Great Theoretical Ideas In Computer Science

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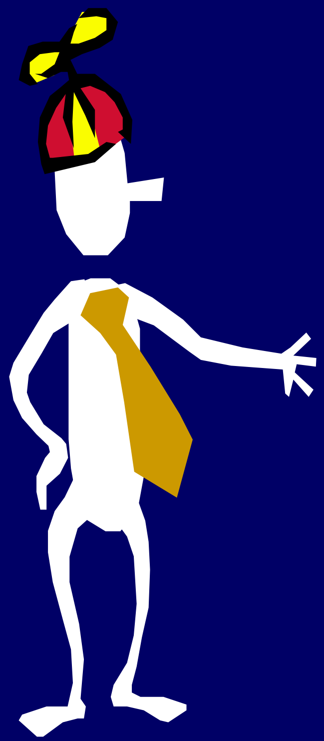
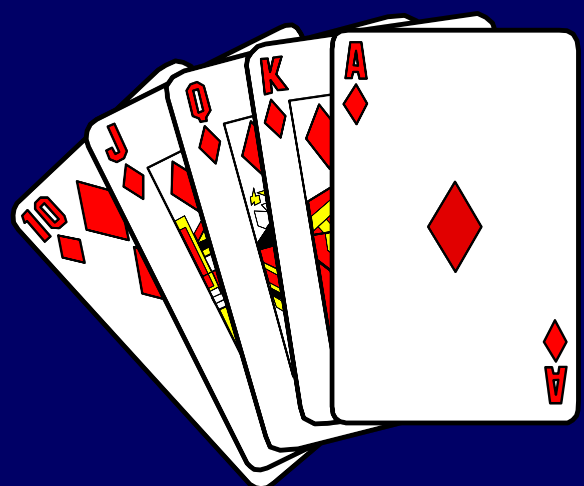
CS 15-251 Fall 2005

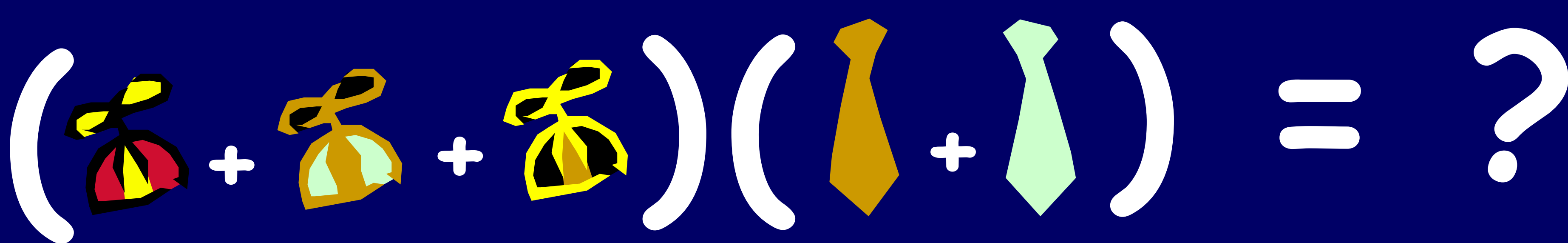
Lecture 7 Sept 20, 2005

Carnegie Mellon University

Counting II: Recurring Problems And Correspondences



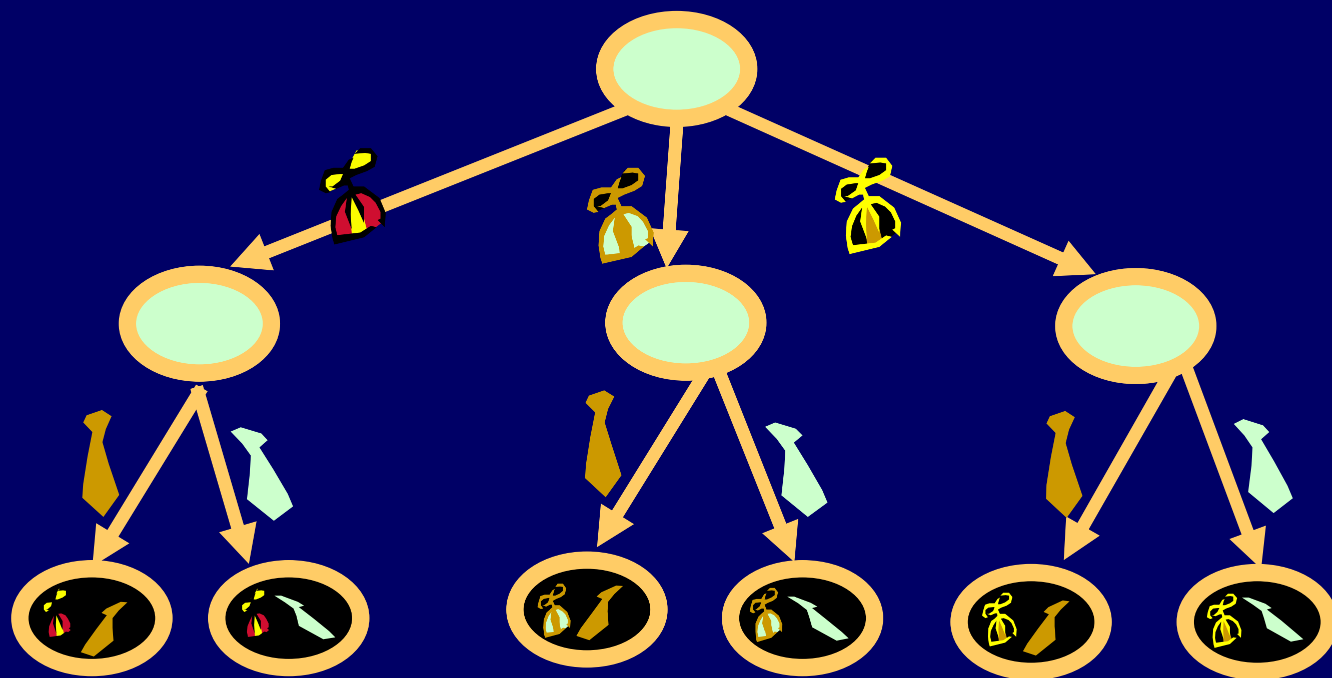




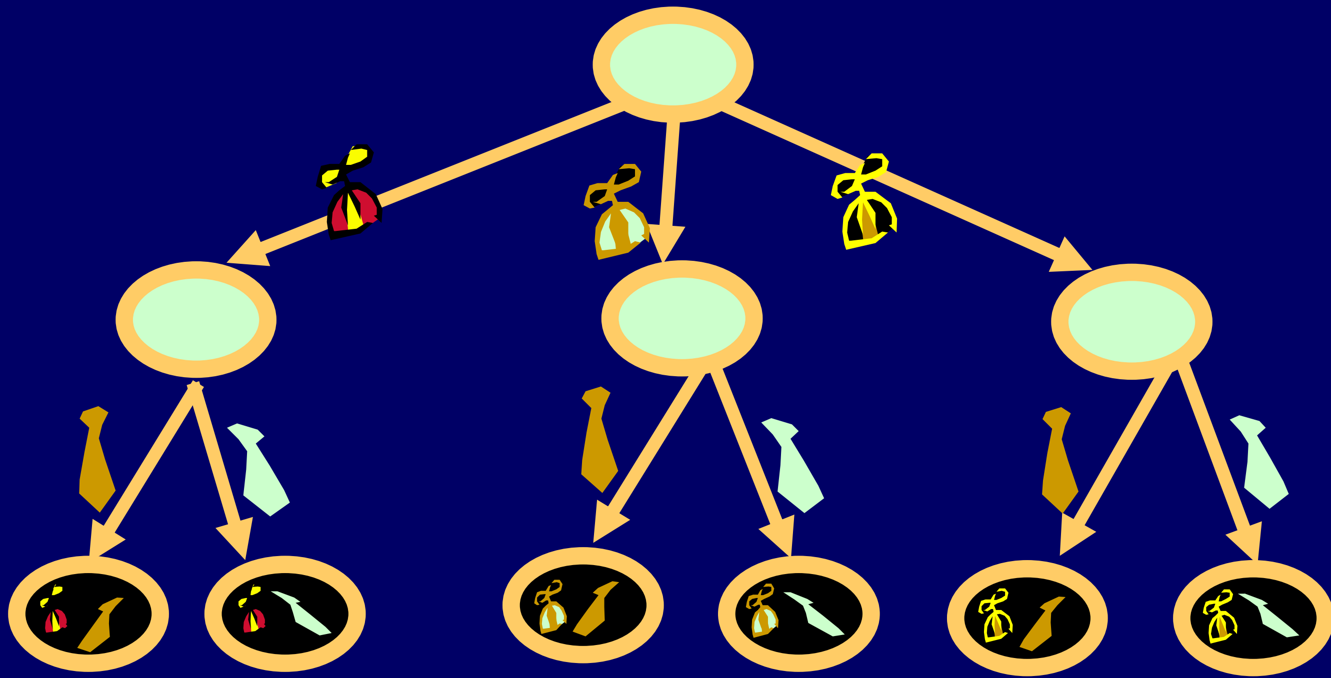
Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

Choice Tree



A choice tree is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.



A choice tree provides a “choice tree representation” of a set S, if

1) Each leaf label is in S, and every element of S is in some leaf
2) No two leaf labels are the same

Product Rule

IF S has a choice tree representation with P_1 possibilities for the first choice, P_2 for the second, and so on,

THEN

there are $P_1P_2P_3 \dots P_n$ objects in S

Proof: The leaves of the choice tree are in 1-1 onto correspondence with the elements of S.

