15-213
Integers
January 22, 2002

Topics

- **Numeric Encodings**
  - Unsigned & Two's complement
- **Programming Implications**
  - C promotion rules
- **Basic operations**
  - Addition, negation, multiplication
- **Programming Implications**
  - Consequences of overflow
  - Using shifts to perform power-of-2 multiply/divide

- **Reading: 2.2 – 2.3**
- **Suggested Practice Problems: 2.40 – 2.45**

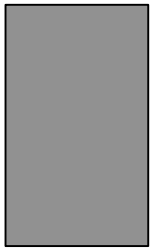
C Puzzles

- **Taken from Exam #2, CS 347, Spring '97**
- **Assume machine with 32 bit word size, two's complement integers**
- **For each of the following C expressions, either:**
  - Argue that is true for all argument values
  - Give example where not true

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

- `x < 0` þ
- `ux >= 0`
- `x & 7 == 7` þ
- `ux > -1`
- `x > y` þ
- `x * x >= 0`
- `x > 0 && y > 0`
- `x >= 0` þ
- `x <= 0` þ



2's Compliment Encoding

Conversion from Binary:

- **Compliment each bit, add 1**
  `3 = 0011 (raw binary)`
  `1100 (after compliment)`
  `1101 (2's compliment, after adding 1)`

Conversion to Binary:

- **Subtract 1 then compliment each bit (reverse example above)**

Signed Representation Using Two's Compliment

- **Encode magnitudes in raw binary, leaving high-order bit for sign**
- **Take 2's compliment of negative numbers, this will set sign bit.** `3 = --3 = 0011 (3 in raw binary)`
  `1100 (after compliment)`
  `1101 (2's compliment, after adding 1)`

Two's Compliment in Math-ese

Unsigned (Raw)

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

The left side is like a switch:
If positive number, no effect. Second part handled as for unsigned number

If negative:
- Realize that $x_{w-1} \cdot 2^{w-1} = \sum_{i=0}^{w-2} 2^i + 1$
- Remember that we added 1 when converting to 2's complement? Subtract it and the +1 goes away.
- This leaves us with a set with a number as wide as the original, with all of the powers of two represented (all bits on)
- Now, think about the right side of that addition. Originally those bits contained the powers of two that were present. But we complemented it. So, now it contains the powers of two that were not present in the original number.
- So, by subtracting those powers of two that were not represented in our original number from a set of all powers of two, we are left with only those that were originally represented.

Encoding Integers

```
short int x =  15213;
short int y = -15213;
```

C `short` 2 bytes long

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
y	-15213	C4 93	11000100 10010011

Sign Bit

- **For 2's complement, most significant bit indicates sign**
  - 0 for nonnegative
  - 1 for negative

Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Numeric Ranges

Unsigned Values

- *UMin* = 0
  000…0
- *UMax* = $2^w - 1$
  111…1

Two's Complement Values

- *TMin* = -2^{w-1}
  100…0
- *TMax* = $2^{w-1} - 1$
  011…1

Other Values

- Minus 1
  111…1

Values for *W* = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	**255**	**65,535**	**4,294,967,295**	**18,446,744,073,709,551,615**
TMax	**127**	**32,767**	**2,147,483,647**	**9,223,372,036,854,775,807**
TMin	**-128**	**-32,768**	**-2,147,483,648**	**-9,223,372,036,854,775,808**

Observations

- |*TMin*| = *TMax* + 1
  - **Asymmetric range**
- *UMax* = 2 * *TMax* + 1

C Programming

- **`#include <limits.h>`**
  - K&R App. B11
- **Declares constants, e.g.,**
  - `ULONG_MAX`
  - `LONG_MAX`
  - `LONG_MIN`
- **Values platform-specific**

Unsigned & Signed Numeric Values

X	B2U(X)	B2T(X)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

Example Values
- W = 4

Equivalence
- Same encodings for nonnegative values

Uniqueness
- Every bit pattern represents unique integer value
- Each representable integer has unique bit encoding

Þ Can Invert Mappings
- **U2B(x) = B2U^{-1}(x)**
  - Bit pattern for unsigned integer
- **T2B(x) = B2T^{-1}(x)**
  - Bit pattern for two's comp integer

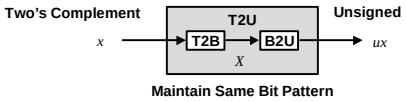
Casting Signed to Unsigned

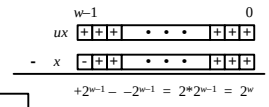
C Allows Conversions from Signed to Unsigned

```
short int          x =  15213;
unsigned short int ux = (unsigned short) x;
short int          y  = -15213;
unsigned short int uy = (unsigned short) y;
```

