightarrow \Phi(p) = p-1$

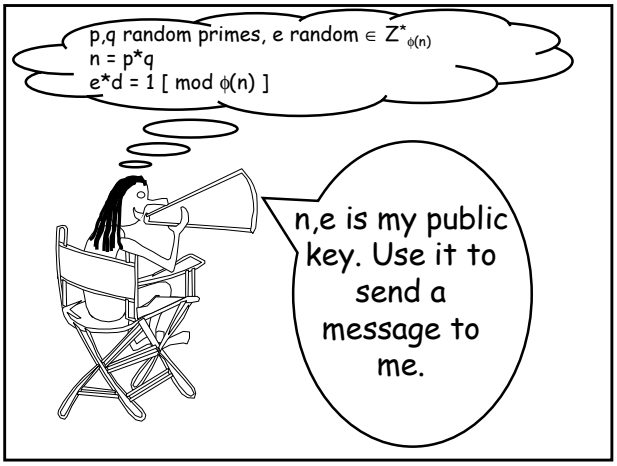
$\phi(pq) = (p-1)(q-1)$
if p,q distinct primes

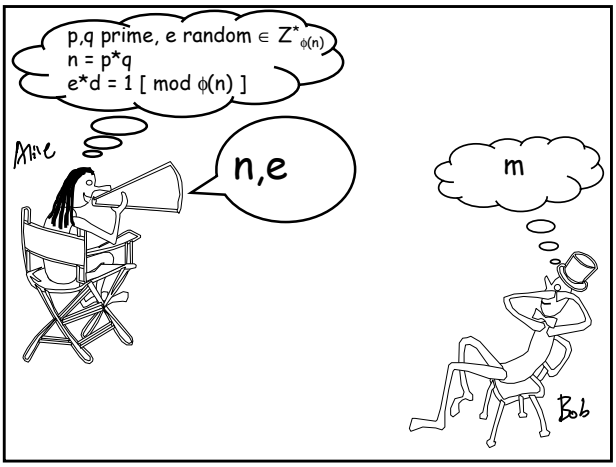
The RSA Cryptosystem

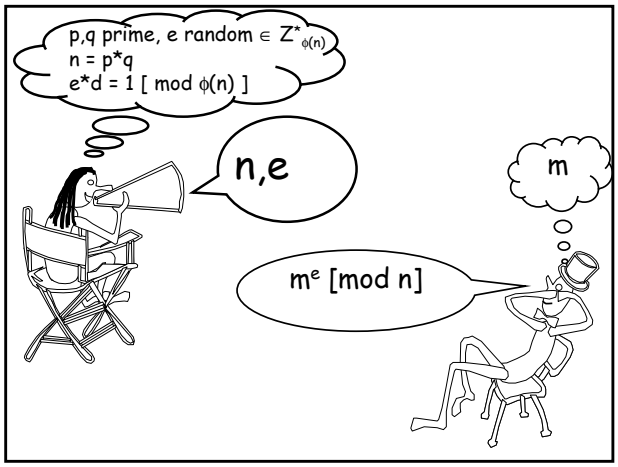
Rivest, Shamir, and Adelman (1978)

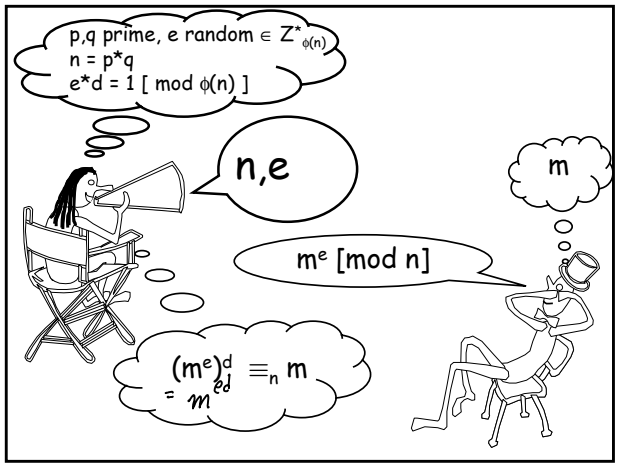
RSA is one of the most used cryptographic protocols on the net. Your browser uses it to establish a secure session with a site.








p,q prime, e random ∈ $Z^*_{\phi(n)}$
n = p*q
e*d = 1 [mod ϕ(n)]
n,e
m
m^e [mod n]


p,q prime, e random ∈ $Z^*_{\phi(n)}$
n = p*q
e*d = 1 [mod ϕ(n)]
n,e
m
m^e [mod n]
$(m^e)^d \equiv_n m$
$= m^{ed}$