Product Rule

Suppose that all objects of a type S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF

1) Each sequence of choices constructs an object of type S

AND

2) No two different sequences create the same object

THEN

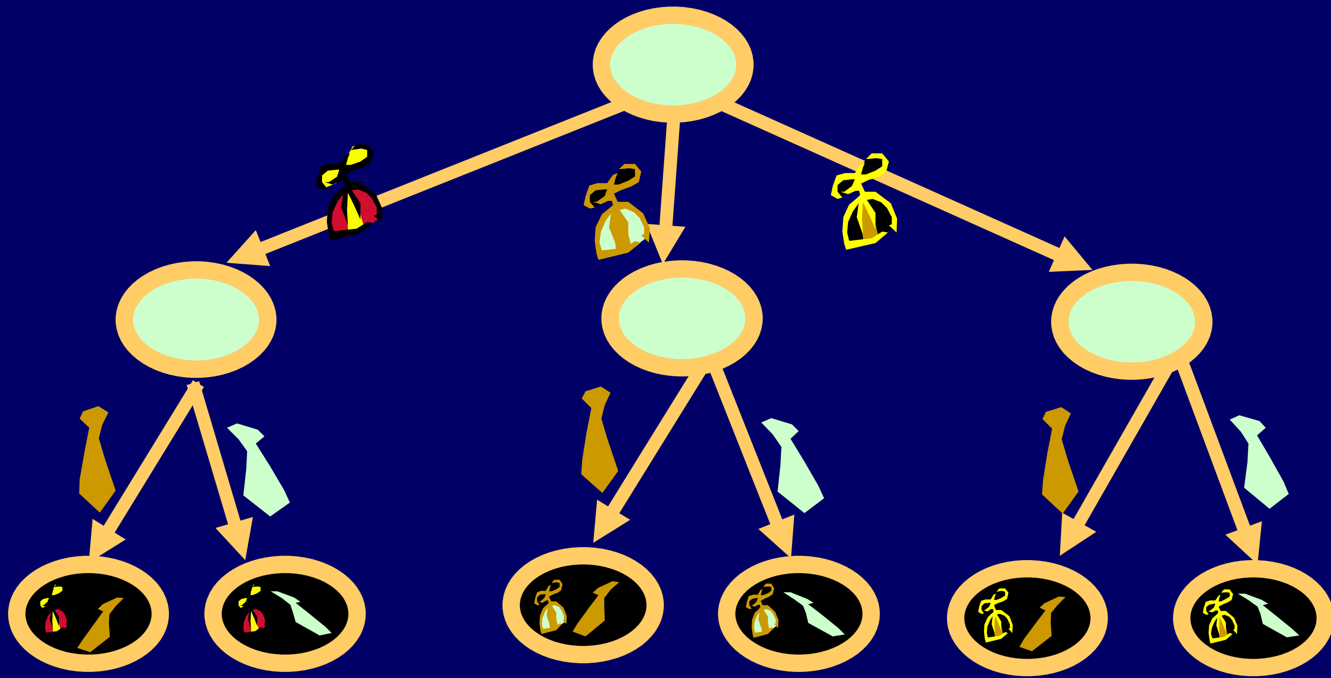
there are $P_1 P_2 P_3 \ldots P_n$ objects of type S.

Condition 2 of the product rule:

No two leaves have the same label.

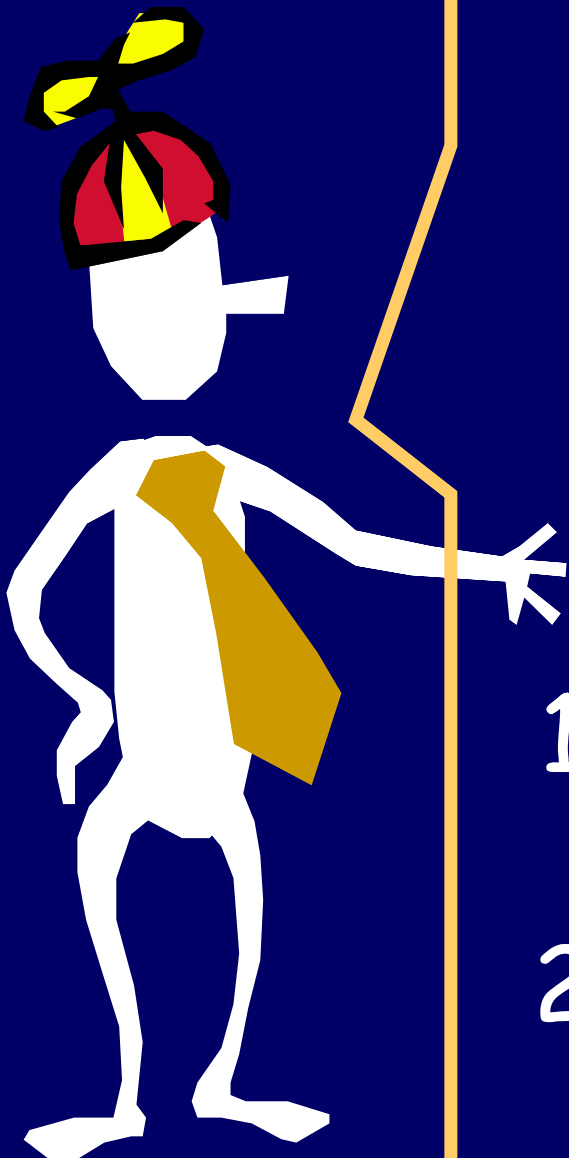
Equivalently,

No object can be created in two different ways.



Reversibility Check:

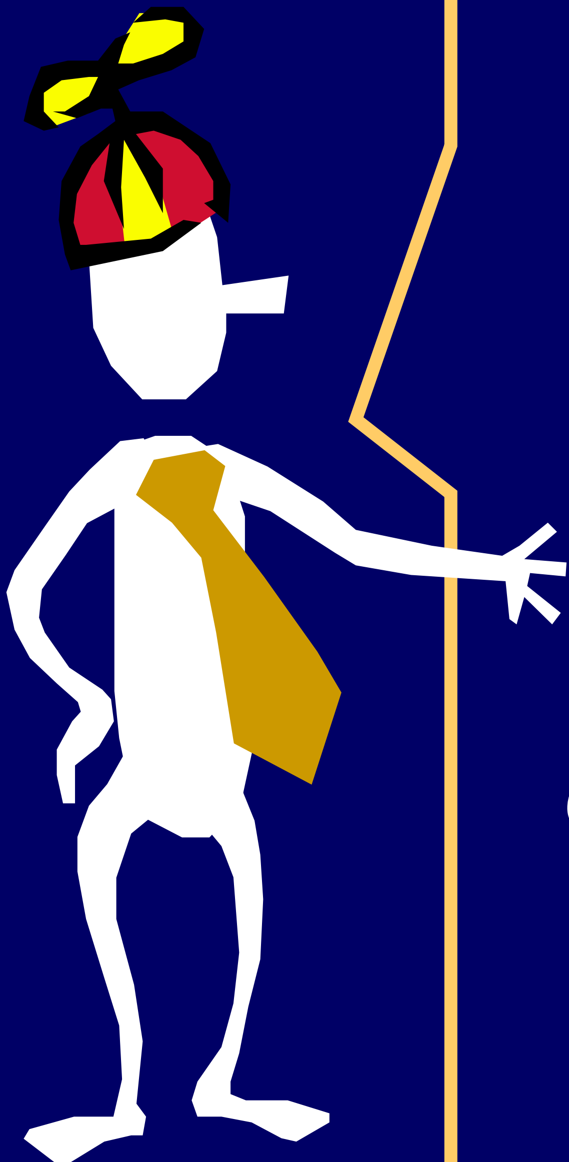
Given an arbitrary object in S, can we reverse engineer the choices that created it?



The two big mistakes people make in associating a choice tree with a set S are:

1) Creating objects not in S

2) Creating the same object two different ways



DEFENSIVE THINKING

Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?

The number of subsets of an n-element set is 2^n

The number of permutations of n distinct objects is

n!

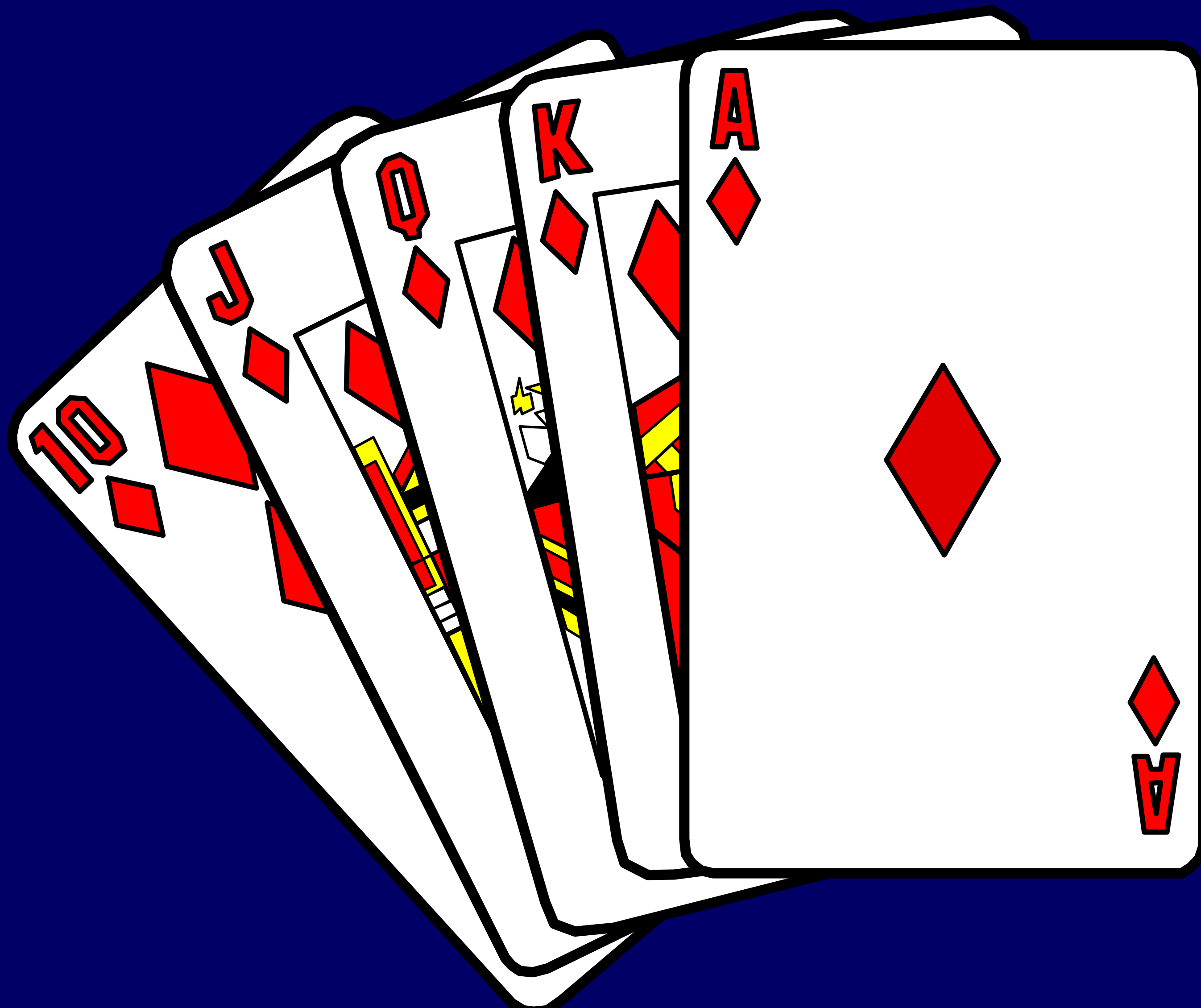
The number of subsets of size r that can be formed from an n-element set is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Sometimes it is easiest to count something by counting its opposite.

Let's use our principles to extend our reasoning to different types of objects.

Counting Poker Hands…



52 Card Deck
5 card hands

4 possible suits:

- ♥ ♦ ♣ ♠

13 possible ranks:

- 2,3,4,5,6,7,8,9,10,J,Q,K,A

Pair: set of two cards of the same rank

Straight: 5 cards of consecutive rank

Flush: set of 5 cards with the same suit

Ranked Poker Hands

Straight Flush
- A straight and a flush

4 of a kind
- 4 cards of the same rank

Full House
- 3 of one kind and 2 of another

Flush
- A flush, *but not a straight*

Straight
- A straight, *but not a flush*

3 of a kind
- 3 of the same rank, *but not a full house or 4 of a kind*

2 Pair
- 2 pairs, *but not 4 of a kind or a full house*

A Pair

Straight Flush

9 choices for rank of lowest card at the start of the straight.