Resulting Value
- No change in bit representation
- Nonnegative values unchanged
  - ux = 15213
- Negative values change into (large) positive values
  - uy = 50323

Relation Between 2's Comp. & Unsigned





$$ux = \begin{cases} x & x \geq 0 \\ x + 2^w & x < 0 \end{cases}$$

Relation Between Signed & Unsigned

Weight	-15213		50323	
1	1	1	1	1
2	1	2	1	2
4	0	0	0	0
8	0	0	0	0
16	1	16	1	16
32	0	0	0	0
64	0	0	0	0
128	1	128	1	128
256	0	0	0	0
512	0	0	0	0
1024	1	1024	1	1024
2048	0	0	0	0
4096	0	0	0	0
8192	0	0	0	0
16384	1	16384	1	16384
32768	1	-32768	1	32768
Sum		**-15213**		**50323**

- uy = y + 2 * 32768 = y + 65536

Signed vs. Unsigned in C

Constants

- By default are considered to be signed integers
- Unsigned if have "U" as suffix

```
0U, 4294967259U
```

Casting

- Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

- Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```


Casting Surprises

Expression Evaluation

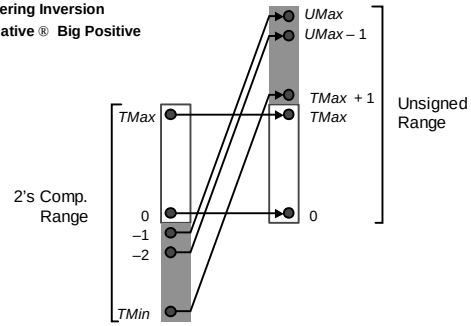
- If mix unsigned and signed in single expression, signed values implicitly cast to unsigned
- Including comparison operations <, >, ==, <=, >=
- Examples for *W* = 32

$Constant_1$	$Constant_2$	Relation	Evaluation
`0`	`0U`	==	unsigned
`-1`	`0`	<	signed
`-1`	`0U`	>	unsigned
`2147483647`	`-2147483648`	>	signed
`2147483647U`	`-2147483648`	<	unsigned
`-1`	`-2`	>	signed
`(unsigned) -1`	`-2`	>	unsigned
`2147483647`	`2147483648U`	<	unsigned
`2147483647`	`(int) 2147483648U`	>	signed

Explanation of Casting Surprises

2's Comp. ® Unsigned

- Ordering Inversion
- Negative ® Big Positive



Why Should I Use Unsigned?

Don't Use Just Because Number Nonzero

Easy to make mistakes with implicit casting

```
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

Do Use When Performing Modular Arithmetic

- Multiprecision arithmetic
- Bit masks, flags structures, and other bitwise data

Do Use When Need Extra Bit's Worth of Range

- Working right up to limit of word size

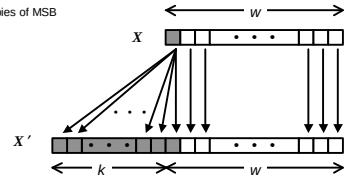
Sign Extension

Task:

- Given *w*-bit signed integer *x*
- Convert it to *w*+*k*-bit integer with same value

Rule:

- Make *k* copies of sign bit:
- $X \Leftarrow \underbrace{x_{w-1},\ldots,x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2},\ldots,x_0$



Sign Extension Example

```
short int x =  15213;
int      ix = (int) x;
short int y = -15213;
int      iy = (int) y;
```

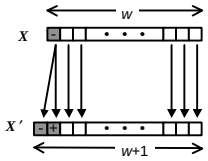
	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 C4 92	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Justification For Sign Extension

Prove Correctness by Induction on *k*

- Induction Step: extending by single bit maintains value



- Key observation: $-2^{w-1} = -2^w + 2^{w-1}$
  $-2^{w-1} - 2^{w-1} = -2^w + 2^{w-1} - 2^{w-1}$
  $2(-2^{w-1}) = -2^w$
  $-2^w = -2^w$
- Look at weight of upper bits:
  x $-2^{w-1} x_{w-1}$
  x' $-2^w x_{w-1} + 2^{w-1} x_{w-1}$ = $-2^{w-1} x_{w-1}$

Negating with Complement & Increment

In C

`~x + 1 == -x`

Complement

- Observation: `~x + x == 1111…11`$_2$ `== -1`

