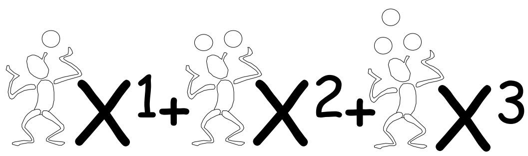
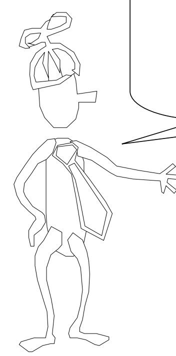
Great Theoretical Ideas In Computer Science		
John Lafferty		CS 15-251 Fall 2005
Lecture 8	Sept 22, 2005	Carnegie Mellon University

Counting III: Pascal's Triangle, Polynomials, and Vector Programs

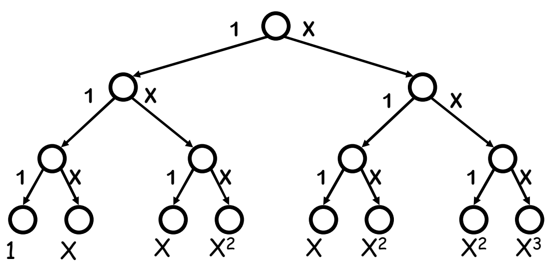




Last time, we saw that

Polynomials Count!

Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

The Binomial Formula

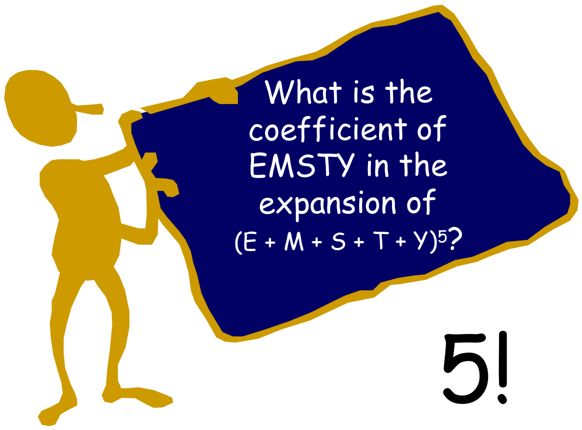
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

One polynomial, two representations

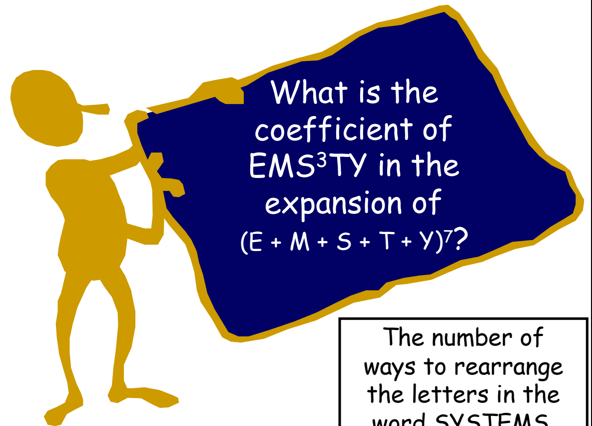
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

"Product form" or "Generating form"

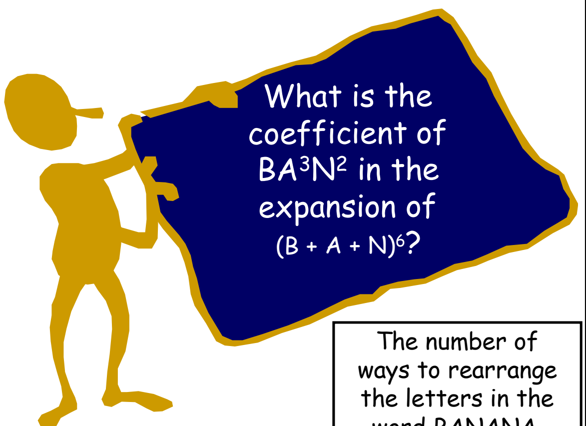
"Additive form" or "Expanded form"



