15-213

"The course that gives CMU its Zip!"

Integer Arithmetic Operations
Sept. 7, 2000

Topics

- **Basic operations**
  - Addition, negation, multiplication
- **Programming Implications**
  - Consequences of overflow
  - Using shifts to perform power-of-2 multiply/divide

C Puzzles

- **Assume machine with 32 bit word size, two's complement integers**
- **For each of the following C expressions, either:**
  - Argue that is true for all argument values
  - Give example where not true

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

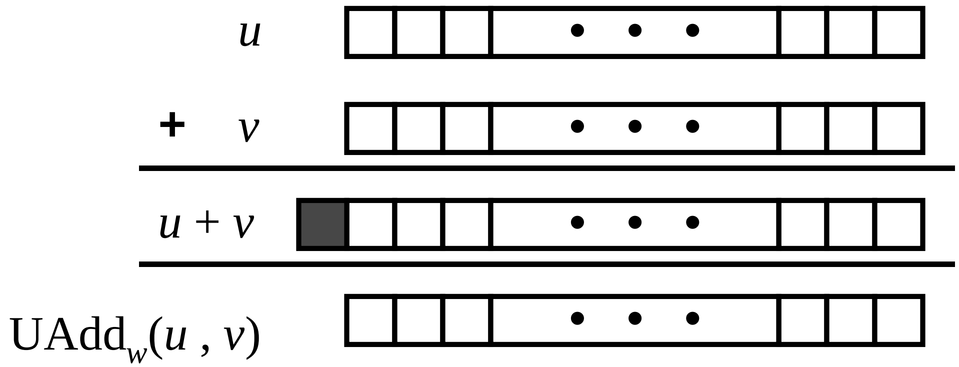
- `x < 0` $\Rightarrow$ `((x*2) < 0)`
- `ux >= 0`
- `x & 7 == 7` $\Rightarrow$ `(x<<30) < 0`
- `ux > -1`
- `x > y` $\Rightarrow$ `-x < -y`
- `x * x >= 0`
- `x > 0 && y > 0` $\Rightarrow$ `x + y > 0`
- `x >= 0` $\Rightarrow$ `-x <= 0`
- `x <= 0` $\Rightarrow$ `-x >= 0`

Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



Standard Addition Function

- **Ignores carry output**

Implements Modular Arithmetic

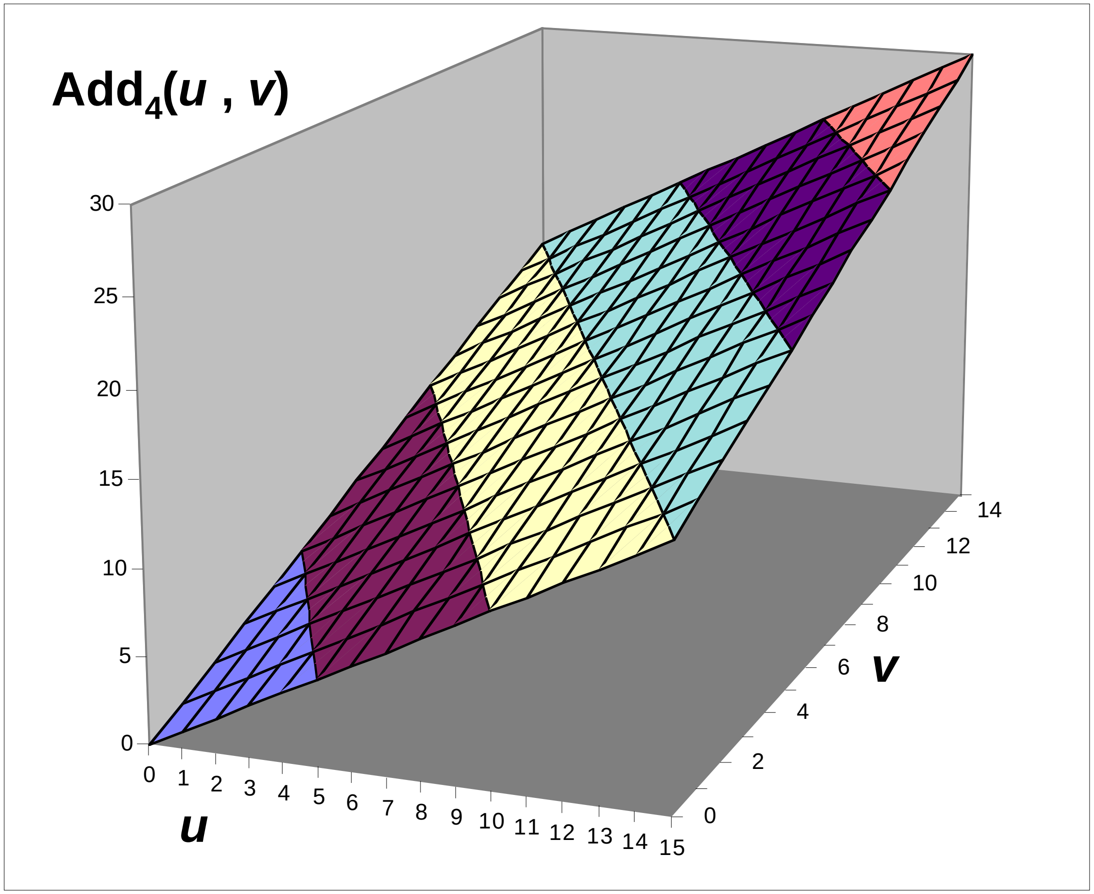
$s \quad = \quad \text{UAdd}_w(u\,,\,v) \quad = \quad u + v \mod 2^w$