4 possible suits for the flush.

$9 \times 4 = 36$

$$\frac{36}{\binom{52}{5}} = \frac{36}{2598960} = 1 \text{ in } 72{,}193.33..$$

4 Of A Kind

13 choices of rank.

48 choices for remaining card.

$$13 \times 48 = 624$$

$$\frac{624}{2598960} = 1 \text{ in } 4165$$

Flush

4 choices of suit.

$\binom{13}{5}$ choices of set of 5 ranks.

$= 5148$

$-$ 36 Straight Flushes

$= 5112$

$$\frac{5112}{2598960} = 1 \text{ in } 508.4$$

Straight

9 choices of lowest rank in the straight.

4^5 choices of suits to each card in sequence.

=9216

- 36 Straight Flushes

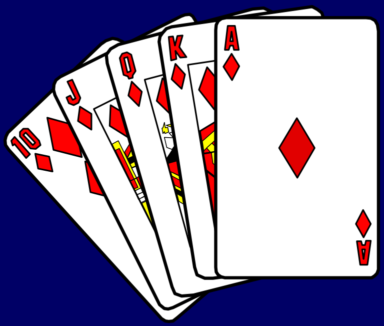
= 9180

$$\frac{9180}{2598960} = 1 \text{ in } 283.11$$

Number of Hands

How many hands does each player need to evaluate in a game of Texas Hold 'em, where there are two cards down and five cards up?

$$\binom{7}{5} = 21$$



Storing Poker Hands
How many bits per hand?

I want to store a 5 card poker hand using the smallest number of bits (space efficient).

2 bits suit
4 bits rank } 6 bits / card

x 5 cards = 30 bits / hand


How can we store a poker hand without storing its order?

Order the 2,598,560 Poker hands lexicographically [or in any fixed manner]

To store a hand all I need is to store its index of size $\lceil \log_2(2{,}598{,}560) \rceil = 22$ bits.

Hand 0000000000000000000000
Hand 0000000000000000000001
Hand 0000000000000000000010

.

.

.

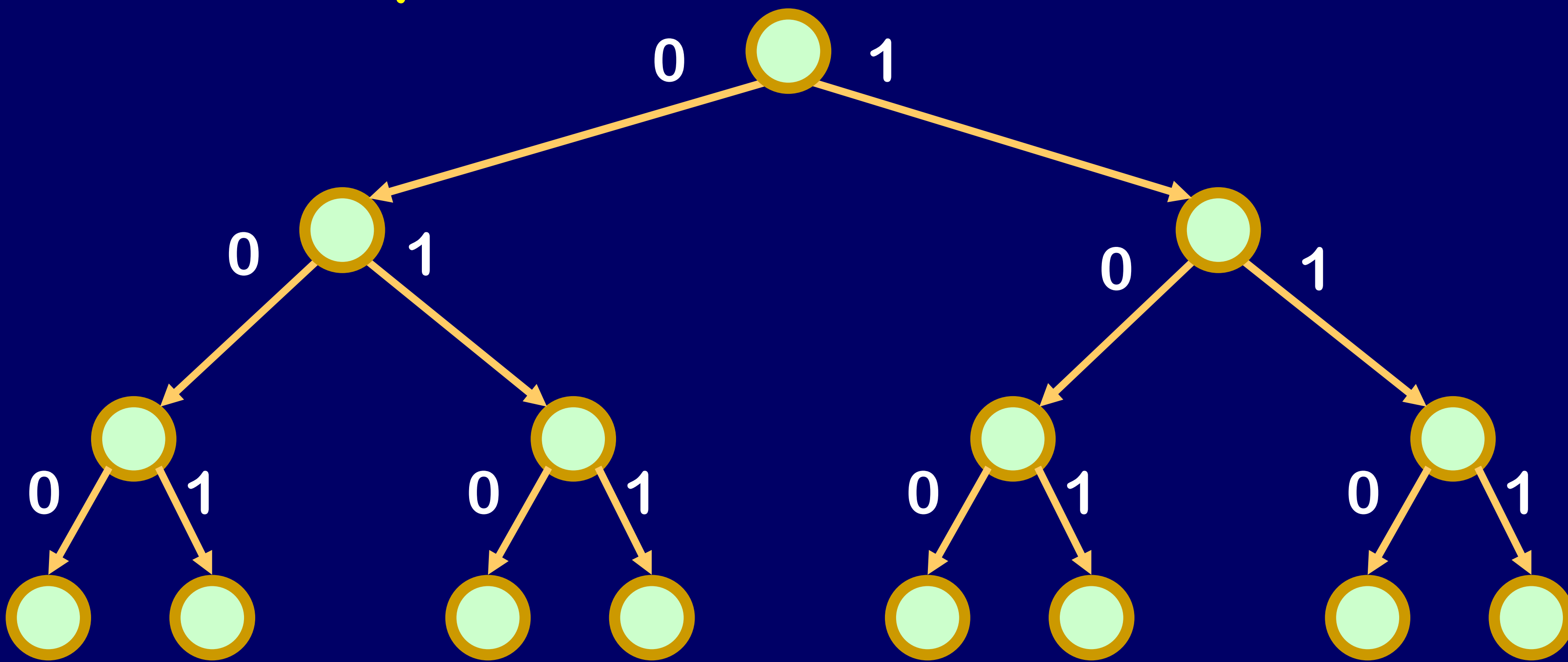
22 Bits Is OPTIMAL

$2^{21} = 2097152 < 2{,}598{,}560$

Thus there are more poker hands than there are 21-bit strings.

Hence, you can't have a 21-bit string for each hand.

Binary (Boolean) Choice Tree



A binary (Boolean) choice tree is a choice tree where each internal node has degree 2. Usually the choices will be labeled 0 and 1.

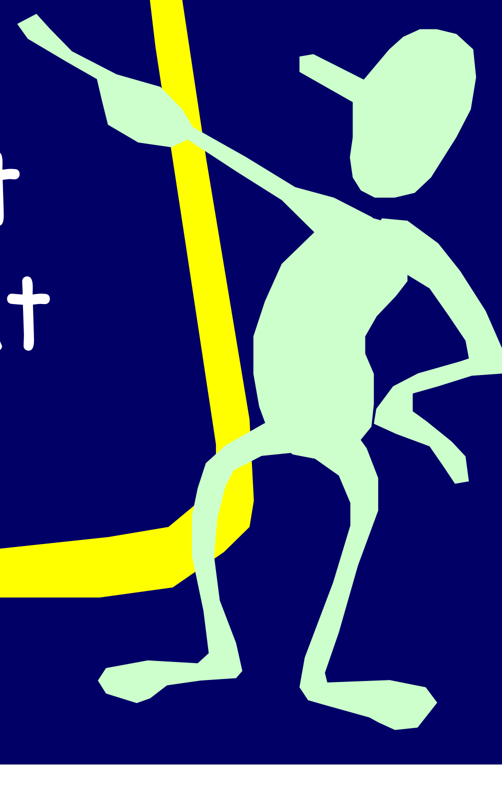
22 Bits Is OPTIMAL

$2^{21} = 2097152 < 2{,}598{,}560$

A binary choice tree of depth 21 can have at most 2^{21} leaves. Hence, there are not enough leaves for all 5-card hands.

An n-element set can be stored so that each element uses $\lceil \log_2(n) \rceil$ bits.

Furthermore, any representation of the set will have some string of at least that length.

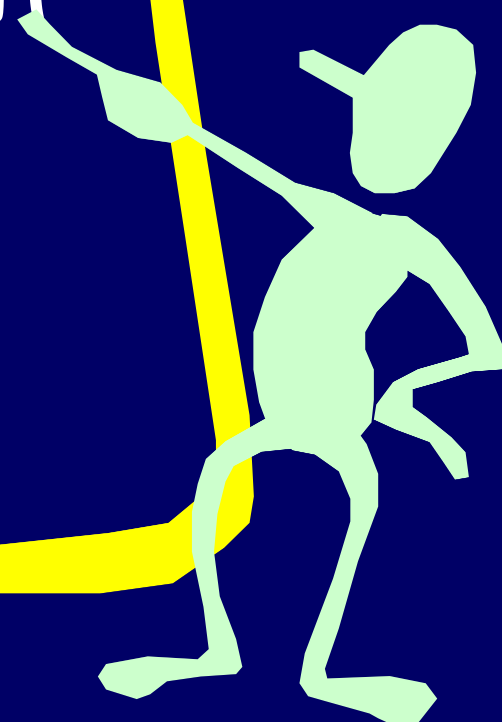


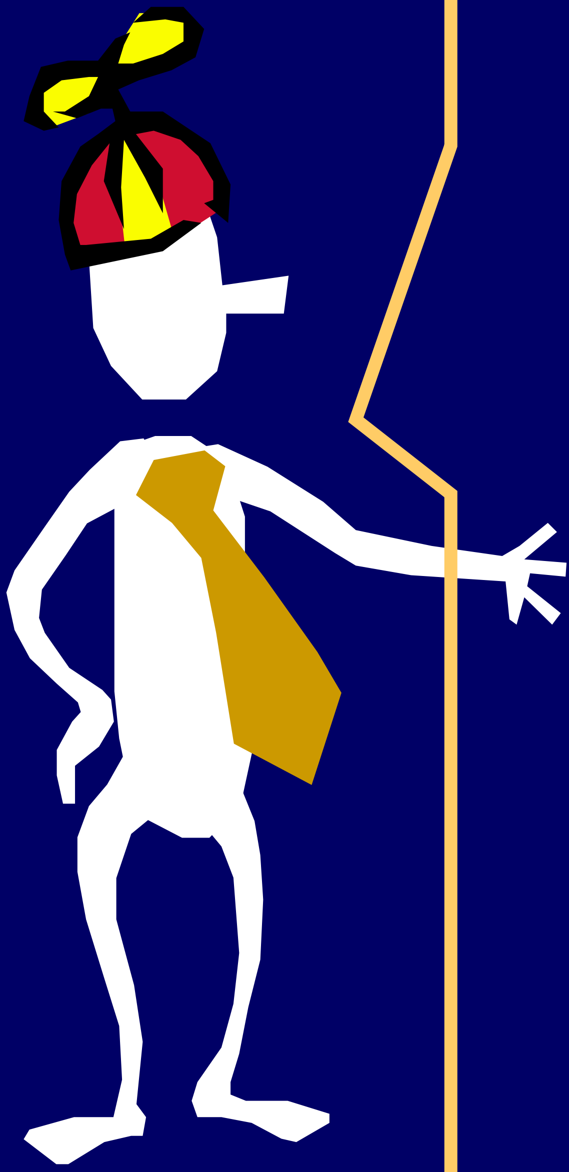
Information Counting Principle:

If each element of a set can be represented using k bits, the size of the set is bounded by 2^k

Information Counting Principle:

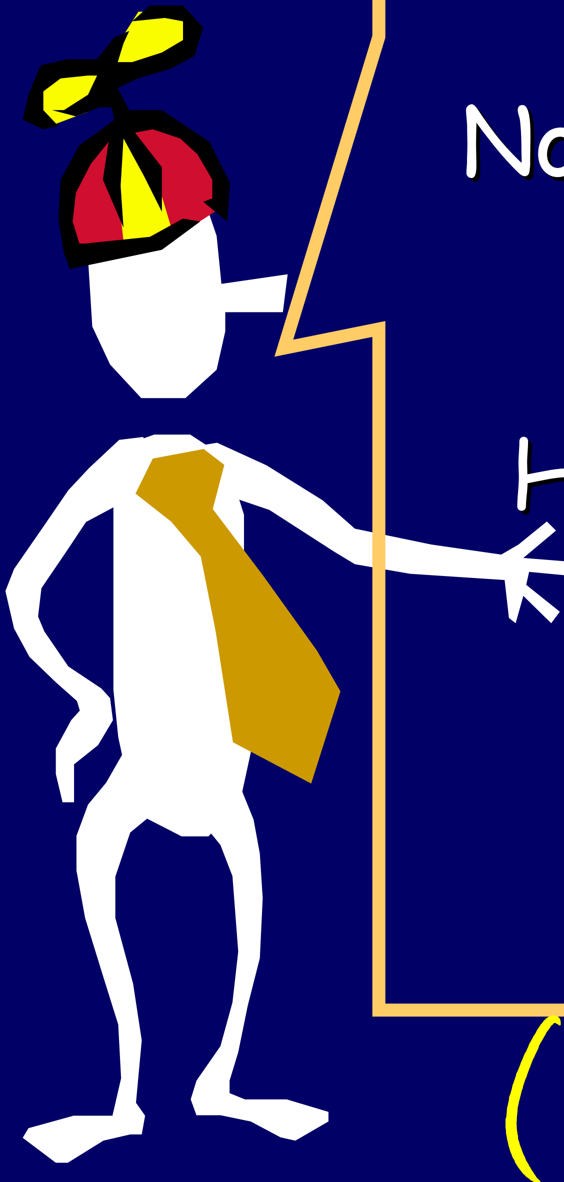
Let S be a set represented by a depth k binary choice tree, the size of the set is bounded by 2^k





ONGOING MEDITATION:

Let S be any set and T be a binary choice tree representation of S. We can think of each element of S being encoded by the binary sequences of choices that lead to its leaf. We can also start with a binary encoding of a set and make a corresponding binary choice tree.



Now, for something completely different…

How many ways to rearrange the letters in the word **"SYSTEMS"**?

$\binom{7}{3} 4!$

SYSTEMS

1) 7 places to put the Y, 6 places to put the T, 5 places to put the E, 4 places to put the M, and the S's are forced.

   $7 \times 6 \times 5 \times 4 = 840$

SYSTEMS

2) Let's pretend that the S's are distinct:

$S_1 Y S_2 TEM S_3$

There are 7! permutations of $S_1 Y S_2 TEM S_3$

But when we stop pretending we see that we have counted each arrangement of SYSTEMS 3! times, once for each of 3! rearrangements of $S_1 S_2 S_3$.

$$\frac{7!}{3!} = 840$$

Arrange n symbols
r_1 of type 1, r_2 of type 2, …, r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{r_k}{r_k}$$

$r_k = n - r_1 - \ldots - r_{k-1}$

$$= \frac{n!}{r_1!(n-r_1)!}\frac{(n-r_1)!}{r_2!(n-r_1-r_2)!}\frac{(n-r_1-r_2)!}{r_3!(n-r_1-r_2-r_3)!}\cdots 1$$

$$= \frac{n!}{r_1!r_2!r_3!\ldots r_k!}$$

CARNEGIEMELLON

14 letters

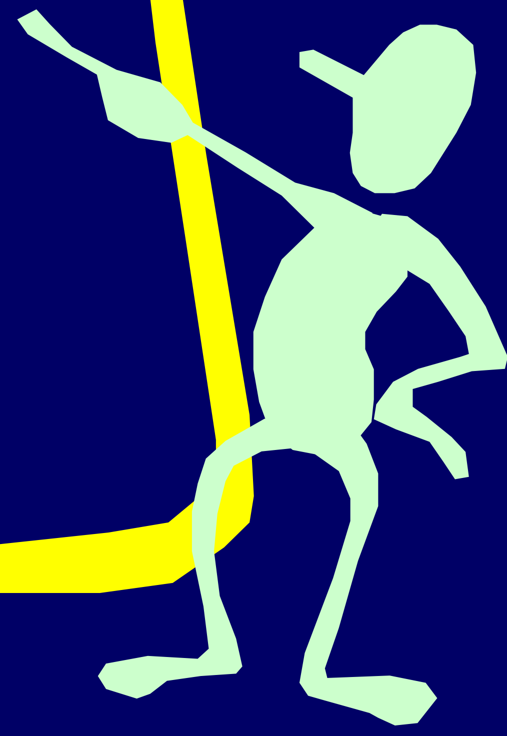
2 L
3 E
2 N

$$\frac{14!}{2!3!2!} = 3,632,428,800$$

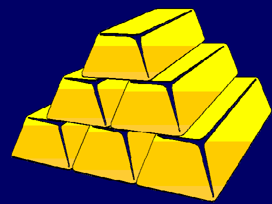
Remember:

The number of ways to arrange n symbols with r_1 of type 1, r_2 of type 2, …, r_k of type k is:

$$\binom{n}{r_1; r_2; \ldots; r_k} = \frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}$$



5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?



Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/

represents the following division among the pirates: 2, 1, 0, 17, 0

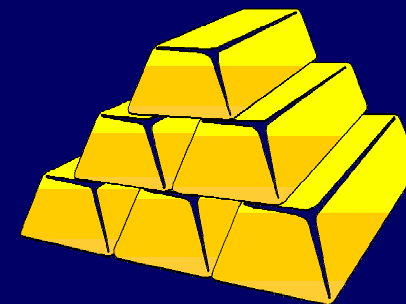
In general, the i^{th} pirate gets the number of G's after the i-1^{st} / and before the i^{th} /.

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

How many different ways to divide up the loot?

Sequences with 20 G's and 4 /'s

$$\binom{24}{4}$$



n+K−1

How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

d partitions

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1 \ , x_2 , x_3 , x_4 , x_5 \geq 0$$

Think of X_k as being the number of gold bars that are allotted to pirate k.

$$\binom{24}{4}$$

How many integer solutions to the following equations?

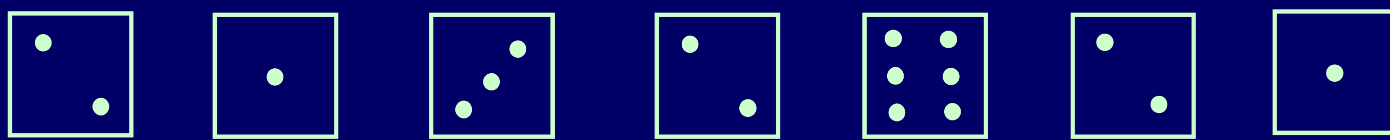
$$x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_{n-1}, x_n \geq 0$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Identical/Distinct Dice

Suppose that we roll seven dice.



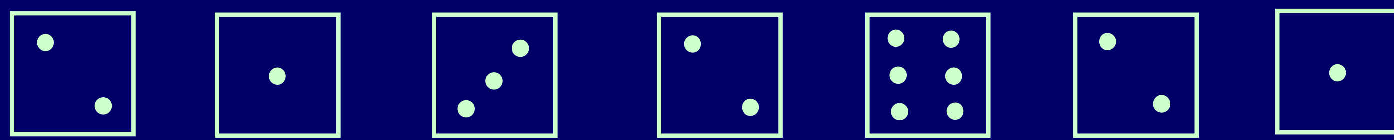
How many different outcomes are there, if order matters?

$$6^7$$

What if order doesn't matter?
(E.g., Yahtzee)

$$\binom{6-1+7}{6-1} = \binom{12}{7}$$

7 Identical Dice



How many different outcomes?

Corresponds to 6 pirates
and 7 bars of gold!

Let X_k be the number of dice showing k.
The k^{th} pirate gets X_k gold bars.

$$\binom{6+7-1}{7}$$

Multisets

" Mathspeak "

A <u>multiset</u> is a set of elements, each of which has a *multiplicity*.

The <u>size</u> of the multiset is the sum of the multiplicities of all the elements.

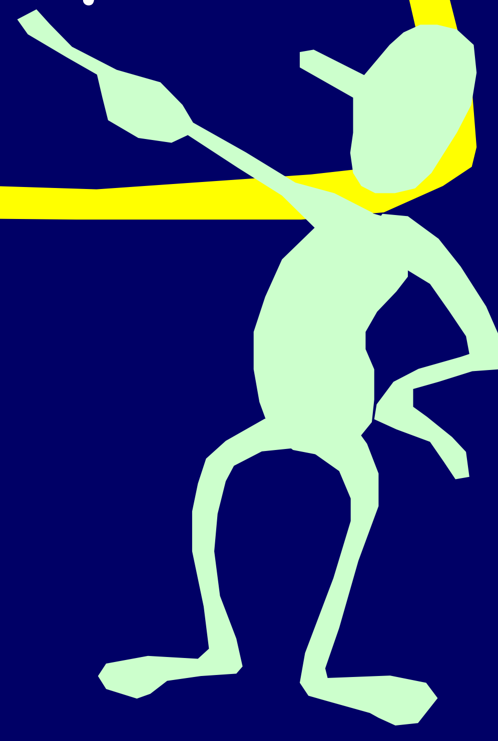
Example:

{X, Y, Z} with m(X)=0 m(Y)=3, m(Z)=2

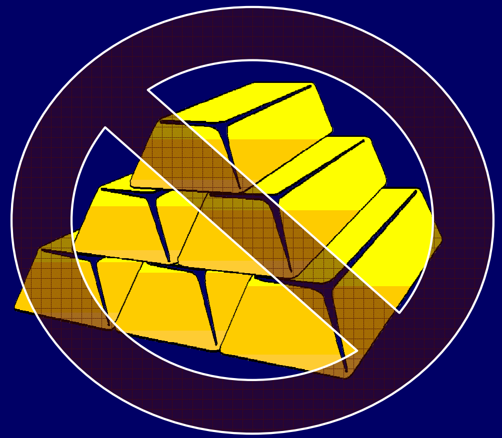
Unary visualization: {Y, Y, Y, Z, Z}

Counting Multisets

There are $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ ways to choose a multiset of size k from n types of elements



Back to the pirates



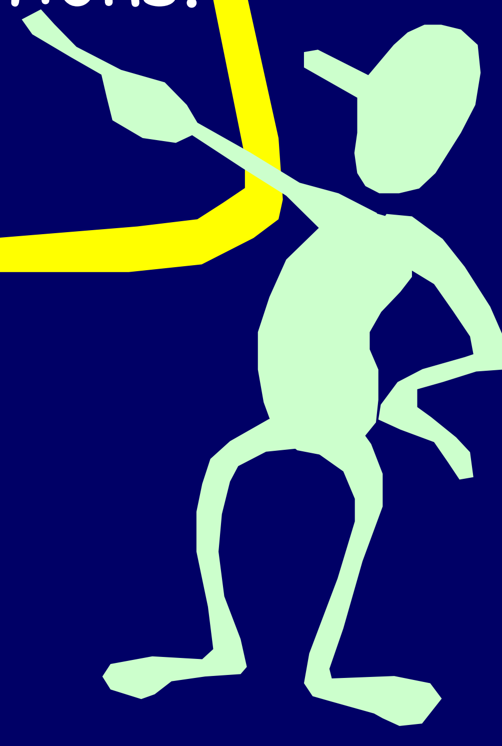
How many ways are there of choosing 20 pirates from a set of 5 pirates, with repetitions allowed?