```
      x  1 0 0 1 1 1 0 1
  +  ~x  0 1 1 0 0 0 1 0
 ------------------------
     -1  1 1 1 1 1 1 1 1
```

Increment

- `~x + x + (-x + 1)  ==  -1 + (-x + 1)`
- `~x + 1  ==  -x`

Warning: Be cautious treating `int`'s as integers

- OK here

Comp. & Incr. Examples

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

0

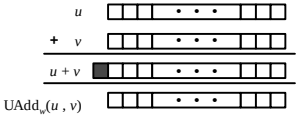
	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

Unsigned Addition

Operands: w bits

True Sum: w+1 bits

Discard Carry: w bits



Standard Addition Function

- **Ignores carry output**

Implements Modular Arithmetic

$s = \text{UAdd}_w(u , v) = u + v \mod 2^w$

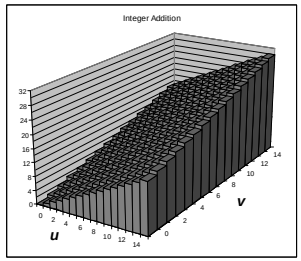
$$UAdd_w(u,v) = \begin{cases} u+v & u+v < 2^w \\ u+v-2^w & u+v \geq 2^w \end{cases}$$

Visualizing Integer Addition

Integer Addition

- **4-bit integers *u* and *v***
- **Compute true sum Add$_4$(*u* , *v*)**
- **Values increase linearly with *u* and *v***
- **Forms planar surface**

Add$_4$(*u* , *v*)

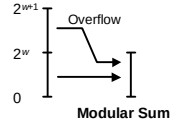


Visualizing Unsigned Addition

Wraps Around

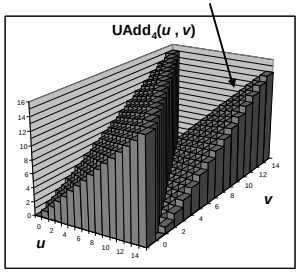
- **If true sum = 2^w**
- **At most once**

True Sum



Overflow

UAdd$_4$(*u* , *v*)



Mathematical Properties

Modular Addition Forms an *Abelian Group*

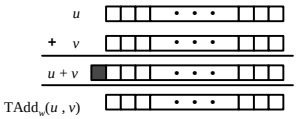
- **Closed under addition**
  $0 \le \text{UAdd}_w(u, v) \le 2^w - 1$
- **Commutative**
  $\text{UAdd}_w(u, v) = \text{UAdd}_w(v, u)$
  (u+v) == (v+u)
- **Associative**
  $\text{UAdd}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UAdd}_w(t, u), v)$
  (t + (u+v)) == ((t+u) + v)
- **0 is additive identity**
  $\text{UAdd}_w(u, 0) = u$
  (u + 0) == u
- **Every element has additive inverse**
  – Let $\text{UComp}_w(u) = 2^w - u$
  $\text{UAdd}_w(u, \text{UComp}_w(u)) = 0$

Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



TAdd and UAdd have Identical Bit-Level Behavior

- **Signed vs. unsigned addition in C:**

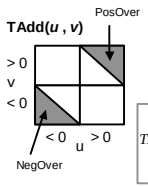
```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

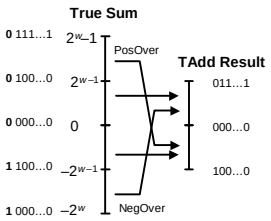
- **Will give `s == t`**

Characterizing TAdd

Functionality

- **True sum requires *w*+1 bits**
- **Drop off MSB**
- **Treat remaining bits as 2's comp. integer**





$$TAdd_w(u,v) = \begin{cases} u+v+2^{w-1} & u+v < TMin_w \quad \text{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^{w-1} & TMax_w < u+v \quad \text{(PosOver)} \end{cases}$$

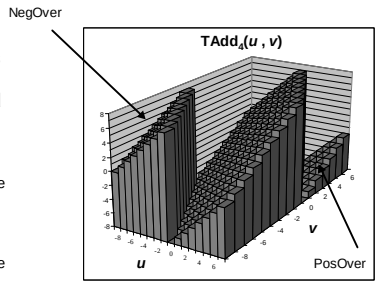
Visualizing 2's Comp. Addition

Values

- **4-bit two's comp.**
- **Range from -8 to +7**

Wraps Around

- **If sum ³ 2^{w-1}**
  – Becomes negative
  – At most once
- **If sum < -2^{w-1}**
  – Becomes positive
  – At most once

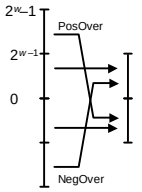


Detecting 2's Comp. Overflow

Task

- **Given** $s = \text{TAdd}_w(u, v)$
- **Determine if** $s = \text{Add}_w(u, v)$
- **Example**

```
int s, u, v;
s = u + v;
```



