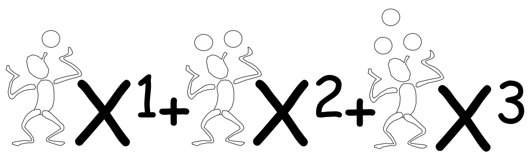
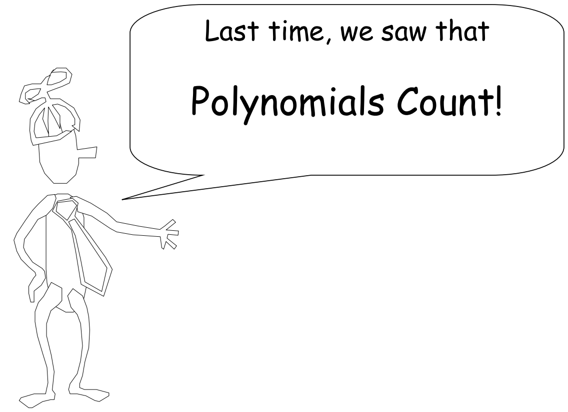
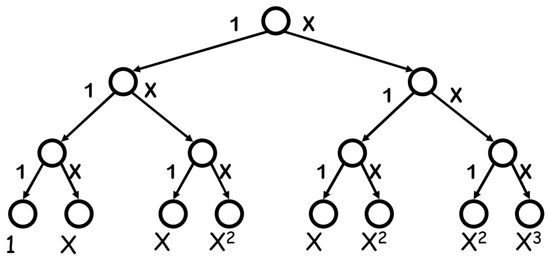
Great Theoretical Ideas In Computer Science		
John Lafferty		CS 15-251 Fall 2005
Lecture 8	Sept 22, 2005	Carnegie Mellon University

Counting III: Pascal's Triangle, Polynomials, and Vector Programs





Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

The Binomial Formula

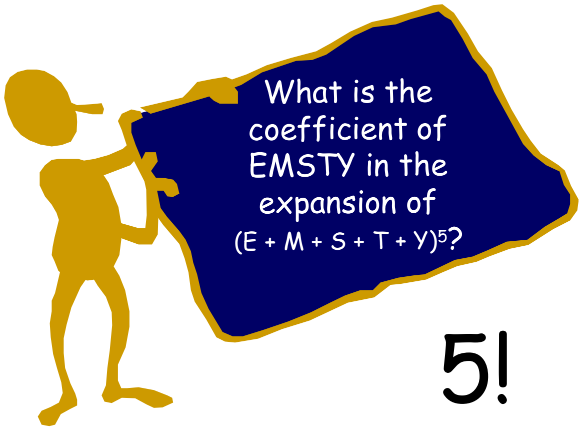
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

One polynomial, two representations

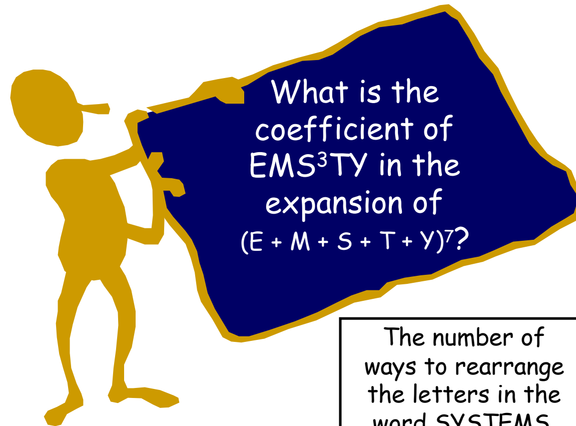
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

"Product form" or "Generating form"

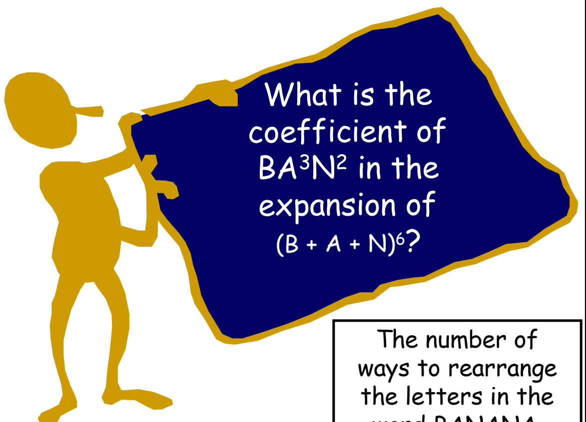
"Additive form" or "Expanded form"



