
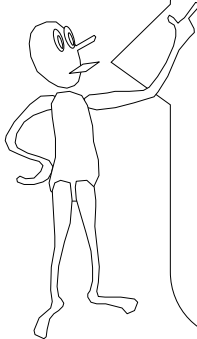
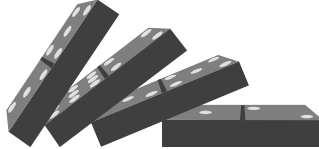
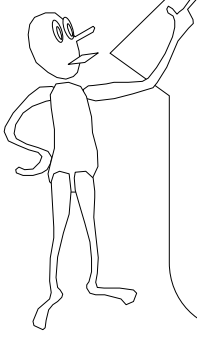


Great Theoretical Ideas In Computer Science		
Anupam Gupta		CS 15-251 Fall 2005
Lecture 2	Sept 01, 2005	Carnegie Mellon University

Induction: One Step At A Time





Today we will talk  
about  
**INDUCTION**


Induction is the  
primary way we:

1. Prove theorems
2. Construct and define objects



Let's start with dominoes

Domino Principle: Line up any  
number of dominos in a row;  
knock the first one over and  
they will all fall.




n dominoes numbered 1 to n

$F_k \equiv$  The  $k^{\text{th}}$  domino falls

If we set them all up in a row then we know that each one is set up to knock over the next one:

For all  $1 \leq k < n$ :

$$F_k \Rightarrow F_{k+1}$$


n dominoes numbered 1 to n

$F_k \equiv$  The  $k^{\text{th}}$  domino falls

For all  $1 \leq k < n$ :

$$F_k \Rightarrow F_{k+1}$$

$F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow \dots$

$F_1 \Rightarrow$  All Dominoes Fall



n dominoes numbered 0 to n-1

$F_k \equiv$  The  $k^{\text{th}}$  domino falls

For all  $0 \leq k < n-1$ :

$$F_k \Rightarrow F_{k+1}$$

$F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \dots$

$F_0 \Rightarrow$  All Dominoes Fall



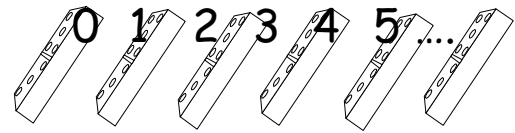
The Natural Numbers

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

The Natural Numbers

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

One domino for each natural number:



Standard Notation/Abbreviation  
"for all" is written " $\forall$ "

Example:

For all  $k > 0$ ,  $P(k)$   
is equivalent to  
 $\forall k > 0, P(k)$

n dominoes numbered 0 to n-1

$F_k \equiv$  The  $k^{\text{th}}$  domino falls

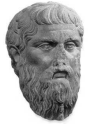
$\forall k, 0 \leq k < n-1$ :

$$F_k \Rightarrow F_{k+1}$$

$F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \dots$

$F_0 \Rightarrow$  All Dominoes Fall





Plato: The Domino Principle works for an infinite row of dominoes

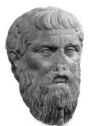
Aristotle: Never seen an infinite number of anything, much less dominoes.



### Plato's Dominoes One for each natural number

An infinite row, 0, 1, 2, ... of dominoes, one domino for each natural number.  
Knock the first domino over and they all will fall.

Proof:  
Suppose they don't all fall. Let  $k > 0$  be the lowest numbered domino that remains standing. Domino  $k-1 \geq 0$  did fall, but  $k-1$  will knock over domino  $k$ . Thus, domino  $k$  must fall and remain standing. Contradiction.



### The Infinite Domino Principle

$F_k \equiv$  The  $k^{\text{th}}$  domino will fall

Assume we know that for every natural number  $k$ ,  
 $F_k \Rightarrow F_{k+1}$

$F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \dots$   
 $F_0 \Rightarrow$  All Dominoes Fall

### Mathematical Induction: statements proved instead of dominoes fallen

Infinite sequence of dominoes.      Infinite sequence of statements:  $S_0, S_1, \dots$

$F_k \equiv$  "domino  $k$  fell"       $F_k \equiv$  " $S_k$  proved"

Establish    1)  $F_0$   
                  2) For all  $k, F_k \Rightarrow F_{k+1}$