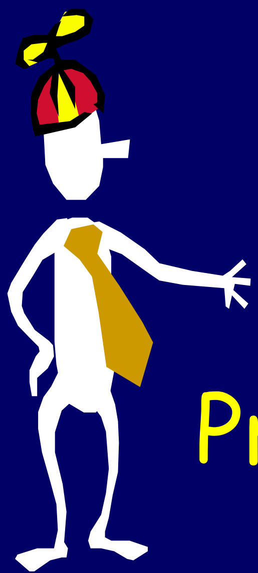
$$\binom{5+20-1}{20} = \binom{24}{20} = \binom{24}{4}$$

$$x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n = k$$

$$x_1\ , x_2, x_3, \ldots, x_{n-1}, x_n \geq 0$$

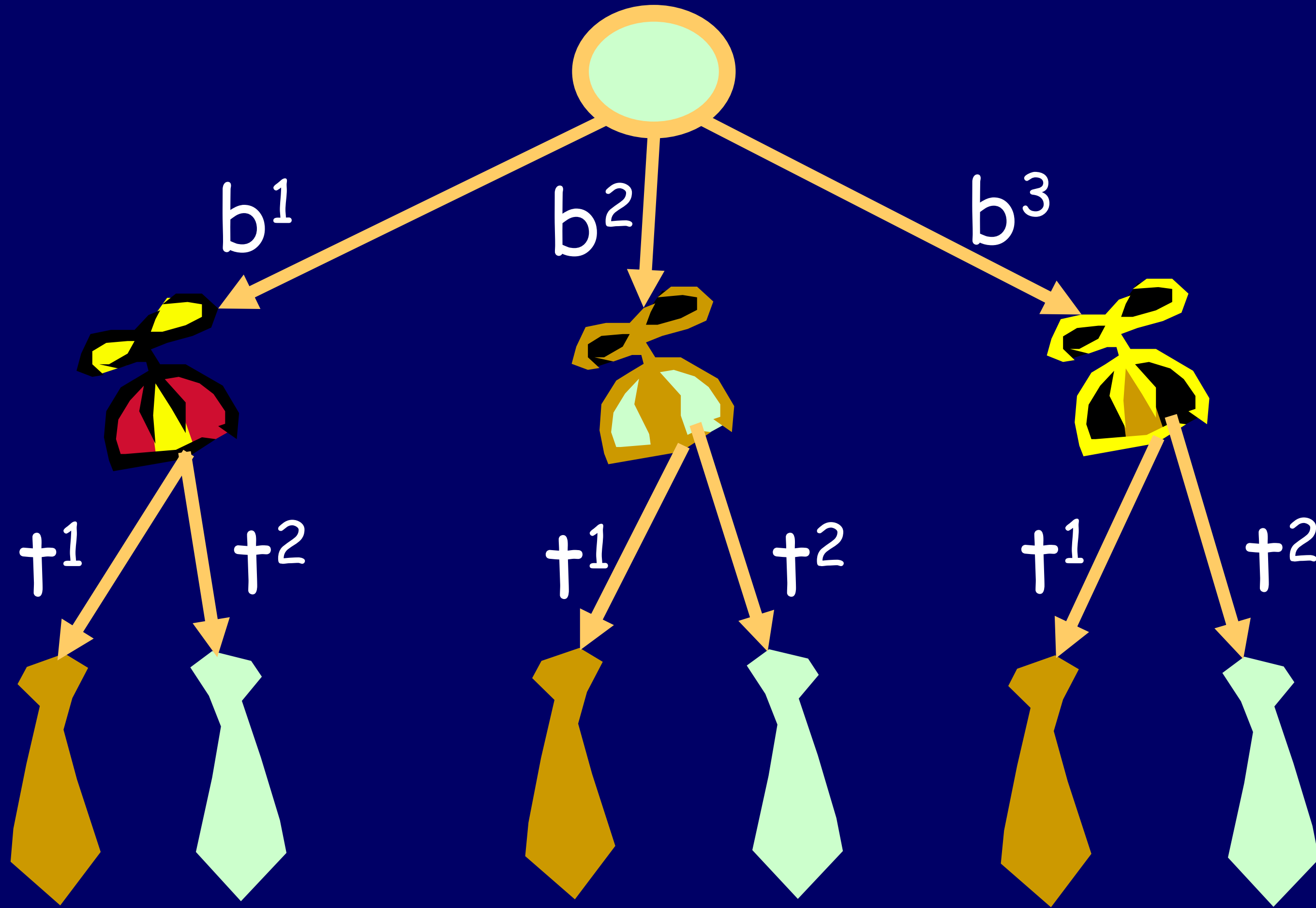
has $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ integer solutions.