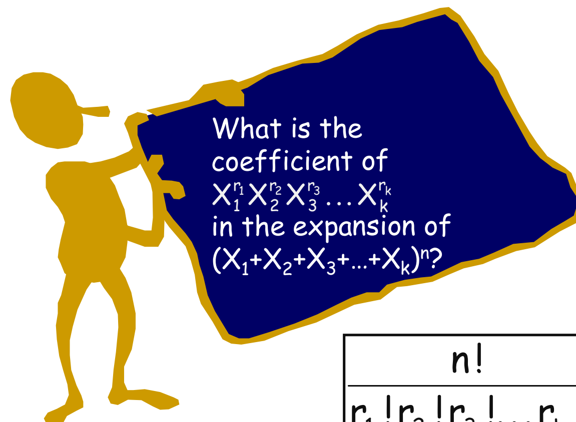
5!



The number of ways to rearrange the letters in the word SYSTEMS.



The number of ways to rearrange the letters in the word BANANA.



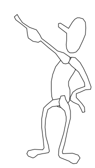
$$\frac{n!}{r_1!r_2!r_3!\ldots r_k!}$$

Multinomial Coefficients

$$\binom{n}{r_1;r_2;\ldots;r_k} \equiv \begin{cases} 0 \text{ if } r_1+r_2+\ldots+r_k \neq n \\ \dfrac{n!}{r_1!r_2!\ldots r_k!} \end{cases}$$

$$\binom{n}{k;\ n-k} = \binom{n}{k}$$

The Multinomial Formula



$$(X_1+X_2+\ldots+X_k)^n = \sum_{\substack{r_1,r_2,\ldots,r_k \\ \sum r_i = n}} \binom{n}{r_1;r_2;\ldots;r_k} X_1^{r_1}X_2^{r_2}X_3^{r_3}\ldots X_k^{r_k}$$

Power Series Representation

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k$$

Since $\binom{n}{k} = 0$ if $k > n$

"Closed form" or "Generating form"

"Power series" ("Taylor series") expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x=1$.

$$2^n = \underbrace{\sum_{k=0}^{n} \binom{n}{k}}$$

The number of subsets of an *n*-element set

By varying x, we can discover new identities

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x = -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

The number of even-sized subsets of an *n* element set is the same as the number of odd-sized subsets.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

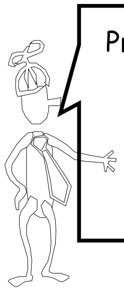
Let $x = -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

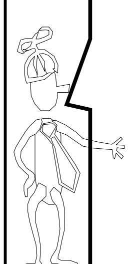
$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



Proofs that work by manipulating algebraic forms are called "algebraic" arguments. Proofs that build a 1-1 onto correspondence are called "combinatorial" arguments.

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$



Let O_n be the set of binary strings of length *n* with an odd number of ones.

Let E_n be the set of binary strings of length *n* with an even number of ones.

We gave an <u>algebraic</u> proof that

$|O_n| = |E_n|$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

A <u>combinatorial</u> proof must construct a one-to-one correspondence between O_n and E_n

An attempt at a correspondence

Let f_n be the function that takes an n-bit string and flips all its bits.

f_n is clearly a one-to-one and onto function

for odd n. *E.g.* in f_7 we have

0010011 à 1101100

1001101 à 0110010

...but do even n work? In f_6 we have

110011 à 001100

101010 à 010101

Uh oh. Complementing maps evens to evens!

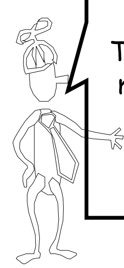
A correspondence that works for all *n*

Let f_n be the function that takes an n-bit string and flips only *the first bit*. For example,

0010011 à 1010011
1001101 à 0001101

110011 à 010011
101010 à 001010

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

The Binomial Formula

$(1+X)^0$ =	1
$(1+X)^1$ =	1 + 1X
$(1+X)^2$ =	1 + 2X + 1X^2
$(1+X)^3$ =	1 + 3X + 3X^2 + 1X^3
$(1+X)^4$ =	1 + 4X + 6X^2 + 4X^3 + 1X^4