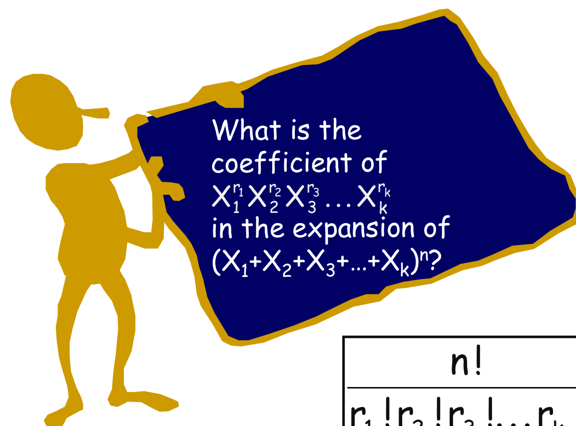
$5!$



The number of ways to rearrange the letters in the word SYSTEMS.



The number of ways to rearrange the letters in the word BANANA.



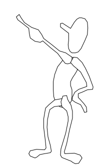
$$\frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}$$

Multinomial Coefficients

$$\binom{n}{r_1;r_2;\ldots;r_k} \equiv \begin{cases} 0 \text{ if } r_1 + r_2 + \ldots + r_k \neq n \\ \dfrac{n!}{r_1!r_2!\ldots r_k!} \end{cases}$$

$$\binom{n}{k;\ n-k} = \binom{n}{k}$$

The Multinomial Formula



$$(X_1+X_2+\ldots+X_k)^n = \sum_{\substack{r_1,r_2,\ldots,r_k \\ \sum r_i = n}} \binom{n}{r_1;r_2;\ldots;r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$$

Power Series Representation

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k$$

Since $\binom{n}{k} = 0$ if $k > n$

"Closed form" or "Generating form"

"Power series" ("Taylor series") expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x=1$.

$$2^n = \underbrace{\sum_{k=0}^{n} \binom{n}{k}}$$

The number of subsets of an *n*-element set

By varying x, we can discover new identities

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x= -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

The number of even-sized subsets of an *n* element set is the same as the number of odd-sized subsets.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

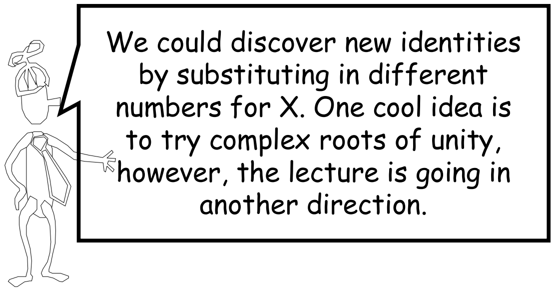
Let $x= -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

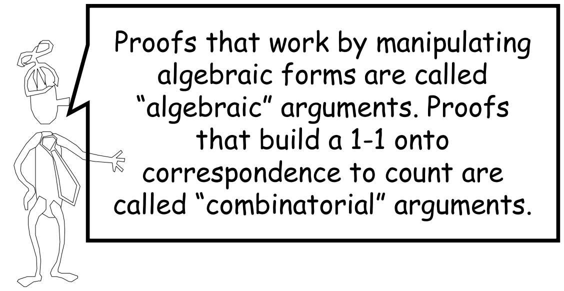
Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

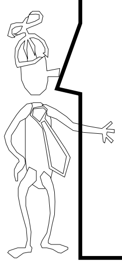
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We gave an <u>algebraic</u> proof that

$|O_n| = |E_n|$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

A <u>combinatorial</u> proof must construct a one-to-one correspondence between O_n and E_n

An attempt at a correspondence

Let f_n be the function that takes an n-bit string and flips all its bits.

f_n is clearly a one-to-one and onto function

for odd n. *E.g.* in f_7 we have

0010011 à 1101100

1001101 à 0110010

...but do even n work? In f_6 we have

110011 à 001100

101010 à 010101

Uh oh. Complementing maps evens to evens!