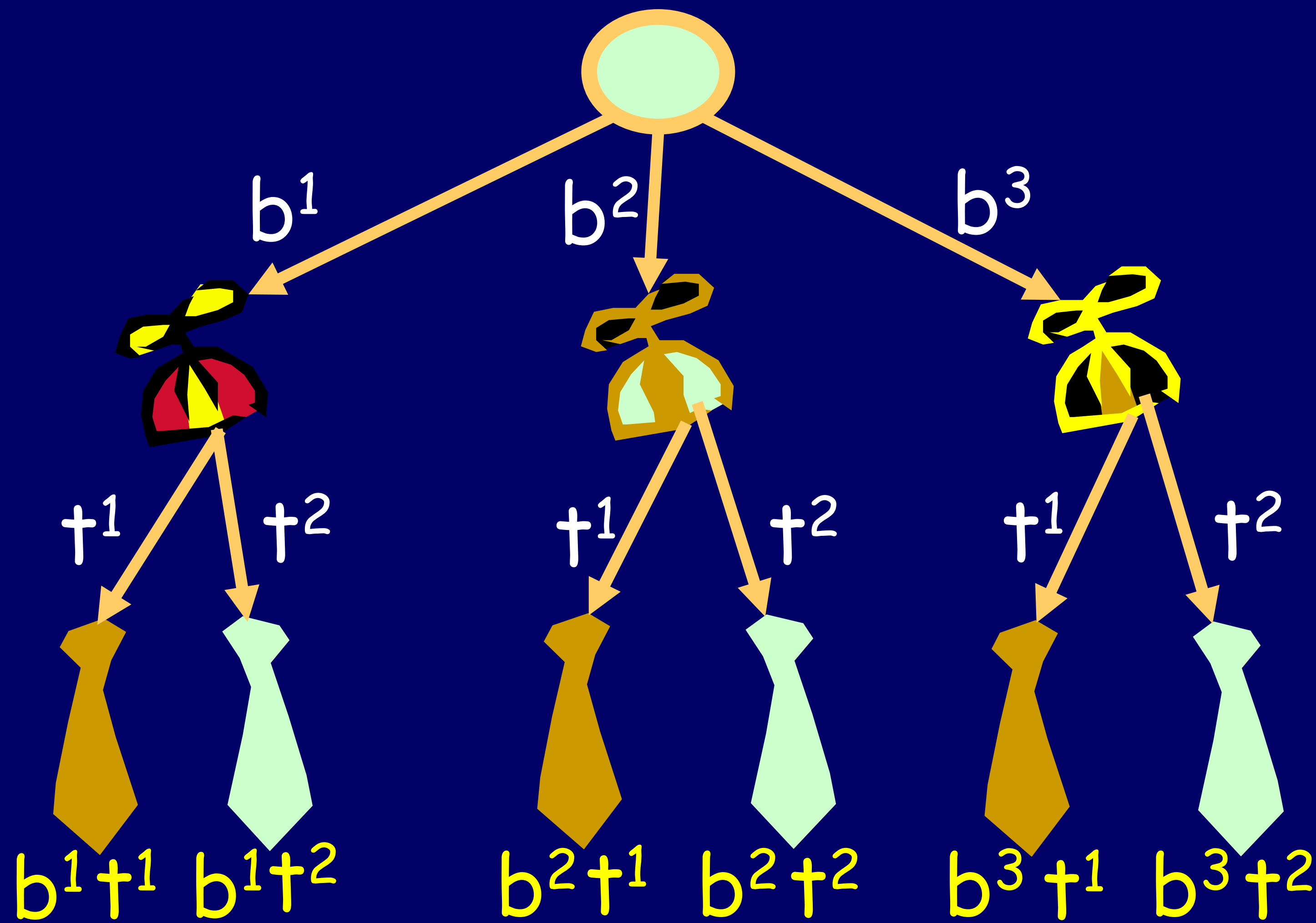


POLYNOMIALS EXPRESS CHOICES AND OUTCOMES

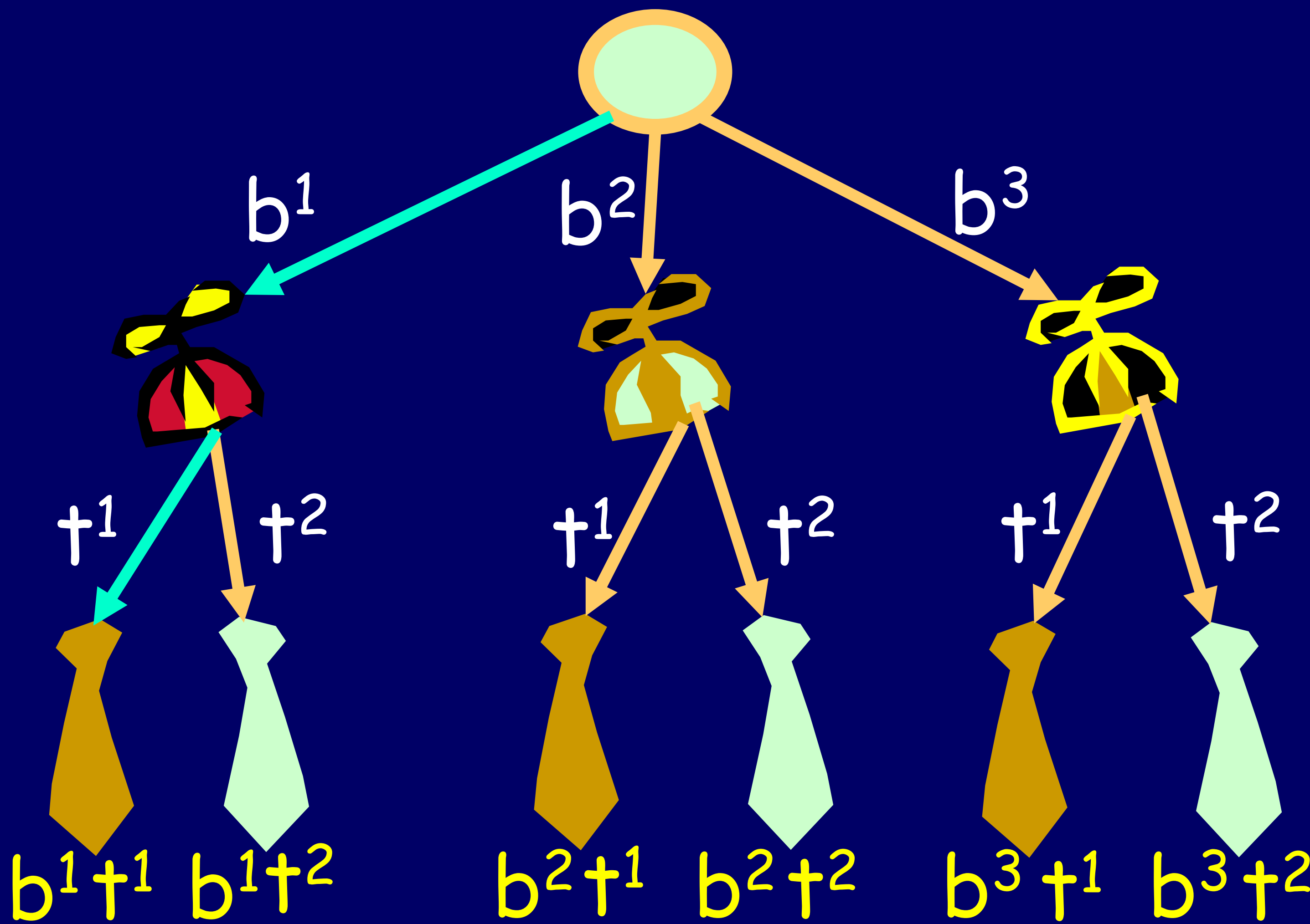
Products of Sum = Sums of Products



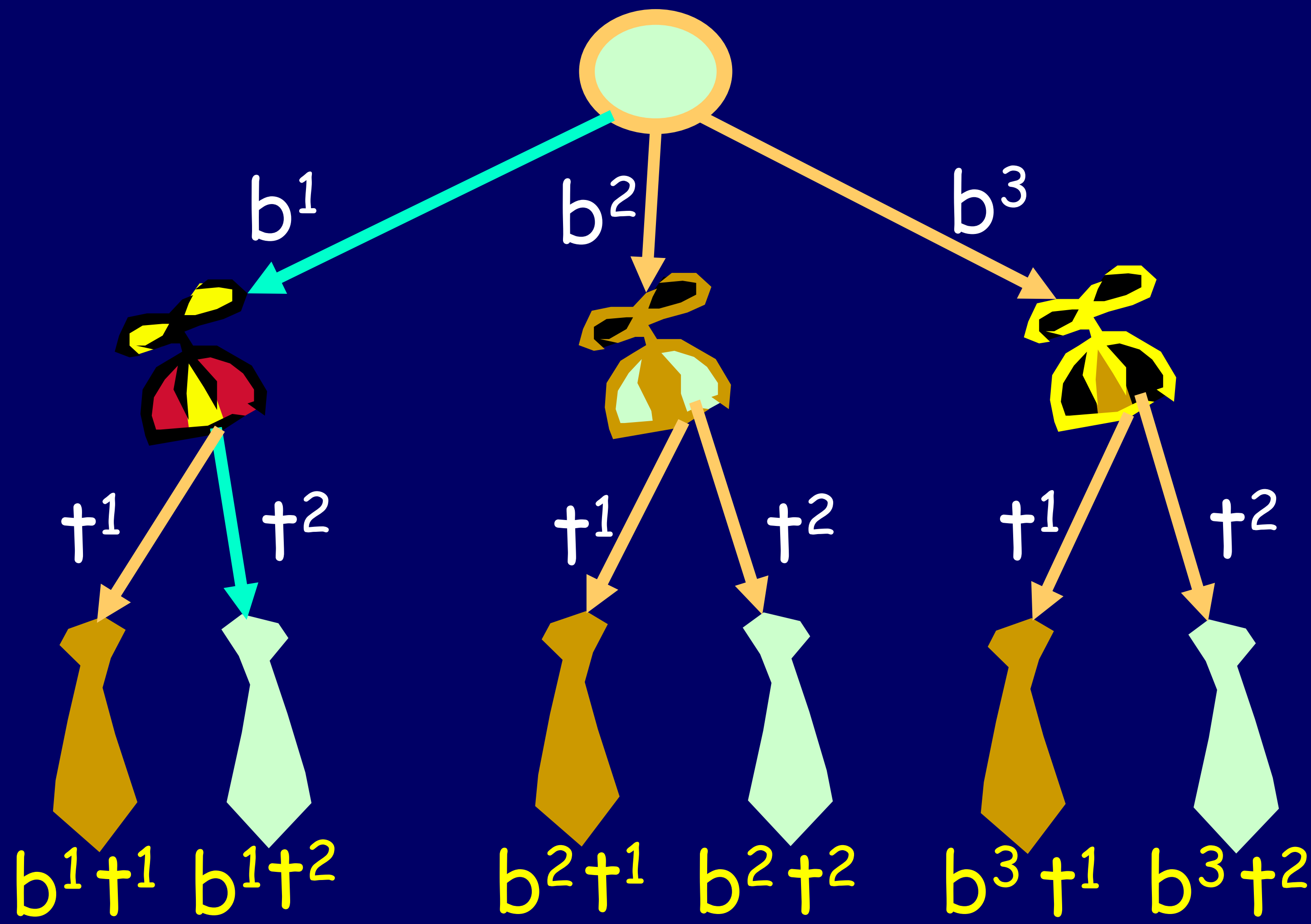
b1