Conclude that  $F_k$  is true for all  $k$

### Inductive Proof / Reasoning To Prove $\forall k \in \mathbb{N}, S_k$

Establish "Base Case":  $S_0$

Establish that  $\forall k, S_k \Rightarrow S_{k+1}$

$\forall k, S_k \Rightarrow S_{k+1}$  { Assume hypothetically that  $S_k$  for any particular  $k$ ;  
                                  Conclude that  $S_{k+1}$

### Inductive Proof / Reasoning To Prove $\forall k \in \mathbb{N}, S_k$

Establish "Base Case":  $S_0$

Establish that  $\forall k, S_k \Rightarrow S_{k+1}$

$\forall k, S_k \Rightarrow S_{k+1}$  { "Induction Hypothesis"  $S_k$   
                                  "Induction Step"  
                                  Use I.H. to show  $S_{k+1}$

**Inductive Proof / Reasoning**  
To Prove  $\forall k \geq b, S_k$

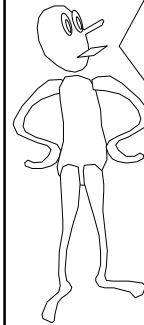
Establish "Base Case":  $S_b$

Establish that  $\forall k \geq b, S_k \Rightarrow S_{k+1}$

Assume  $k \geq b$

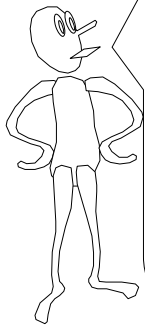
"Inductive Hypothesis": Assume  $S_k$

"Inductive Step:" Prove that  $S_{k+1}$  follows



Theorem?

The sum of the first  $n$  odd numbers is  $n^2$ .

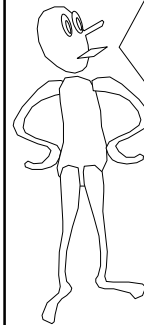


Theorem:?

The sum of the first  $n$  odd numbers is  $n^2$ .

Check on small values:

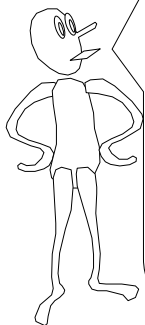
1	=	1
1+3	=	4
1+3+5	=	9
1+3+5+7	=	16



Theorem:?

The sum of the first  $n$  odd numbers is  $n^2$ .

The  $k^{\text{th}}$  odd number is expressed by the formula  $(2k - 1)$ , when  $k > 0$ .



$S_n \equiv$  "The sum of the first  $n$  odd numbers is  $n^2$ ."

Equivalently,

$S_n$  is the statement that:  
"1 + 3 + 5 + (2k-1) + ... + (2n-1) =  $n^2$  "



$S_n \equiv$  "The sum of the first  $n$  odd numbers is  $n^2$ ."  
"1 + 3 + 5 + (2k-1) + ... + (2n-1) =  $n^2$ "

Trying to establish that:  $\forall n \geq 1 S_n$

Assume "Induction Hypothesis":  $S_k$

(for any particular  $k \geq 1$ )

1+3+5+...+ (2k-1) =  $k^2$

Add (2k+1) to both sides.

1+3+5+...+ (2k-1)+(2k+1) =  $k^2 + (2k+1)$

Sum of first  $k+1$  odd numbers =  $(k+1)^2$

CONCLUDE:  $S_{k+1}$

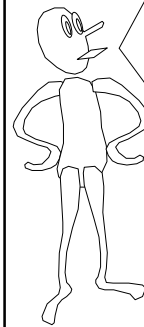
$S_n \equiv$  "The sum of the first  $n$  odd numbers is  $n^2$ ."  
 $"1 + 3 + 5 + (2k-1) + \dots + (2n-1) = n^2"$

Trying to establish that:  $\forall n \geq 1 S_n$

In summary:

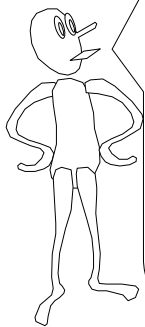
- 1) Establish base case:  $S_1$
- 2) Establish domino property:  $\forall k \geq 0 S_k \Rightarrow S_{k+1}$

By induction on  $n$ , we conclude that:  $\forall k \geq 0 S_k$



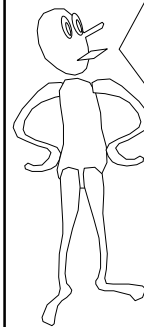
**THEOREM:**

The sum of the first  $n$  odd numbers is  $n^2$ .