$$UAdd_w(u,v) = \begin{cases} u+v & u+v < 2^w \\ u+v-2^w & u+v \geq 2^w \end{cases}$$

Visualizing Integer Addition

Integer Addition

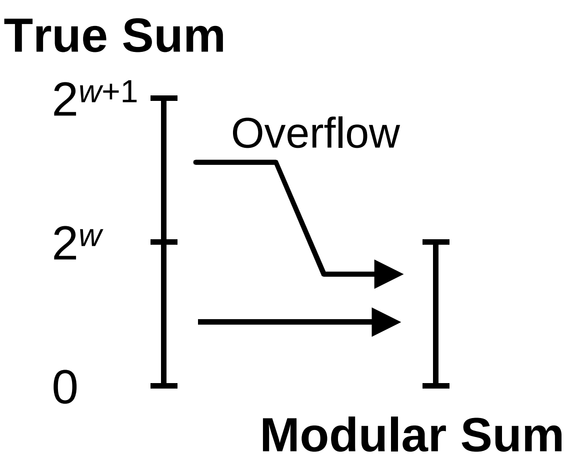
- 4-bit integers *u* and *v*
- Compute true sum $Add_4(u, v)$
- Values increase linearly with *u* and *v*
- Forms planar surface

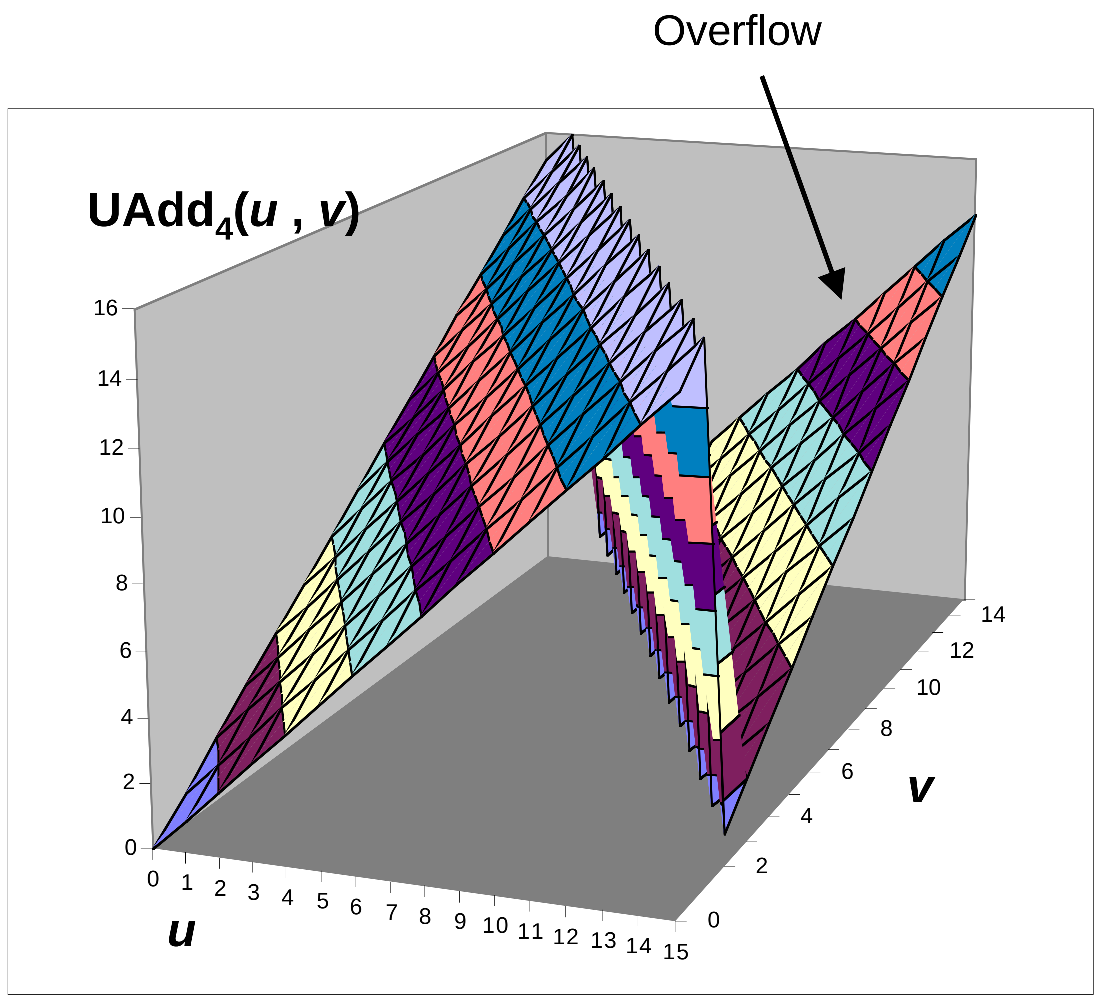


Visualizing Unsigned Addition

Wraps Around

- **If true sum ≥ 2^w**
- **At most once**





Mathematical Properties

Modular Addition Forms an *Abelian Group*

- **Closed under addition**

  $0 \leq \mathrm{UAdd}_w(u, v) \leq 2^w - 1$

- **Commutative**

  $\mathrm{UAdd}_w(u, v) = \mathrm{UAdd}_w(v, u)$

- **Associative**

  $\mathrm{UAdd}_w(t, \mathrm{UAdd}_w(u, v)) = \mathrm{UAdd}_w(\mathrm{UAdd}_w(t, u), v)$