Claim

- **Overflow iff either:**
  - $u, v < 0, s \geq 0$ (NegOver)
  - $u, v \geq 0, s < 0$ (PosOver)

```
ovf = (u<0 == v<0) && (u<0 != s<0);
```


Mathematical Properties of TAdd

Isomorphic Algebra to UAdd

- $\text{TAdd}_w(u, v) = \text{U2T}(\text{UAdd}_w(\text{T2U}(u), \text{T2U}(v)))$
  - Since both have identical bit patterns

Two's Complement Under TAdd Forms a Group

- **Closed, Commutative, Associative, 0 is additive identity**
- **Every element has additive inverse**

  Let $\text{TComp}_w(u) = \text{U2T}(\text{UComp}_w(\text{T2U}(u)))$

  $\text{TAdd}_w(u, \text{TComp}_w(u)) = 0$

$$TComp_w(u) = \begin{cases} -u & u \neq TMin_w \\ TMin_w & u = TMin_w \end{cases}$$

Negating with Complement & Increment

In C

```
~x + 1 == -x
```

Complement

- **Observation:** $\sim x + x = 1111\ldots11_2 = -1$

```
      x  1 0 0 1 1 1 0 1
    + ~x 0 1 1 0 0 0 1 0
    ------------------
     -1  1 1 1 1 1 1 1 1
```