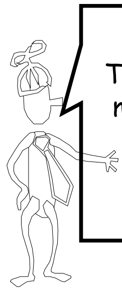
A correspondence that works for all n

Let f_n be the function that takes an n-bit string and flips only *the first bit*. For example,

0010011 à 1010011
1001101 à 0001101

110011 à 010011
101010 à 001010

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

The Binomial Formula

$(1+X)^0 =$ 1

$(1+X)^1 =$ 1 + 1X

$(1+X)^2 =$ 1 + 2X + 1X^2

$(1+X)^3 =$ 1 + 3X + 3X^2 + 1X^3

$(1+X)^4 =$ 1 + 4X + 6X^2 + 4X^3 + 1X^4

Pascal's Triangle:

n^{th} row are the coefficients of $(1+X)^n$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

---

n^{th} Row Of Pascal's Triangle:

$\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{k},\ldots\binom{n}{n}$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Inductive definition of kth entry of nth row:
Pascal(n,0) = Pacal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

"Pascal's Triangle"

$$\binom{0}{0}=1$$

$$\binom{1}{0}=1 \quad \binom{1}{1}=1$$

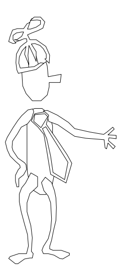
$$\binom{2}{0}=1 \quad \binom{2}{1}=2 \quad \binom{2}{2}=1$$

$$\binom{3}{0}=1 \quad \binom{3}{1}=3 \quad \binom{3}{2}=3 \quad \binom{3}{3}=1$$

Brahmagupta, 628
Al-Karaji, Baghdad 953-1029
Yanghui, 1261, Chu Shin-Chieh 1303
The Precious Mirror of the Four Elements
. . . Known in Europe by 1529

Blaise Pascal 1654

Pascal's Triangle



```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
```

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

Summing The Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

```
              1                  =1
            1 + 1                =2
          1 + 2 + 1              =4
        1 + 3 + 3 + 1            =8
      1 + 4 + 6 + 4 + 1          =16
   1 + 5 + 10 + 10 + 5 + 1       =32
1 + 6 + 15 + 20 + 15 + 6 + 1     =64
```

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

6+20+6 = 1+15+15+1

Summing on 1^{st} Avenue

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n\cdot(n+1)}{2}$$

Summing on k^{th} Avenue

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

1
1 1 =2
=3
1 2 1 =5
=8
1 3 3 1 =13
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1
1 1
1^2 2^2 1^2
1^2 3^2 3^2 1^2
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Al-Karaji Squares

1 =0
1 1 =1
1 2 + 2*1 =4
1 3 + 2*3 1 =9
1 4 + 2*6 4 1 =16
1 5 + 2*10 10 5 1 =25
1 6 + 2*15 20 15 6 1=36

Al-Karaji Squares

1 =0

1 1 =1

1 2 + 2*1 =4

1 3 + 2*3 1 =9

1 4 + 2*6 4 1 =16

1 5 + 2*10 10 5 1 =25

1 6 + 2*15 20 15 6 1=36

Application (Al-Karaji):

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$
$$= (1\cdot 0+1) + (2\cdot 1+2) + (3\cdot 2+3) + \cdots + (n(n-1)+n)$$
$$= 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + n(n-1) + \sum_{i=1}^{n} i$$
$$= 2\left[\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\cdots+\binom{n}{2}\right]+\binom{n+1}{2}$$
$$= 2\binom{n+1}{3}+\binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}$$

Al-Karaji Squares

1 =0

1 1 =1

1 2 + 2*1 =4

1 3 + 2*3 1 =9

1 4 + 2*6 4 1 =16

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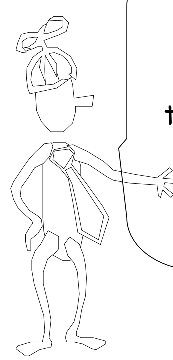
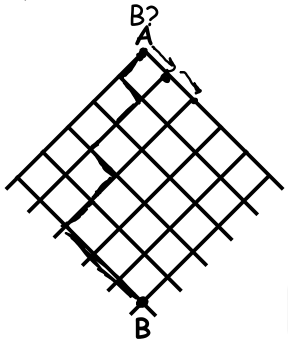
1 6 + 2*15 20 15 6 1=36