**Theorem?**

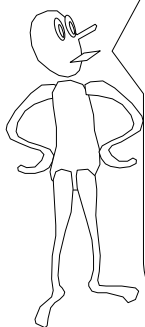
The sum of the first  $n$  numbers is  $\frac{1}{2}n(n+1)$ .



Theorem? The sum of the first  $n$  numbers is  $\frac{1}{2}n(n+1)$ .

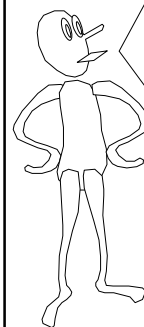
Try it out on small numbers!

$$\begin{aligned} 1 &= 1 = \frac{1}{2}1(1+1). \\ 1+2 &= 3 = \frac{1}{2}2(2+1). \\ 1+2+3 &= 6 = \frac{1}{2}3(3+1). \\ 1+2+3+4 &= 10 = \frac{1}{2}4(4+1). \end{aligned}$$



Theorem? The sum of the first  $n$  numbers is  $\frac{1}{2}n(n+1)$ .

$$\begin{aligned} &= 0 = \frac{1}{2}0(0+1). \\ 1 &= 1 = \frac{1}{2}1(1+1). \\ 1+2 &= 3 = \frac{1}{2}2(2+1). \\ 1+2+3 &= 6 = \frac{1}{2}3(3+1). \\ 1+2+3+4 &= 10 = \frac{1}{2}4(4+1). \end{aligned}$$



**Notation:**

$$\Delta_0 = 0$$

$$\Delta_n = 1 + 2 + 3 + \dots + n-1 + n$$

Let  $S_n$  be the statement  
 $"\Delta_n = n(n+1)/2"$



$S_n \equiv \Delta_n = n(n+1)/2$   
 Use induction to prove  $\forall k \geq 0, S_k$

Establish "Base Case":  $S_0$

$\Delta_0$  = The sum of the first 0 numbers = 0.

Setting  $n=0$ , the formula gives  $0(0+1)/2 = 0$ .

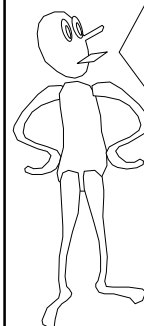
Establish that  $\forall k \geq 0, S_k \Rightarrow S_{k+1}$

"Inductive Hypothesis"  $S_k: \Delta_k = k(k+1)/2$

$$\Delta_{k+1} = \Delta_k + (k+1)$$

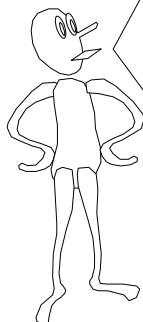
$$= k(k+1)/2 + (k+1) \quad [\text{Using I.H.}]$$

$$= (k+1)(k+2)/2 \quad [\text{which proves } S_{k+1}]$$



Theorem:

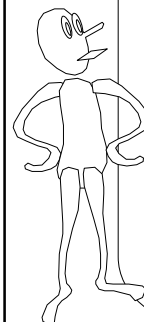
The sum of the first  $n$  numbers is  $\frac{1}{2}n(n+1)$ .



Primes:

A natural number  $n > 1$  is called prime if it has no divisors besides 1 and itself.

n.b. 1 is not considered prime.



Theorem:?

Every natural number  $> 1$  can be factored into primes.

$S_n \equiv$  "n can be factored into primes"

Base case:

2 is prime  $\Rightarrow S_2$  is true.

Trying to prove  $S_{k-1} \Rightarrow S_k$

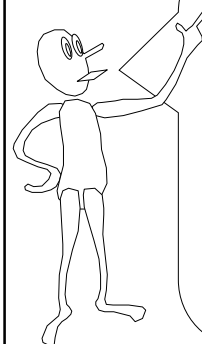
How do we use the fact

$S_{k-1} \equiv$  "k-1 can be factored into primes"

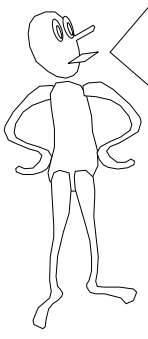
to prove that

$S_k \equiv$  "k can be factored into primes"

Hmm!?