b2
b3
t1
t2
t1
t2
t1
t2



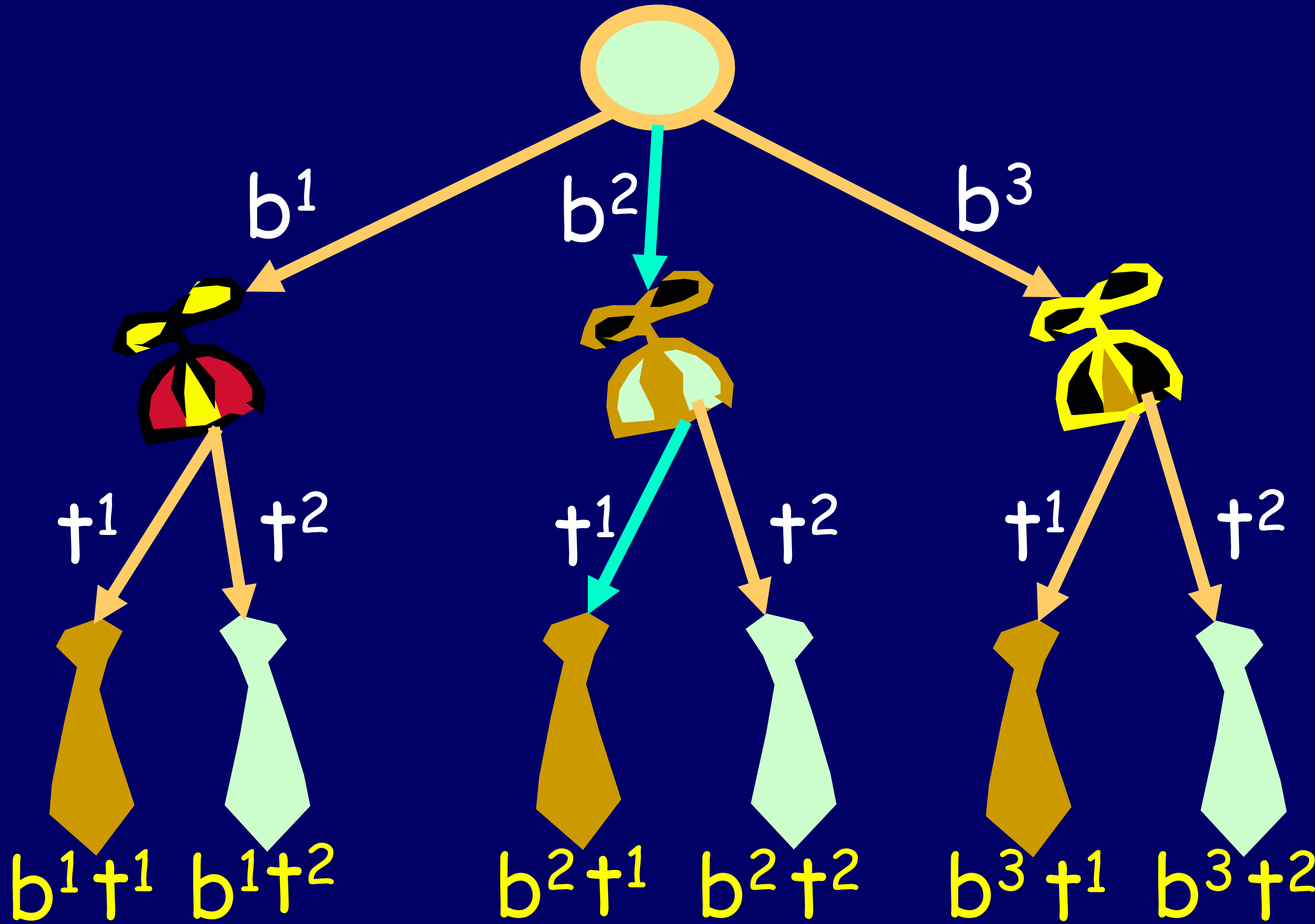
$(b^1 + b^2 + b^3)(t^1 + t^2) =$



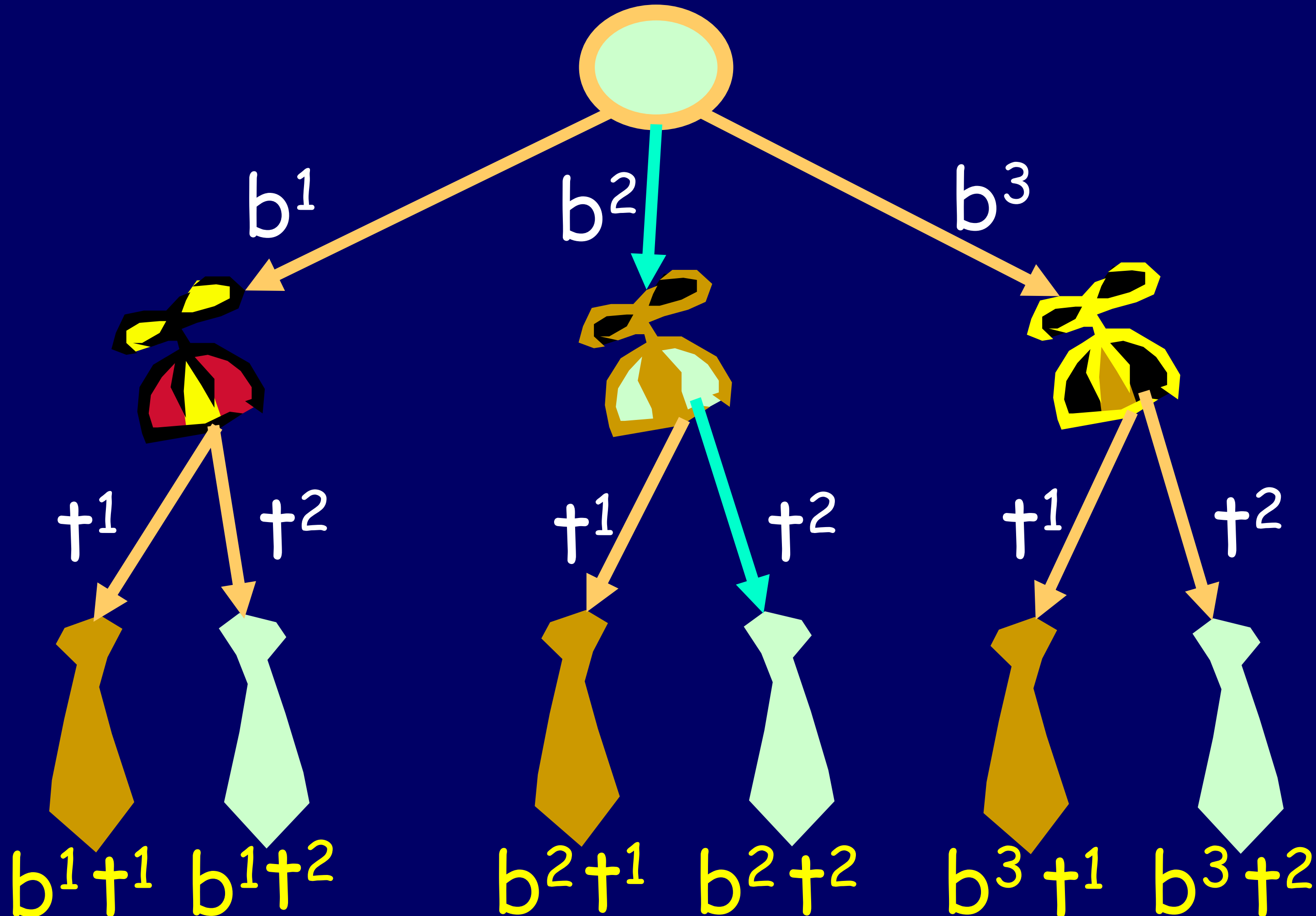
$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 +$$