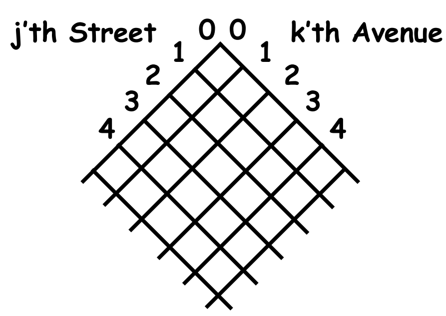


All these properties can be proved inductively and algebraically. We will give *combinatorial* proofs using the Manhattan block walking representation of binomial coefficients.

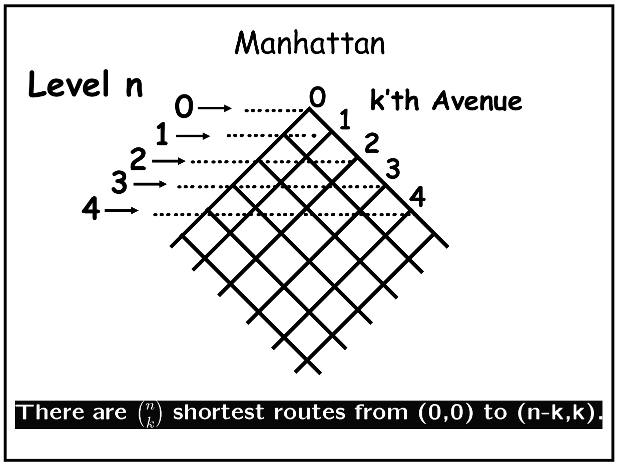
How many shortest routes from A to B?