This illustrates a technical point about using and defining mathematical induction.



**Theorem:?**  
Every natural number  $>1$  can be factored into primes.

**A different approach:**

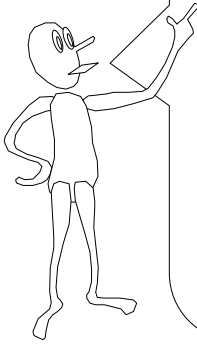
Assume  $2, 3, \dots, k-1$  all can be factored into primes.

Then show that  $k$  can be factored into primes.

**All Previous Induction  
To Prove  $\forall k, S_k$**

Establish Base Case:  $S_0$

Establish that  $\forall k, S_k \Rightarrow S_{k+1}$   
Let  $k$  be any natural number.  
Induction Hypothesis:  
Assume  $\forall j < k, S_j$   
Use that to derive  $S_k$




**Also called  
"Strong Induction"**

**"All Previous" Induction  
Can Be Repackaged As  
Standard Induction**

Establish "Base Case": $S_0$  Establish that $\forall k, S_k \Rightarrow S_{k+1}$ Let $k$ be any number. Assume $\forall j < k, S_j$ Prove $S_k$	}	Define $T_i = \forall j \leq i, S_j$ Establish "Base Case": $T_0$  Establish that $\forall k, T_k \Rightarrow T_{k+1}$ Let $k$ be any number. Assume $T_{k-1}$ Prove $T_k$
---	---	--



**And there are more  
ways to do inductive  
proofs**



**Aristotle's  
Contrapositive**

Let  $S$  be a sentence of the form " $A \Rightarrow B$ ".

The Contrapositive of  $S$  is the sentence " $\neg B \Rightarrow \neg A$ ".

$A \Rightarrow B$ : When  $A$  is true,  $B$  is true.  
 $\neg B \Rightarrow \neg A$ : When  $B$  is false,  $A$  is false.



### Aristotle's Contrapositive

Logically equivalent:

A	B	"A⇒B"	"¬B ⇒ ¬A"
False	False	True	True
False	True	True	True
True	False	False	False
True	True	True	True

### Contrapositive or Least Counter-Example Induction to Prove $\forall k, S_k$

Establish "Base Case":  $S_0$

Establish that  $\forall k, S_k \Rightarrow S_{k+1}$

Let  $k > 0$  be the least number such that  $S_k$  is false.

Prove that  $\neg S_k \Rightarrow \neg S_{k-1}$

Contradiction of  $k$  being the least counter-example!

### Least Counter-Example Induction to Prove $\forall k, S_k$

Establish "Base Case":  $S_0$

Establish that  $\forall k, S_k \Rightarrow S_{k+1}$

Assume that  $S_k$  is the least counter-example.

Derive the existence of a smaller counter-example  $S_j$  (for  $j < k$ )



### Rene Descartes [1596-1650] "Method Of Infinite Decent"

Show that for any counter-example you find a smaller one. If a counter-example exists there would be an infinite sequence of smaller and smaller counter examples.

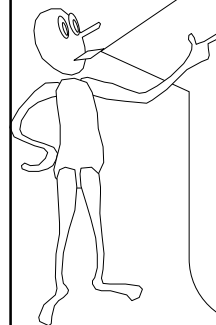
### Each number $> 1$ has a prime factorization.

Let  $n$  be the least counter-example.

Hence  $n$  is not prime  
 $\Rightarrow$  so  $n = ab$ .

If both  $a$  and  $b$  had prime factorizations, then  $n$  would too.

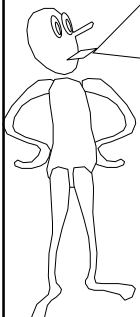
Thus  $a$  or  $b$  is a smaller counter-example.



Inductive reasoning is the high level idea:

"Standard" Induction  
"All Previous" Induction  
"Least Counter-example"  
all just different packaging.





Euclid's theorem on the unique factorization of a number into primes.

Assume there is a least counter-example. Derive the existence of a smaller counter-example.