Pascal's Triangle: k^{th} row are the coefficients of $(1+X)^k$

$(1+X)^0$ =	1
$(1+X)^1$ =	1 + 1X
$(1+X)^2$ =	1 + 2X + 1X^2
$(1+X)^3$ =	1 + 3X + 3X^2 + 1X^3
$(1+X)^4$ =	1 + 4X + 6X^2 + 4X^3 + 1X^4

k^{th} Row Of Pascal's Triangle:

$$\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{k},\ldots\binom{n}{n}$$

$(1+X)^0 =$	1
$(1+X)^1 =$	$1 + 1X$
$(1+X)^2 =$	$1 + 2X + 1X^2$
$(1+X)^3 =$	$1 + 3X + 3X^2 + 1X^3$
$(1+X)^4 =$	$1 + 4X + 6X^2 + 4X^3 + 1X^4$

Inductive definition of kth entry of nth row:
Pascal(n,0) = Pacal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

$(1+X)^0 =$	1
$(1+X)^1 =$	$1 + 1X$
$(1+X)^2 =$	$1 + 2X + 1X^2$
$(1+X)^3 =$	$1 + 3X + 3X^2 + 1X^3$
$(1+X)^4 =$	$1 + 4X + 6X^2 + 4X^3 + 1X^4$

"Pascal's Triangle"

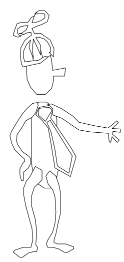
$$\binom{0}{0}=1$$
$$\binom{1}{0}=1 \quad \binom{1}{1}=1$$
$$\binom{2}{0}=1 \quad \binom{2}{1}=2 \quad \binom{2}{2}=1$$
$$\binom{3}{0}=1 \quad \binom{3}{1}=3 \quad \binom{3}{2}=3 \quad \binom{3}{3}=1$$

Al-Karaji, Baghdad 953-1029

Chu Shin-Chieh 1303
The Precious Mirror of the Four Elements
. . . Known in Europe by 1529

Blaise Pascal 1654

Pascal's Triangle



```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

Summing The Rows

$$2^n = \sum_{k=0}^{n}\binom{n}{k}$$

```
                 1                    =1
               1 + 1                  =2
             1 + 2 + 1                =4
           1 + 3 + 3 + 1              =8
         1 + 4 + 6 + 4 + 1            =16
      1 + 5 + 10 + 10 + 5 + 1         =32
1 + 6 + 15 + 20 + 15 + 6 + 1          =64
```

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

6+20+6 = 1+15+15+1

Summing on 1st Avenue

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5   10  10  5   1
1   6   15  20  15  6   1
```

(15 circled)

$$\sum_{i=k}^{n} i = \sum_{i=k}^{n}\binom{i}{1} = \binom{n+1}{2} = \frac{n\cdot(n+1)}{2}$$

Summing on kth Avenue

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5   10  10  5   1
1   6   15  20  15  6   1
```

(20 circled)

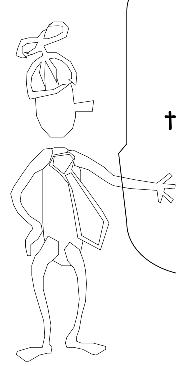
$$\sum_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1}$$

```
            1
          1   1      =2
                      =3
        1   2   1    =5
                      =8
      1   3   3   1  =13
    1   4   6   4   1
  1   5   10  10  5   1
1   6   15  20  15  6   1
```

```
            1
          1   1
        1²  2²  1²
      1²  3²  3²  1²
    1   4   6   4   1
  1   5   10  10  5   1
1   6   15  20  15  6   1
```

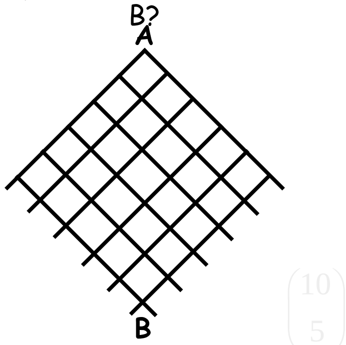
Al-Karaji Squares