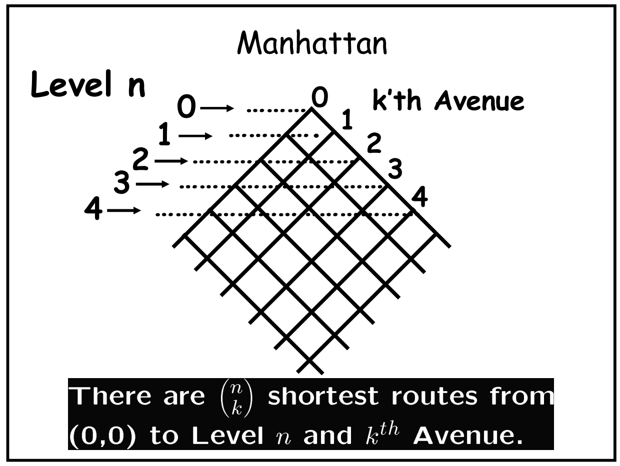


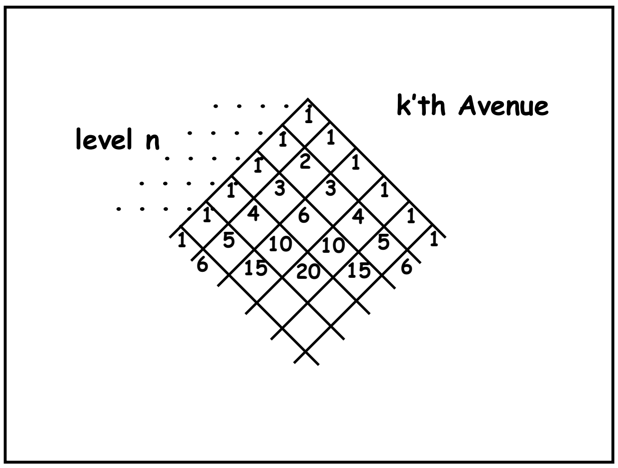
Manhattan

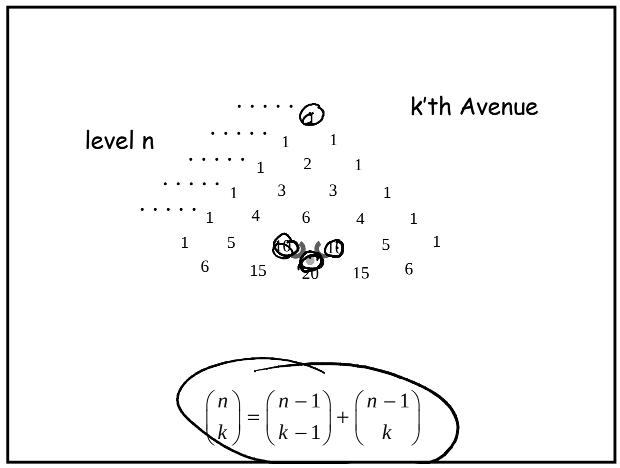


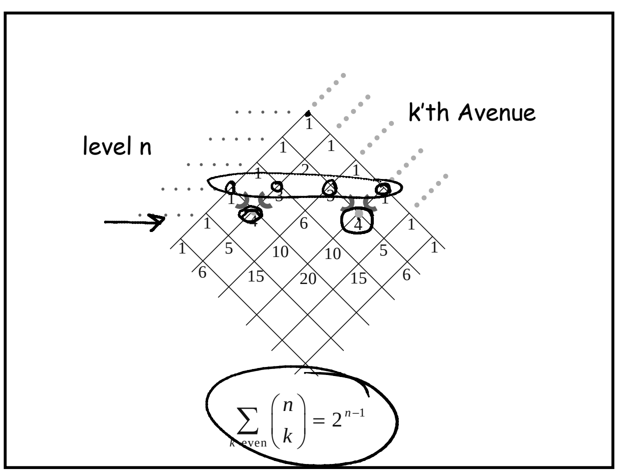
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k).

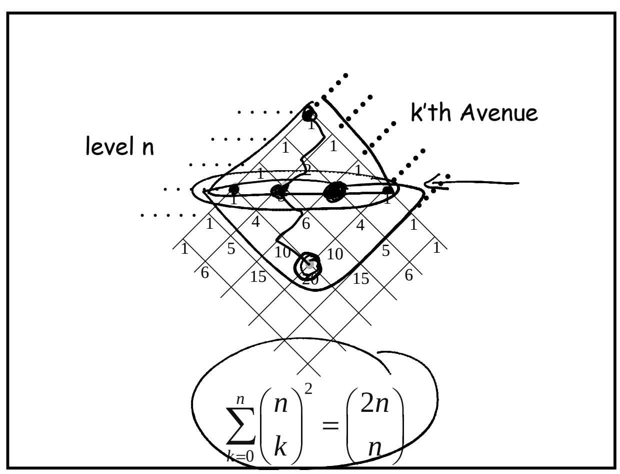

Manhattan
Level n
0 1 2 3 4
k'th Avenue
There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k).


Manhattan
Level n
0 1 2 3 4
k'th Avenue
There are $\binom{n}{k}$ shortest routes from (0,0) to Level n and k^{th} Avenue.


k'th Avenue
level n
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6


k'th Avenue
level n
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6
$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$


k'th Avenue
level n
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6
$\sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}$

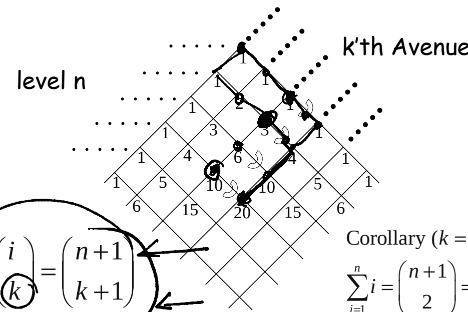

k'th Avenue
level n
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6
$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

By convention:

$0! = 1$ (empty product = 1)

$\binom{n}{k} = 1 \quad \text{if } k = 0$

$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n$



$$\sum_{i=1}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Corollary $(k = 1)$