Theorem: Each natural has a unique factorization into primes written in non-decreasing order.

Let  $n$  be the least counter-example.  $n$  has at least two ways of being written as a product of primes:

$$n = p_1 p_2 \dots p_k = q_1 q_2 \dots q_r$$

The  $p$ 's must be totally different primes than the  $q$ 's or else we could divide both sides by one of a common prime and get a smaller counter-example. Without loss of generality, assume  $p_1 > q_1$ .

Theorem: Each natural has a unique factorization into primes written in non-decreasing order.

Let  $n$  be the least counter-example.

$$n = p_1 p_2 \dots p_k = q_1 q_2 \dots q_r \quad [p_1 > q_1]$$

$$n \geq p_1 p_1 > p_1 q_1 + 1 \quad [\text{Since } p_1 > q_1]$$

.

Theorem: Each natural has a unique factorization into primes written in non-decreasing order.

Let  $n$  be the least counter-example.

$$n = p_1 p_2 \dots p_k = q_1 q_2 \dots q_r \quad [p_1 > q_1]$$

$$n \geq p_1 p_1 > p_1 q_1 + 1 \quad [\text{Since } p_1 > q_1]$$

$$m = n - p_1 q_1 \quad [\text{Thus } 1 < m < n]$$

Notice:  $m = p_1(p_2 \dots p_k - q_1) = q_1(q_2 \dots q_r - p_1)$

Thus,  $p_1 | m$  and  $q_1 | m$

By unique factorization of  $m$ ,  $p_1 q_1 | m$ , thus  $m = p_1 q_1 z$

Theorem: Each natural has a unique factorization into primes written in non-decreasing order.

Let  $n$  be the least counter-example.

$$n = p_1 p_2 \dots p_k = q_1 q_2 \dots q_r \quad [p_1 > q_1]$$

$$n \geq p_1 p_1 > p_1 q_1 + 1 \quad [\text{Since } p_1 > q_1]$$

$$m = n - p_1 q_1 \quad [\text{Thus } 1 < m < n]$$

Notice:  $m = p_1(p_2 \dots p_k - q_1) = q_1(q_2 \dots q_r - p_1)$

Thus,  $p_1 | m$  and  $q_1 | m$

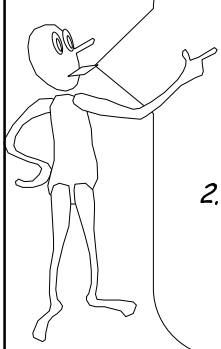
By unique factorization of  $m$ ,  $p_1 q_1 | m$ , thus  $m = p_1 q_1 z$

We have:  $m = n - p_1 q_1 = p_1(p_2 \dots p_k - q_1) = p_1 q_1 z$

Dividing by  $p_1$  we obtain:  $(p_2 \dots p_k - q_1) = q_1 z$

$p_2 \dots p_k = q_1 z + q_1 = q_1(z+1)$  so  $q_1 | p_2 \dots p_k$

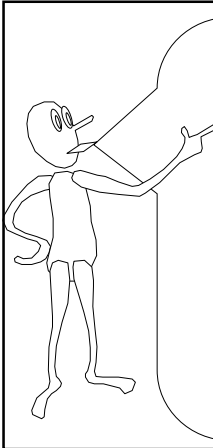
And hence, by unique factorization of  $p_2 \dots p_k$ ,  $q_1$  must be one of the primes  $p_2, \dots, p_k$ . Contradiction of  $q_1 < p_1$ .



Yet another way of packaging inductive reasoning is to define "invariants".

Invariant:

1. Not varying; constant.
2. Mathematics. Unaffected by a designated operation, as a transformation of coordinates.



Yet another way of packaging inductive reasoning is to define "invariants".

**Invariant:**  
 3. programming A rule, such as the ordering an ordered list or heap, that applies throughout the life of a data structure or procedure. Each change to the data structure must maintain the correctness of the invariant.

**Invariant Induction**  
 Suppose we have a time varying world state:  $W_0, W_1, W_2, \dots$   
 Each state change is assumed to come from a list of permissible operations. We seek to prove that statement  $S$  is true of all future worlds.

Argue that  $S$  is true of the initial world.