```
                1                =0
            1       1            =1
        1     2 + 2*1            =4
    1     3 + 2*3     1          =9
  1     4 + 2*6    4     1       =16
1     5 + 2*10    10    5    1   =25
1    6 + 2*15    20    15    6    1=36
```

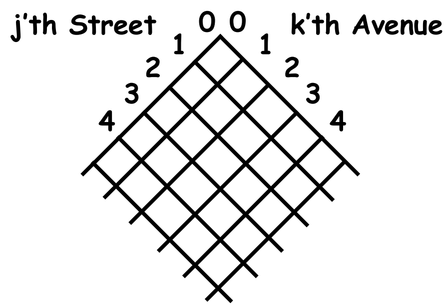


All these properties can be proved inductively and algebraically. We will give *combinatorial* proofs using the Manhattan block walking representation of binomial coefficients.

How many shortest routes from A to B?

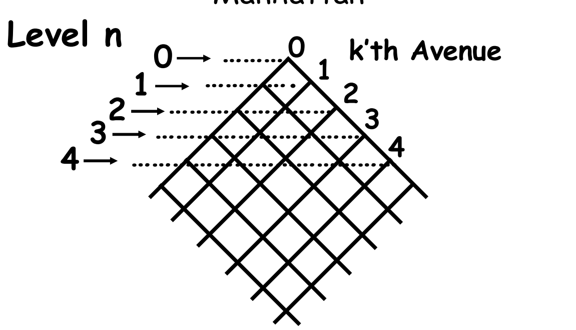


Manhattan



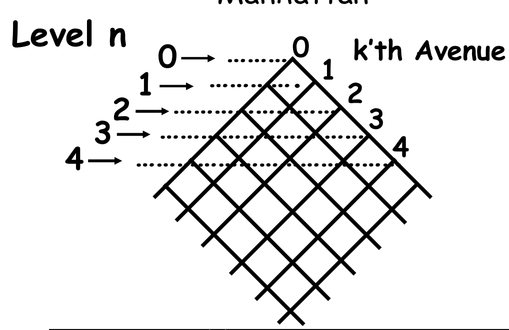
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k).

Manhattan

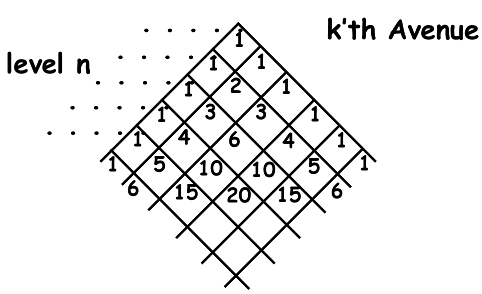


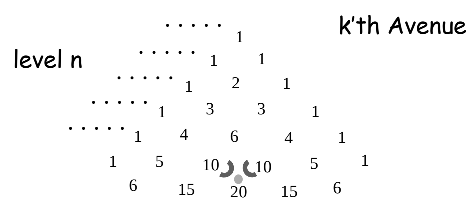
There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k).

Manhattan

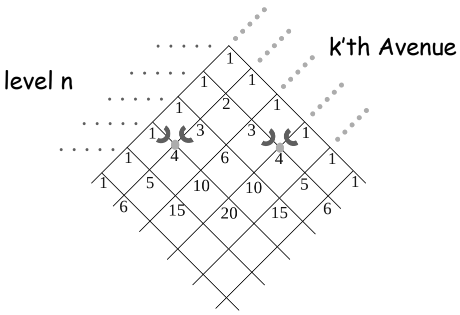


There are $\binom{n}{k}$ shortest routes from (0,0) to Level n and k^{th} Avenue.

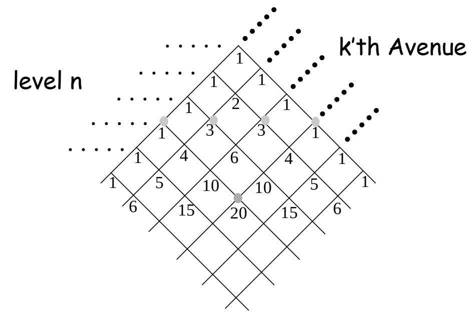




$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$$\sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}$$



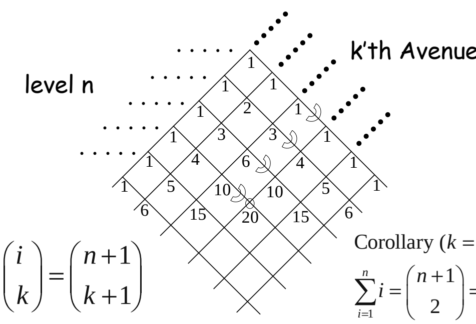
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

By convention:

$$0! = 1 \quad \text{(empty product = 1)}$$

$$\binom{n}{k} = 1 \quad \text{if } k = 0$$

$$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n$$



$$\sum_{i=1}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Corollary ($k = 1$)

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

Application (Al-Karaji):

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$
$$= (1\cdot 0 + 1) + (2\cdot 1 + 2) + (3\cdot 2 + 3) + \cdots + (n(n-1) + n)$$
$$= 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + n(n-1) + \sum_{i=1}^{n} i$$
$$= 2\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2}\right] + \binom{n+1}{2}$$
$$= 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}$$

Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable V→ can be thought of as:

< * , * , * , * , * , *, ... >

0 1 2 3 4 5 . . . .

Vector Programs

Let k stand for a scalar constant
<k> will stand for the vector <k,0,0,0,…>

<0> = <0,0,0,0,….>
<1> = <1,0,0,0,…>