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

Al-Karaji Squares

```
                 1                   =0
              1   1                  =1
           1   2 + 2*1               =4
        1   3 + 2*3    1             =9
      1   4 + 2*6    4    1          =16
    1   5 + 2*10    10    5    1     =25
 1   6 + 2*15    20    15    6    1=36
```

Application (Al-Karaji):

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Application (Al-Karaji):

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$= (1\cdot 0 + 1) + (2\cdot 1 + 2) + (3 \cdot 2 + 3) + \cdots + (n(n-1) + n)$$

$$= 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + n(n-1) + \sum_{i=1}^{n} i$$

$$= 2\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2}\right] + \binom{n+1}{2}$$

$$= 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}$$

Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable V→ can be thought of as:

< * , * , * , * , * , *, ... >

0 1 2 3 4 5 . . . .

Vector Programs

Let k stand for a scalar constant
<k> will stand for the vector <k,0,0,0,…>

<0> = <0,0,0,0,….>
<1> = <1,0,0,0,…>

V→ + T→ means to add the vectors position-wise.

<4,2,3,…> + <5,1,1,….> = <9,3,4,…>

Vector Programs

RIGHT(V→) means to shift every number in V→ one position to the right and to place a 0 in position 0.

RIGHT(<1,2,3, …>) = <0,1,2,3,. …>

Vector Programs

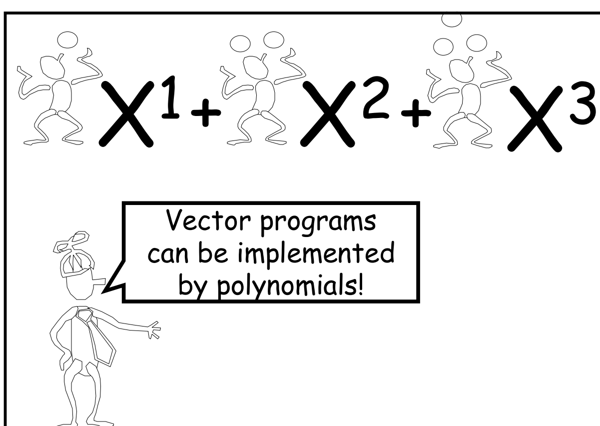
Example:	Store
V→ := <6>;	V→ = <6,0,0,0,..>
V→ := RIGHT(V→) + <42>;	V→ = <42,6,0,0,..>
V→ := RIGHT(V→) + <2>;	V→ = <2,42,6,0,..>
V→ := RIGHT(V→) + <13>;	V→= <13,2,42,6,.>

V→ = < 13, 2, 42, 6, 0, 0, 0, . . . >

Vector Programs

Example:	Store
V→ := <1>;	V→ = <1,0,0,0,..>
Loop n times:	V→ = <1,1,0,0,..>
V→ := V→ + RIGHT(V→);	V→ = <1,2,1,0,..>
	V→ = <1,3,3,1,.>

V→ = n^{th} row of Pascal's triangle.



Programs -----> Polynomials

The vector $\vec{V}$ = < $a_0, a_1, a_2, \ldots$ > will be represented by the polynomial:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

Formal Power Series

The vector $\vec{V}$ = < $a_0, a_1, a_2, \ldots$ > will be represented by the formal power series:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

$\vec{V}$ = < $a_0, a_1, a_2, \ldots$ >

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

<0> is represented by	0
<k> is represented by	k
$\vec{V} + \vec{T}$ is represented by	$(P_V + P_T)$
RIGHT($\vec{V}$) is represented by	$(X\,P_V)$

Vector Programs

Example:

$\vec{V}$:= <1>; P_V := 1;

Loop n times:
$\vec{V}$:= $\vec{V}$ + RIGHT($\vec{V}$); **P_V := P_V + X P_V;**

$\vec{V}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

$\vec{V}$:= <1>; P_V := 1;

Loop n times:
$\vec{V}$:= $\vec{V}$ + RIGHT($\vec{V}$); P_V := P_V (1+ X);