- **0 is additive identity**

  $\mathrm{UAdd}_w(u, 0) = u$

- **Every element has additive inverse**

  – Let $\mathrm{UComp}_w(u) = 2^w - u$

  $\mathrm{UAdd}_w(u, \mathrm{UComp}_w(u)) = 0$

Detecting Unsigned Overflow

Task

- **Given** $s = \text{UAdd}_w(u, v)$
- **Determine if** $s = u + v$

Application

```
unsigned s, u, v;
s = u + v;
```

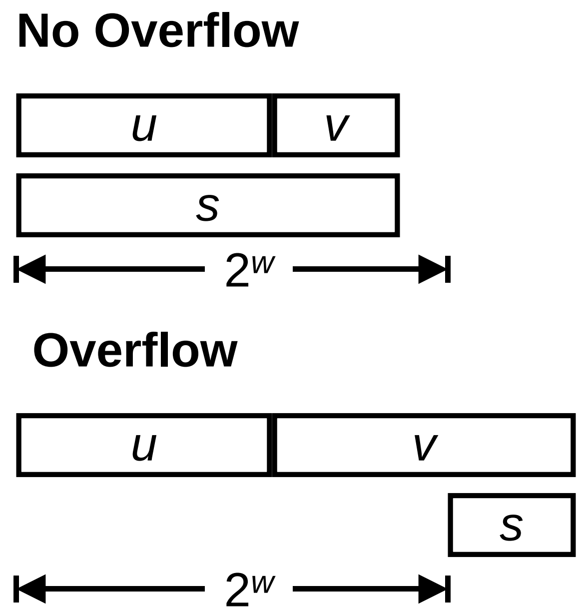
- **Did addition overflow?**

Claim

- **Overflow iff** $s < u$

```
ovf = (s < u)
```

- **Or symmetrically iff** $s < v$



Proof

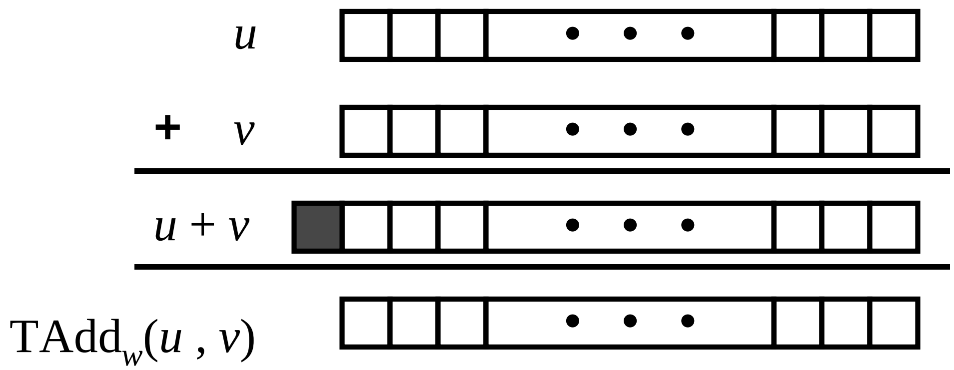
- **Know that** $0 \leq v < 2^w$
- **No overflow** $\Rightarrow \quad s = u + v \quad \geq \quad u + 0 = u$