V→ + T→ means to add the vectors position-wise.

<4,2,3,…> + <5,1,1,….> = <9,3,4,…>

Vector Programs

RIGHT(V→) means to shift every number in V→ one position to the right and to place a 0 in position 0.

RIGHT(<1,2,3, …>) = <0,1,2,3,. …>

Vector Programs

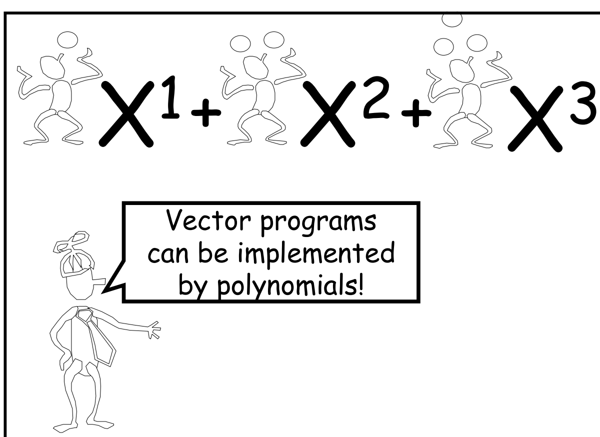
Example:	Store
V→ := <6>;	V→ = <6,0,0,0,..>
V→ := RIGHT(V→) + <42>;	V→ = <42,6,0,0,..>
V→ := RIGHT(V→) + <2>;	V→ = <2,42,6,0,..>
V→ := RIGHT(V→) + <13>;	V→= <13,2,42,6,.>

V→ = < 13, 2, 42, 6, 0, 0, 0, . . . >

Vector Programs

Example:	Store
V→ := <1>;	V→ = <1,0,0,0,..>
Loop n times:	V→ = <1,1,0,0,..>
V→ := V→ + RIGHT(V→);	V→ = <1,2,1,0,..>
	V→ = <1,3,3,1,.>

V→ = n^{th} row of Pascal's triangle.



Programs -----> Polynomials

The vector V→ = < a_0, a_1, a_2, . . . > will be represented by the polynomial:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

Formal Power Series

The vector $\vec{V} = \langle a_0, a_1, a_2, \ldots \rangle$ will be represented by the formal power series:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

$\vec{V} = \langle a_0, a_1, a_2, \ldots \rangle$

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

<0> is represented by	0
<k> is represented by	k
$\vec{V} + \vec{T}$ is represented by	$(P_V + P_T)$
RIGHT($\vec{V}$) is represented by	$(P_V X)$

Vector Programs

Example:

$\vec{V}$:= <1>; $P_V := 1;$

Loop n times:
$\vec{V}$:= $\vec{V}$ + RIGHT($\vec{V}$); $P_V := P_V + P_V X;$

$\vec{V}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

$\vec{V}$:= <1>; $P_V := 1;$

Loop n times:
$\vec{V}$:= $\vec{V}$ + RIGHT($\vec{V}$); $P_V := P_V (1+ X);$

$\vec{V}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

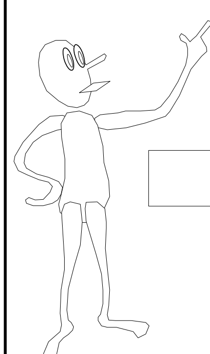
$\vec{V}$:= <1>;

Loop n times:
$\vec{V}$:= $\vec{V}$ + RIGHT($\vec{V}$);

} $P_V = (1+ X)^n$

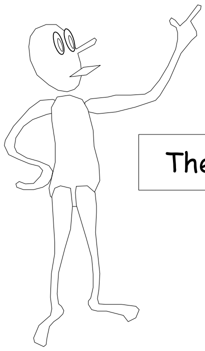
$\vec{V}$ = n^{th} row of Pascal's triangle.

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$



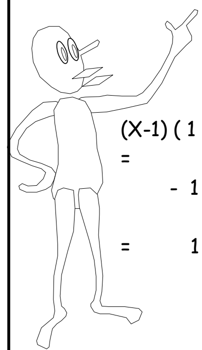
The Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



The Infinite Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



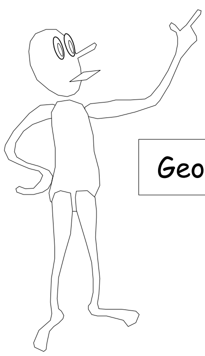
$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots)$