$\vec{V}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

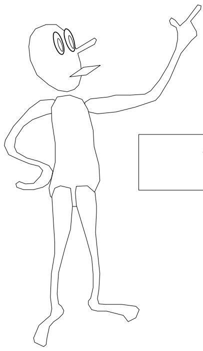
$\vec{V}$:= <1>;

Loop n times:
$\vec{V}$:= $\vec{V}$ + RIGHT($\vec{V}$);

} $P_V = (1+ X)^n$

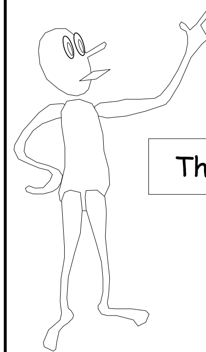
$\vec{V}$ = n^{th} row of Pascal's triangle.

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$



The Geometric Series

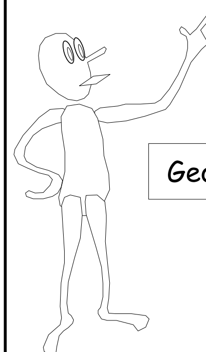
$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots.. = \frac{1}{1 - X}$$



The Infinite Geometric Series

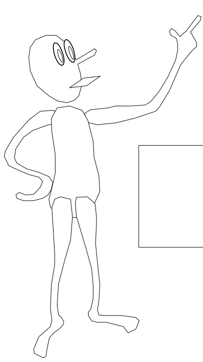
$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots.. = \frac{1}{1 - X}$$

$$1 + aX^1 + a^2X^2 + a^3X^3 + \ldots + a^nX^n + \ldots.. = \frac{1}{1 - aX}$$



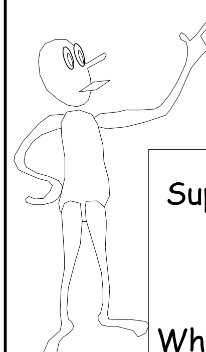
Geometric Series (Linear Form)

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) = \frac{1}{(1 - aX)(1-bX)}$$



Geometric Series (Quadratic Form)

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) = 1 + c_1X^1 + .. + c_k X^k + \ldots$$



Suppose we multiply this out to get a single, infinite polynomial.

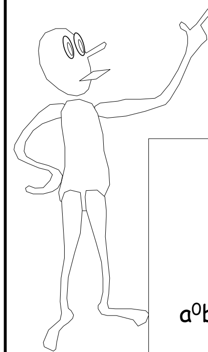
What is an expression for C_n?

$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots)\,(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots) =$

$1 + c_1X^1 + .. + c_k X^k + \dots$

$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots)\,(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots) =$

$1 + c_1X^1 + .. + c_k X^k + \dots$



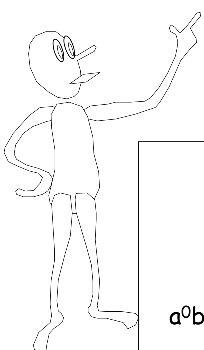
if $a \neq b$ then

$$c_n = \frac{a^{n+1} - b^{n+1}}{a - b}$$

$a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0$

$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots)\,(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots) =$

$1 + c_1X^1 + .. + c_k X^k + \dots$

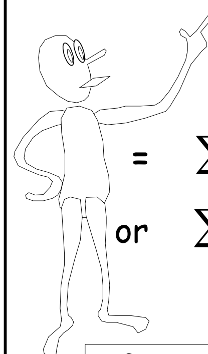


If $a = b$ then

$c_n = (n+1)(a^n)$

$a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0$

$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots)\,(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots) =$



$$= \frac{1}{(1 - aX)(1-bX)}$$

$$= \sum_{n=0..\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

or $\sum_{n=0..\infty} (n+1)a^n X^n$ when $a=b$

Geometric Series (Quadratic Form)

Another way to derive this:

Study Bee

- Polynomials count
- Binomial formula
- Multinomial coefficients
- Combinatorial proofs of binomial identities
- Vector programs
- Geometric series