Show that if  $S$  is true of some world - then  $S$  remains true after one permissible operation is performed.

**Invariant Induction**  
 Suppose we have a time varying world state:  $W_0, W_1, W_2, \dots$   
 Each state change is assumed to come from a list of permissible operations.

Let  $S$  be a statement true of  $W_0$ .

Let  $W$  be any possible future world state.  
 Assume  $S$  is true of  $W$ .  
 Show that  $S$  is true of any world  $W'$  obtained by applying a permissible operation to  $W$ .

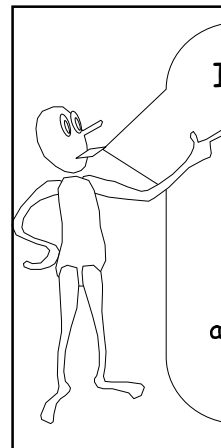
**Odd/Even Handshaking Theorem:**  
 At any party at any point in time define a person's parity as ODD/EVEN according to the number of hands they have shaken.

Statement: The number of people of odd parity must be even.

Statement: The number of people of odd parity must be even.

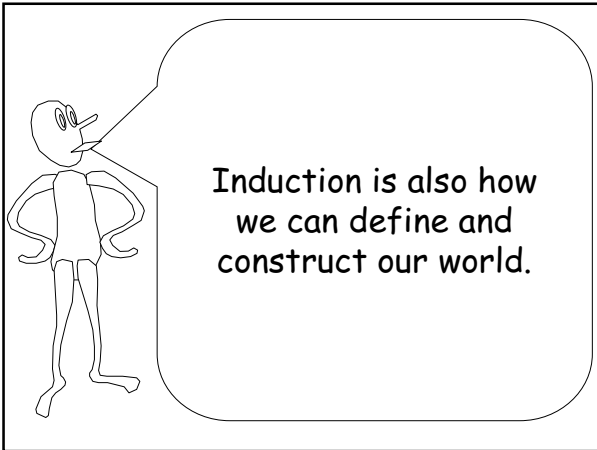
Initial case: Zero hands have been shaken at the start of a party, so zero people have odd parity.

If 2 people of different parities shake, then they both swap parities and the odd parity count is unchanged. If 2 people of the same parity shake, they both change and hence the odd parity count changes by 2 - and remains even.

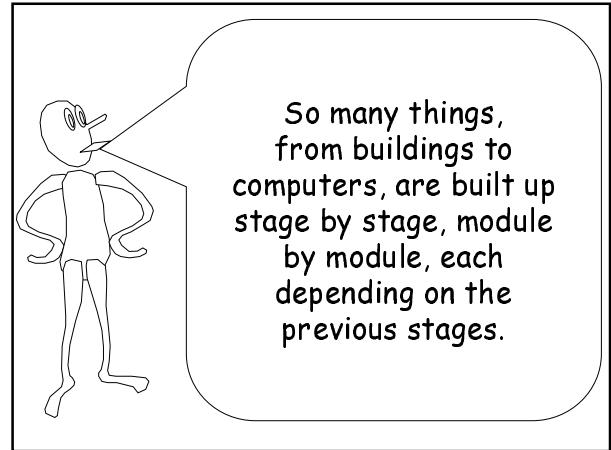


Inductive reasoning is the high level idea:

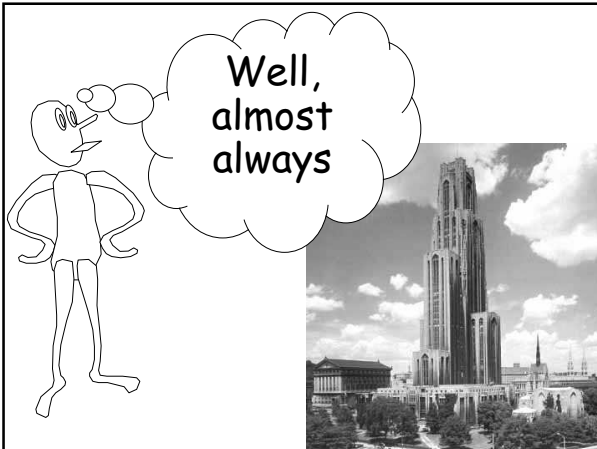
"Standard" Induction  
 "Least Counter-example"  
 "All-Previous" Induction  
 and "Invariants"  
 all just different packaging.