$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 +$

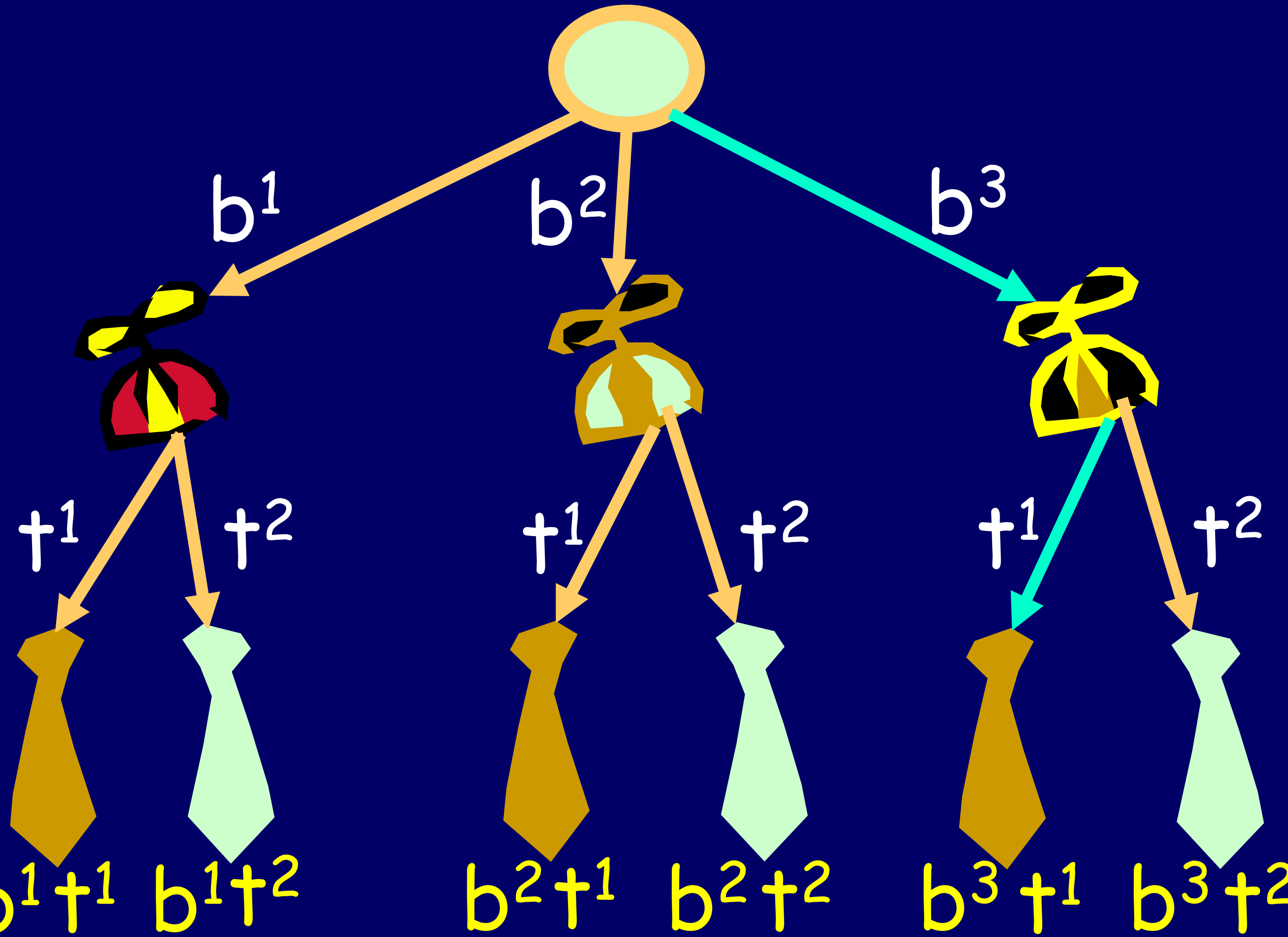


$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 +$$

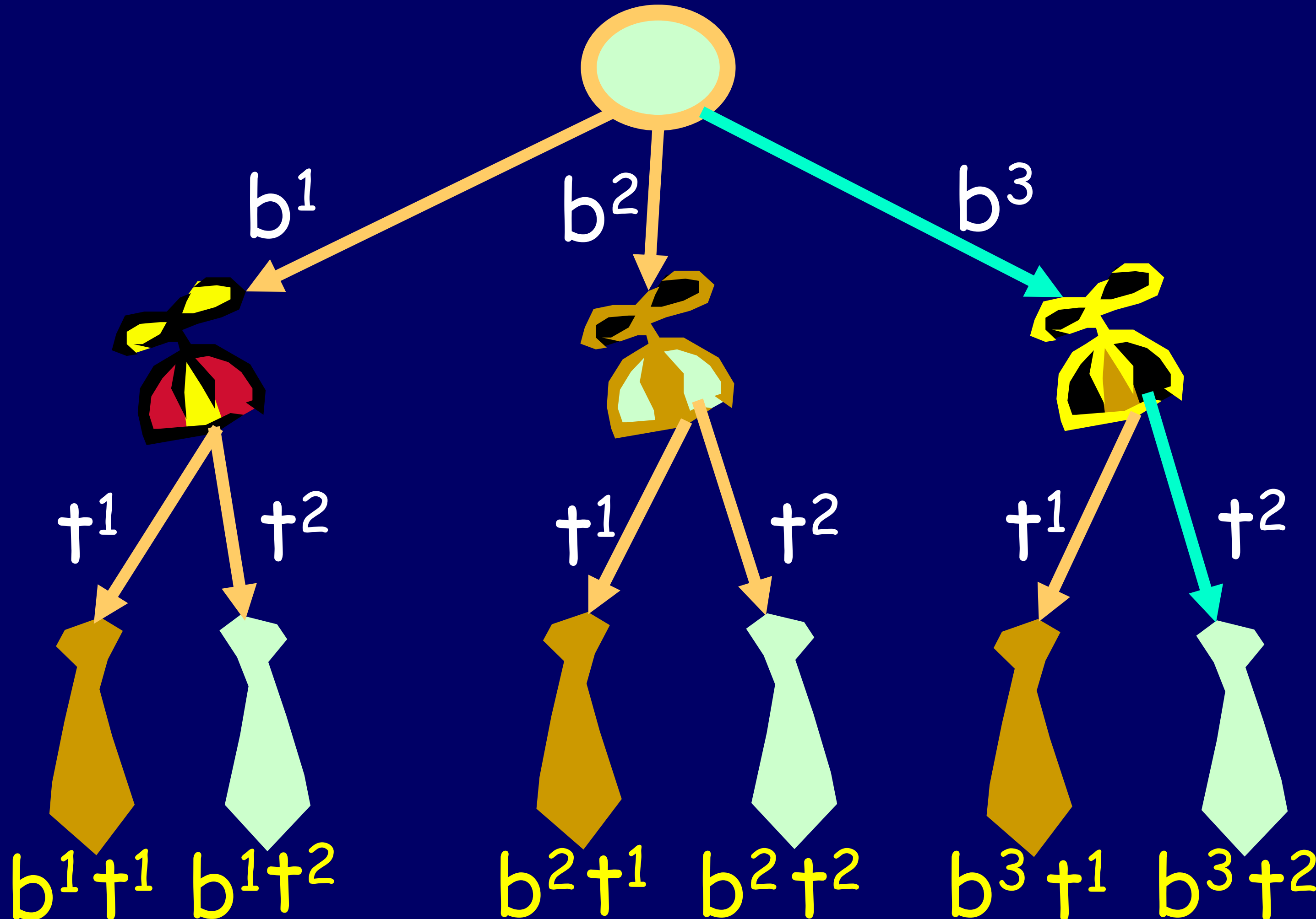


$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$$

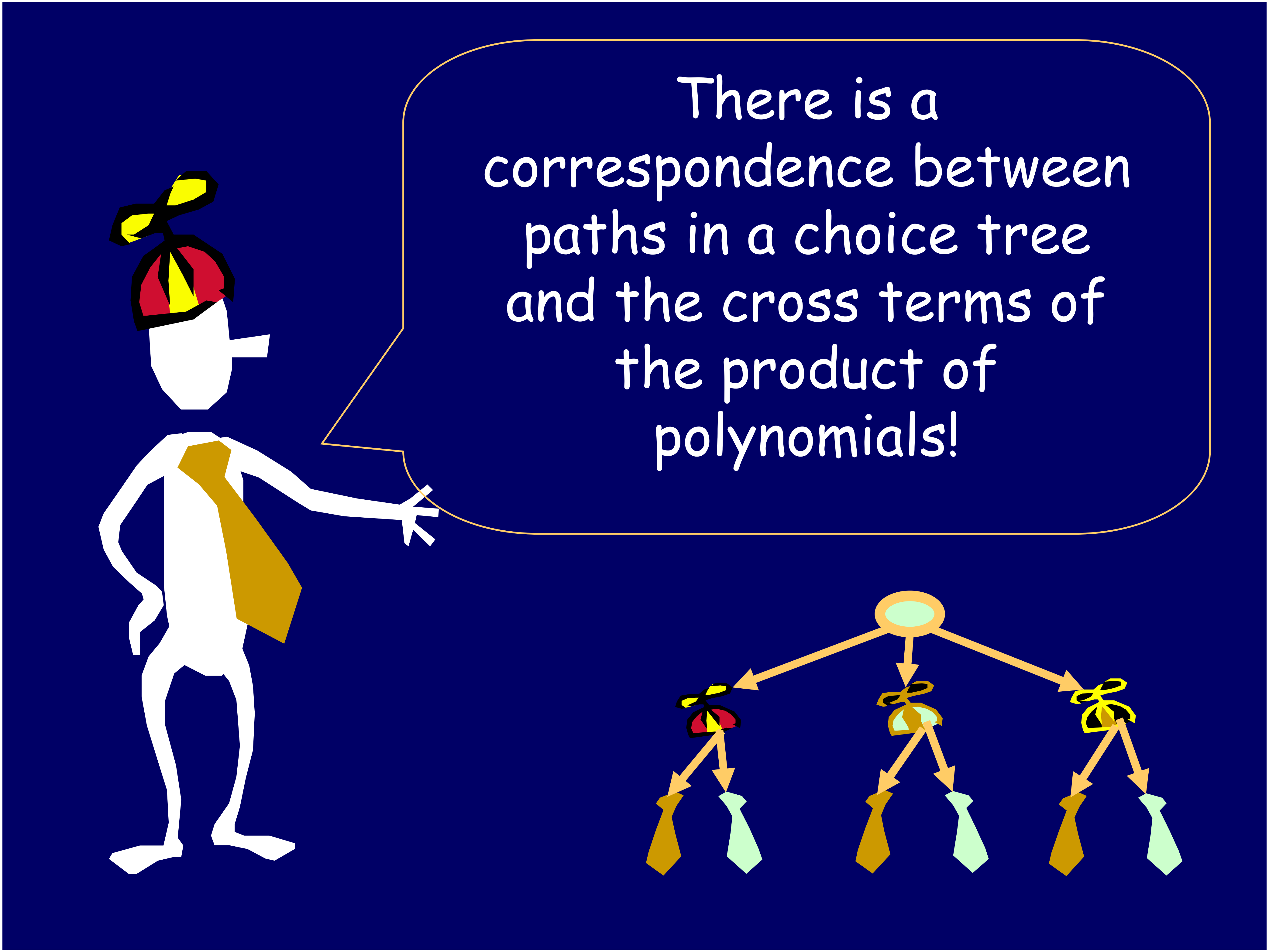
+



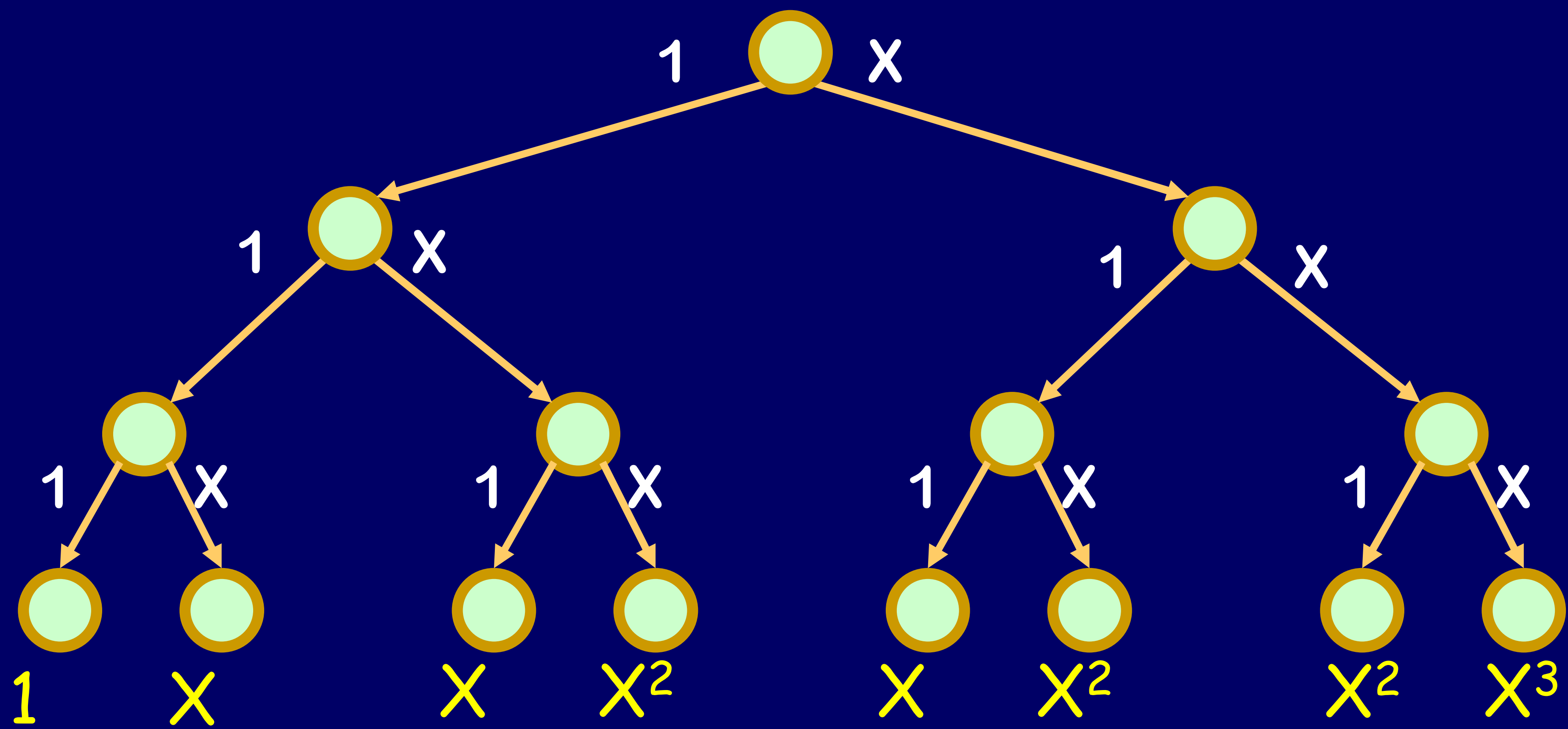
$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$$
$$+ b^3t^1 +$$



$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2$$


There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

What is a closed form expression for c_k?

$$(1+X)^n = c_0 + c_1 X + c_2 X^2 + \ldots + c_n X^n$$

What is a closed form expression for c_n?

$$(1+X)^n$$

$$= \overbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}^{\text{n times}}$$

After multiplying things out, but *before combining like terms*, we get 2^n cross terms, each corresponding to a path in the choice tree.

c_k, the coefficient of X^k, is the number of paths with *exactly k X's*.

$$c_k = \binom{n}{k}$$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

The Binomial Formula

$$(1+X)^0 = 1$$

$$(1+X)^1 = 1 + 1X$$

$$(1+X)^2 = 1 + 2X + 1X^2$$

$$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$$

$$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$$

The Binomial Formula

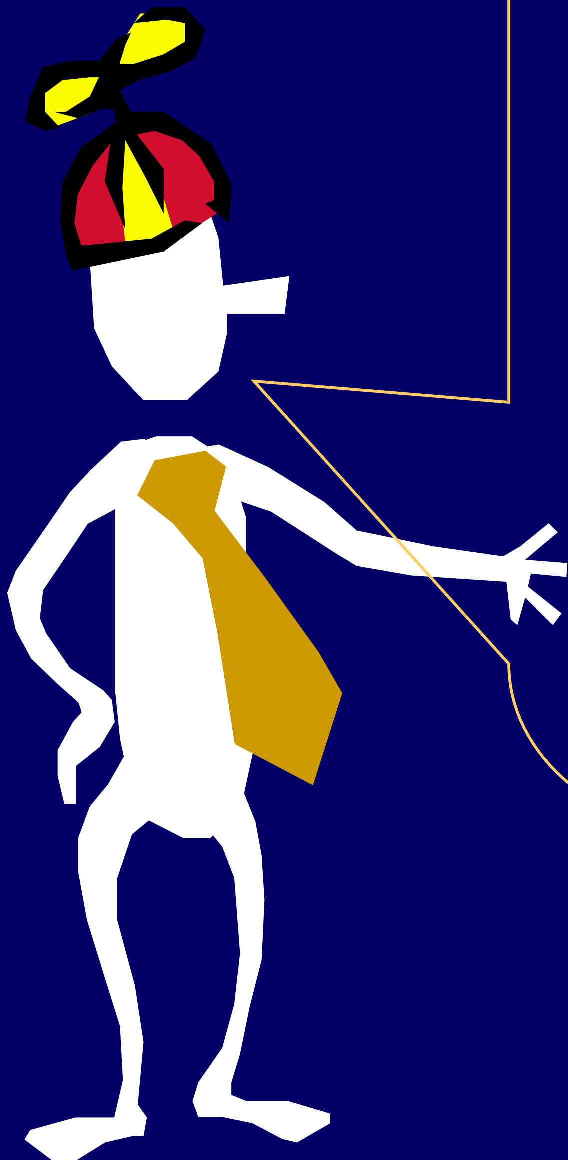
$$(x+y)^n$$

$$= \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \ldots + \binom{n}{k}x^ky^{n-k} + \ldots + \binom{n}{n}x^ny^0$$

The Binomial Formula

$$(X+Y)^n = \sum_{k=0}^{k=n} \binom{n}{k} X^k Y^{n-k}$$

$$x^n\Big(\underbrace{1+\frac{y}{x}}_{z}\Big)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$



There is much, much more to be said about how polynomials encode counting questions!

References

Applied Combinatorics, by Alan Tucker