$= \quad X^1 + X^2 + X^3 + \ldots \quad + X^n + X^{n+1} + \ldots.$

$\quad - \; 1 - X^1 - X^2 - X^3 - \ldots \; - X^{n-1} - X^n - X^{n+1} - \ldots$

$= \quad 1$

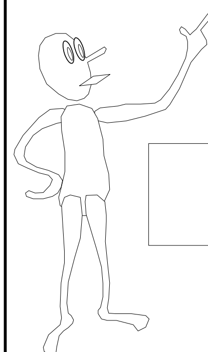
$$1 + aX^1 + a^2X^2 + a^3X^3 + \ldots + a^nX^n + \ldots\ldots = \frac{1}{1 - aX}$$



Geometric Series (Linear Form)

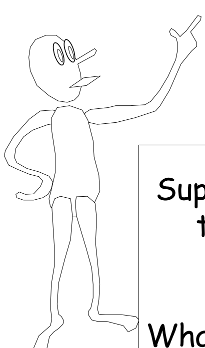
$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots) =$$

$$\frac{1}{(1 - aX)(1-bX)}$$



Geometric Series
(Quadratic Form)

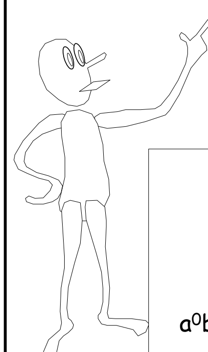
$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots) =$$
$$1 + c_1X^1 + \ldots + c_k X^k + \ldots$$



Suppose we multiply this out to get a single, infinite polynomial.

What is an expression for C_n?

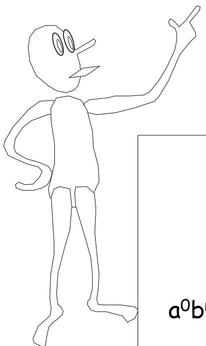
$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots) =$$
$$1 + c_1X^1 + \ldots + c_k X^k + \ldots$$



$c_n =$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)\ (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$
$1 + c_1X^1 + .. + c_k X^k + \ldots$

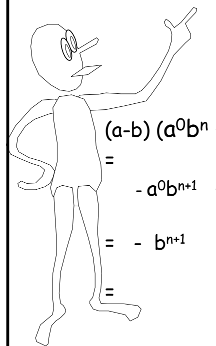


If a = b then

$c_n = (n+1)(a^n)$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0 = \dfrac{a^{n+1} - b^{n+1}}{a - b}$

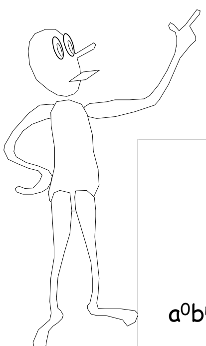


$(a-b)\ (a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0)$
$= \quad a^1b^n + \ldots a^{i+1}b^{n-i} \ldots \quad + a^nb^1 + a^{n+1}b^0$
$\quad - a^0b^{n+1} - a^1b^n \ldots \quad a^{i+1}b^{n-i} \ldots \quad - a^{n-1}b^2 - a^nb^1$

$= \ - b^{n+1} \quad + \quad a^{n+1}$

$= \quad a^{n+1} - b^{n+1}$

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)\ (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$
$1 + c_1X^1 + .. + c_k X^k + \ldots$

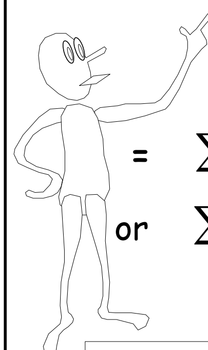


if $a \neq b$ then

$c_n = \dfrac{a^{n+1} - b^{n+1}}{a - b}$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)\ (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$



$= \dfrac{1}{(1 - aX)(1-bX)}$

$= \sum_{n=0..\infty} \dfrac{a^{n+1} - b^{n+1}}{a - b} X^n$

or $\sum_{n=0..\infty} (n+1)a^n\ X^n$ when a=b

Geometric Series (Quadratic Form)

Study Bee

- Polynomials count
- Binomial formula
- Multinomial coefficients
- Combinatorial proofs of binomial identities
- Vector programs
- Geometric series