Induction is also how we can define and construct our world.



So many things, from buildings to computers, are built up stage by stage, module by module, each depending on the previous stages.



Well, almost always

### Inductive Definition Of Functions

Stage 0, Initial Condition, or Base Case:  
Declare the value of the function on some subset of the domain.

Inductive Rules  
Define new values of the function in terms of previously defined values of the function

$F(x)$  is defined if and only if it is implied by finite iteration of the rules.

### Inductive Definition Recurrence Relation for $F(X)$

Initial Condition, or Base Case:  
 $F(0) = 1$

Inductive Rule  
For  $n > 0$ ,  $F(n) = F(n-1) + F(n-1)$

n	0	1	2	3	4	5	6	7
F(n)	1							

### Inductive Definition Recurrence Relation for $F(X)$

Initial Condition, or Base Case:  
 $F(0) = 1$

Inductive Rule  
For  $n > 0$ ,  $F(n) = F(n-1) + F(n-1)$

n	0	1	2	3	4	5	6	7
F(n)	1	2						

### Inductive Definition Recurrence Relation for $F(X)$

Initial Condition, or Base Case:  
 $F(0) = 1$

Inductive Rule  
For  $n > 0$ ,  $F(n) = F(n-1) + F(n-1)$

n	0	1	2	3	4	5	6	7
F(n)	1	2	4					

### Inductive Definition Recurrence Relation for $F(X)$

Initial Condition, or Base Case:  
 $F(0) = 1$

Inductive Rule  
For  $n > 0$ ,  $F(n) = F(n-1) + F(n-1)$

n	0	1	2	3	4	5	6	7
F(n)	1	2	4	8	16	32	64	128

### Inductive Definition Recurrence Relation for $F(X) = 2^X$

Initial Condition, or Base Case:  
 $F(0) = 1$

Inductive Rule  
For  $n > 0$ ,  $F(n) = F(n-1) + F(n-1)$

n	0	1	2	3	4	5	6	7
F(n)	1	2	4	8	16	32	64	128

### Inductive Definition Recurrence Relation

Initial Condition, or Base Case:  
 $F(1) = 1$

Inductive Rule  
For  $n > 1$ ,  $F(n) = F(n/2) + F(n/2)$

n	0	1	2	3	4	5	6	7
F(n)		1						

### Inductive Definition Recurrence Relation

Initial Condition, or Base Case:  
 $F(1) = 1$

Inductive Rule  
For  $n > 1$ ,  $F(n) = F(n/2) + F(n/2)$

n	0	1	2	3	4	5	6	7
F(n)		1	2					

### Inductive Definition Recurrence Relation

Initial Condition, or Base Case:  
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For  $n > 1$ ,  $F(n) = F(n/2) + F(n/2)$

n	0	1	2	3	4	5	6	7
F(n)		1	2		4			

### Inductive Definition Recurrence Relation

Initial Condition, or Base Case:  
 $F(1) = 1$

Inductive Rule  
 For  $n > 1$ ,  $F(n) = F(n/2) + F(n/2)$

n	0	1	2	3	4	5	6	7
F(n)	%	1	2	%	4	%	%	%

### Inductive Definition Recurrence Relation $F(X) = X$ for X a whole power of 2.

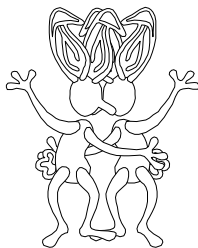
Initial Condition, or Base Case:  
 $F(1) = 1$

Inductive Rule  
 For  $n > 1$ ,  $F(n) = F(n/2) + F(n/2)$

n	0	1	2	3	4	5	6	7
F(n)	%	1	2	%	4	%	%	%

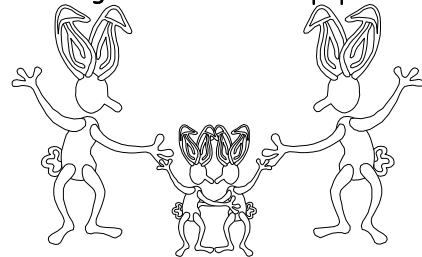
### Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.