Increment

- ~~~x + x~~ + (~~-x~~ + 1) == ~~-1~~ + (-x + ~~1~~)
- ~x + 1 == -x

Warning: Be cautious treating `int`'s as integers

- **OK here: We are using group properties of** TAdd **and** TComp

Multiplication

Computing Exact Product of *w*-bit numbers *x*, *y*

- **Either signed or unsigned**

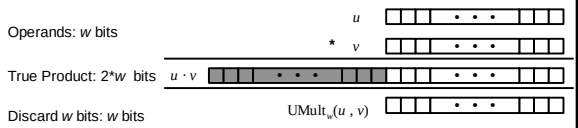
Ranges

- **Unsigned:** $0 = x * y = (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
  - Up to $2w$ bits
- **Two's complement min:** $x * y = (-2^{w-1}) * (2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}$
  - Up to $2w-1$ bits
- **Two's complement max:** $x * y = (-2^{w-1})^2 = 2^{2w-2}$
  - Up to $2w$ bits, but only for $TMin_w^2$

Maintaining Exact Results

- **Would need to keep expanding word size with each product computed**
- **Done in software by "arbitrary precision" arithmetic packages**
- **Also implemented in Lisp, and other "advanced" languages**

Unsigned Multiplication in C



Standard Multiplication Function

- **Ignores high order *w* bits**

Implements Modular Arithmetic

$\text{UMult}_w(u , v) = u \cdot v \mod 2^w$

Unsigned vs. Signed Multiplication

Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

- **Truncates product to *w*-bit number** $up = \text{UMult}_w(ux, uy)$
- **Simply modular arithmetic**

  $up = ux \cdot uy \mod 2^w$

Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

- **Compute exact product of two *w*-bit numbers *x*, *y***
- **Truncate result to *w*-bit number** $p = \text{TMult}_w(x, y)$

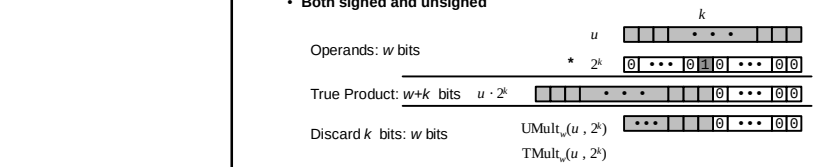
Relation

- **Signed multiplication gives same bit-level result as unsigned**
- **`up == (unsigned) p`**

Power-of-2 Multiply with Shift

Operation

- **`u << k` gives `u *`** 2^k
- **Both signed and unsigned**



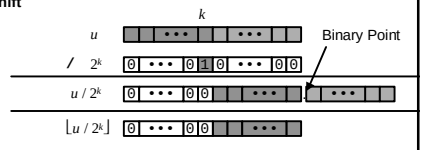
Examples

- **`u << 3 == u * 8`**
- **`u << 5 - u << 3 == u * 24`**
- **Most machines shift and add much faster than multiply**
  - Compiler will generate this code automatically

Unsigned Power-of-2 Divide with Shift

Quotient of Unsigned by Power of 2

- **`u >> k` gives** $\lfloor u / 2^k \rfloor$
- **Uses logical shift**

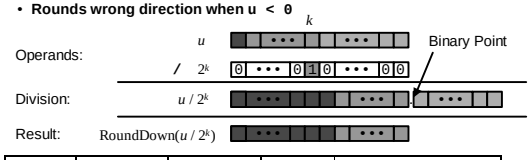


	Division	Computed	Hex	Binary
x	15213	15213	3B 6D	00111011 01101101
x >> 1	7606.5	7606	1D B6	**0**0011101 10110110
x >> 4	950.8125	950	03 B6	**0000**0011 10110110
x >> 8	59.4257813	59	00 3B	**00000000** 00111011

2's Comp Power-of-2 Divide with Shift

Quotient of Signed by Power of 2

- **u >> k gives** $\lfloor u / 2^k \rfloor$
- **Uses arithmetic shift**
- **Rounds wrong direction when u < 0**



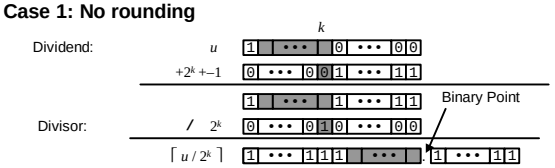
	Division	Computed	Hex	Binary
y	-15213	-15213	C4 93	11000100 10010011
y >> 1	-7606.5	-7607	E2 49	**1**1100010 01001001
y >> 4	-950.8125	-951	FC 49	**1111**1100 01001001
y >> 8	-59.4257813	-60	FF C4	**11111111** 11000100

Correct Power-of-2 Divide

Quotient of Negative Number by Power of 2

- **Want** $\lceil u / 2^k \rceil$ **(**Round Toward 0)
- **Compute as** $\lfloor (u+2^k-1) / 2^k \rfloor$
  - In C: `(u + (1<<k)-1) >> k`
  - Biases dividend toward 0

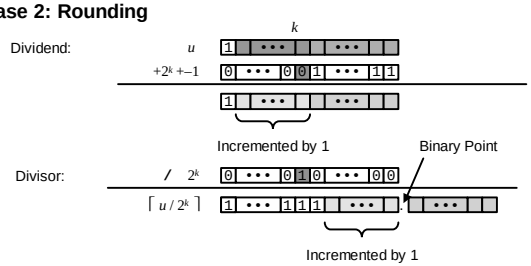
Case 1: No rounding



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

Properties of Unsigned Arithmetic

Unsigned Multiplication with Addition Forms Commutative Ring

- **Addition is commutative group**
- **Closed under multiplication**
  $0 \le \text{UMult}_w(u, v) \le 2^w - 1$
- **Multiplication Commutative**
  $\text{UMult}_w(u, v) = \text{UMult}_w(v, u)$
- **Multiplication is Associative**
  $\text{UMult}_w(t, \text{UMult}_w(u, v)) = \text{UMult}_w(\text{UMult}_w(t, u), v)$
- **1 is multiplicative identity**
  $\text{UMult}_w(u, 1) = u$
- **Multiplication distributes over addtion**
  $\text{UMult}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UMult}_w(t, u), \text{UMult}_w(t, v))$

Properties of Two's Comp. Arithmetic

Isomorphic Algebras

- **Unsigned multiplication and addition**
  - Truncating to *w* bits
- **Two's complement multiplication and addition**
  - Truncating to *w* bits

Both Form Rings

- **Isomorphic to ring of integers mod 2^w**

Comparison to Integer Arithmetic

- **Both are rings**
- **Integers obey ordering properties, e.g.,**

  $u > 0 \quad \Rightarrow \quad u + v > v$

  $u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$
- **These properties are not obeyed by two's complement arithmetic**

  TMax `+ 1` `==` *TMin*

  `15213 * 30426 == -10030` **(16-bit words)**

C Puzzle Answers

- **Assume machine with 32 bit word size, two's complement integers**
- ***TMin* makes a good counterexample in many cases**

• `x < 0`	Þ	`((x*2) < 0)`	**False:**	*TMin*
• `ux >= 0`			**True:**	0 = *UMin*
• `x & 7 == 7`	Þ	`(x<<30) < 0`	**True:**	$x_1 = 1$
• `ux > -1`			**False:**	0
• `x > y`	Þ	`-x < -y`	**False:**	`-1`, *TMin*
• `x * x >= 0`			**False:**	`30426`
• `x > 0 && y > 0`	Þ	`x + y > 0`	**False:**	*TMax*, *TMax*
• `x >= 0`	Þ	`-x <= 0`	**True:**	$-TMax < 0$
• `x <= 0`	Þ	`-x >= 0`	**False:**	*TMin*