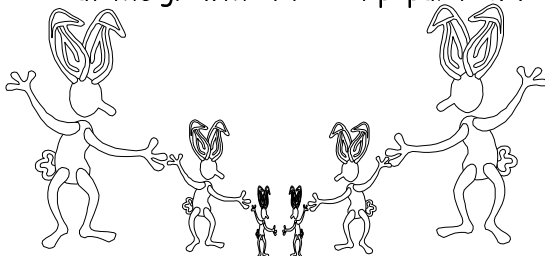
### Leonardo Fibonacci

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### Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.



### The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n =$  # of rabbit pairs at the beginning of the  $n^{\text{th}}$  month

month	1	2	3	4	5	6	7
rabbits							

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rabbits	1						

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month	1	2	3	4	5	6	7
rabbits	1	1	2				

### The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n =$  # of rabbit pairs at the beginning of the  $n^{\text{th}}$  month

month	1	2	3	4	5	6	7
rabbits	1	1	2	3			

### The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n =$  # of rabbit pairs at the beginning of the  $n^{\text{th}}$  month

month	1	2	3	4	5	6	7
rabbits	1	1	2	3	5		

### The rabbit reproduction model

- A rabbit lives forever
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$F_n =$  # of rabbit pairs at the beginning of the  $n^{\text{th}}$  month

month	1	2	3	4	5	6	7
rabbits	1	1	2	3	5	8	13

### Inductive Definition or Recurrence Relation for the Fibonacci Numbers

Stage 0, Initial Condition, or Base Case:  
 $Fib(1) = 1; Fib(2) = 1$

Inductive Rule  
 For  $n > 3$ ,  $Fib(n) = Fib(n-1) + Fib(n-2)$

n	0	1	2	3	4	5	6	7
Fib(n)	%	1	1	2	3	5	8	13

### Inductive Definition or Recurrence Relation for the Fibonacci Numbers

Stage 0, Initial Condition, or Base Case:  
 $Fib(0) = 0; Fib(1) = 1$

Inductive Rule  
 For  $n > 1$ ,  $Fib(n) = Fib(n-1) + Fib(n-2)$

n	0	1	2	3	4	5	6	7
Fib(n)	0	1	1	2	3	5	8	13

### Programs to compute Fib(n)?

Stage 0, Initial Condition, or Base Case:  
 $Fib(0) = 0; Fib(1) = 1$

Inductive Rule  
 For  $n > 1$ ,  $Fib(n) = Fib(n-1) + Fib(n-2)$

n	0	1	2	3	4	5	6	7
Fib(n)								

Inductive Definition:  
 $Fib(0)=0, Fib(1)=1, k > 1, Fib(k)=Fib(k-1)+Fib(k-2)$

#### Bottom-Up, Iterative Program:

$Fib(0) = 0; Fib(1) = 1;$   
 Input x:  
 For k= 2 to x do  $Fib(k)=Fib(k-1)+Fib(k-2);$   
 Return  $Fib(x);$

#### Top-Down, Recursive Program:

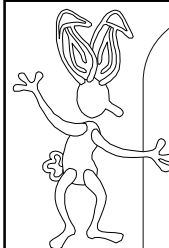
Return  $Fib(x);$   
 Procedure  $Fib(k)$   
 If  $k=0$  return 0  
 If  $k=1$  return 1  
 Otherwise return  $Fib(k-1)+Fib(k-2);$

### What is a closed form formula for Fib(n) ????

Stage 0, Initial Condition, or Base Case:  
 $Fib(0) = 0; Fib(1) = 1$

Inductive Rule  
 For  $n > 1$ ,  $Fib(n) = Fib(n-1) + Fib(n-2)$

n	0	1	2	3	4	5	6	7
Fib(n)	0	1	1	2	3	5	8	13



Leonhard Euler (1765)  
 J. P. M. Binet (1843)  
 August de Moivre (1730)

$$Fib(n) = \frac{\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n}{\sqrt{5}}$$



Inductive Proof  
Standard Form  
All Previous Form  
Least-Counter Example Form  
Invariant Form

Inductive Definition  
Bottom-Up Programming  
Top-Down Programming  
Recurrence Relations  
Fibonacci Numbers

Study Bee<sub>Logic</sub>

Contrapositive Form of  $S$