- **Overflow** $\Rightarrow \quad s = u + v - 2^w \quad < \quad u + 0 = u$

Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



TAdd and UAdd have Identical Bit-Level Behavior

- **Signed vs. unsigned addition in C:**

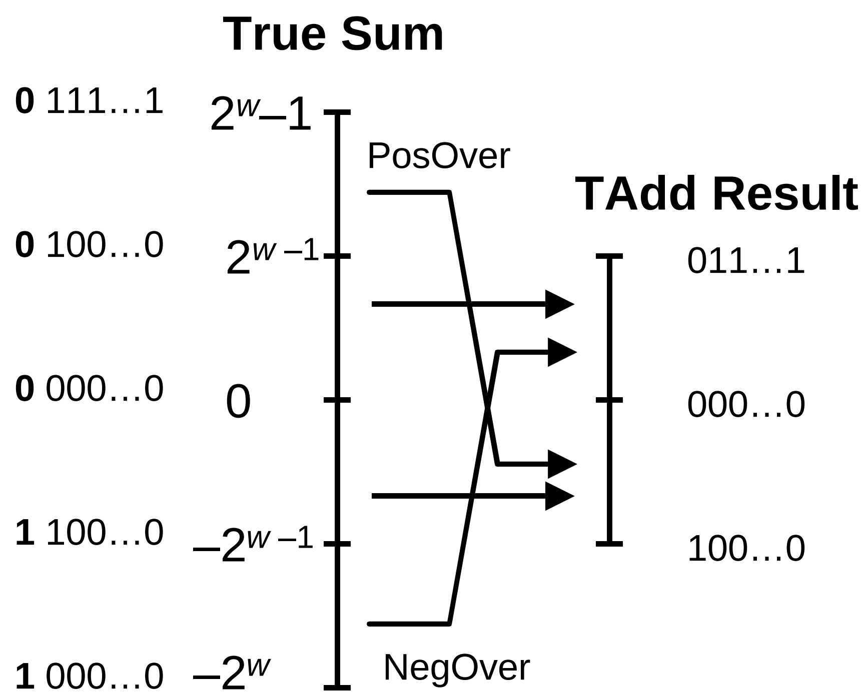
```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
 t = u + v
```

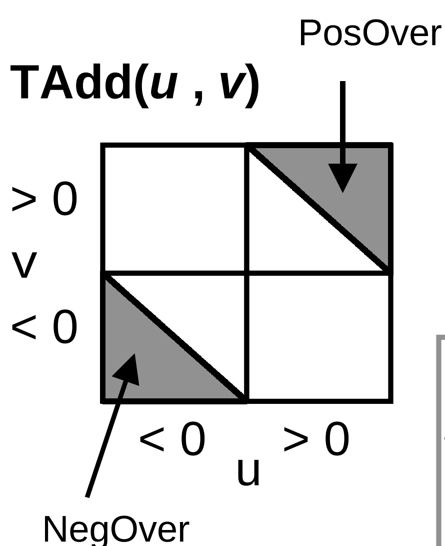
- **Will give** `s == t`

Characterizing TAdd

Functionality

- **True sum requires** *w*+1 **bits**
- **Drop off MSB**
- **Treat remaining bits as 2's comp. integer**





$$TAdd_w(u,v) = \begin{cases} u+v+2^w & u+v < TMin_w \text{ (NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^w & TMax_w < u+v \text{ (PosOver)} \end{cases}$$

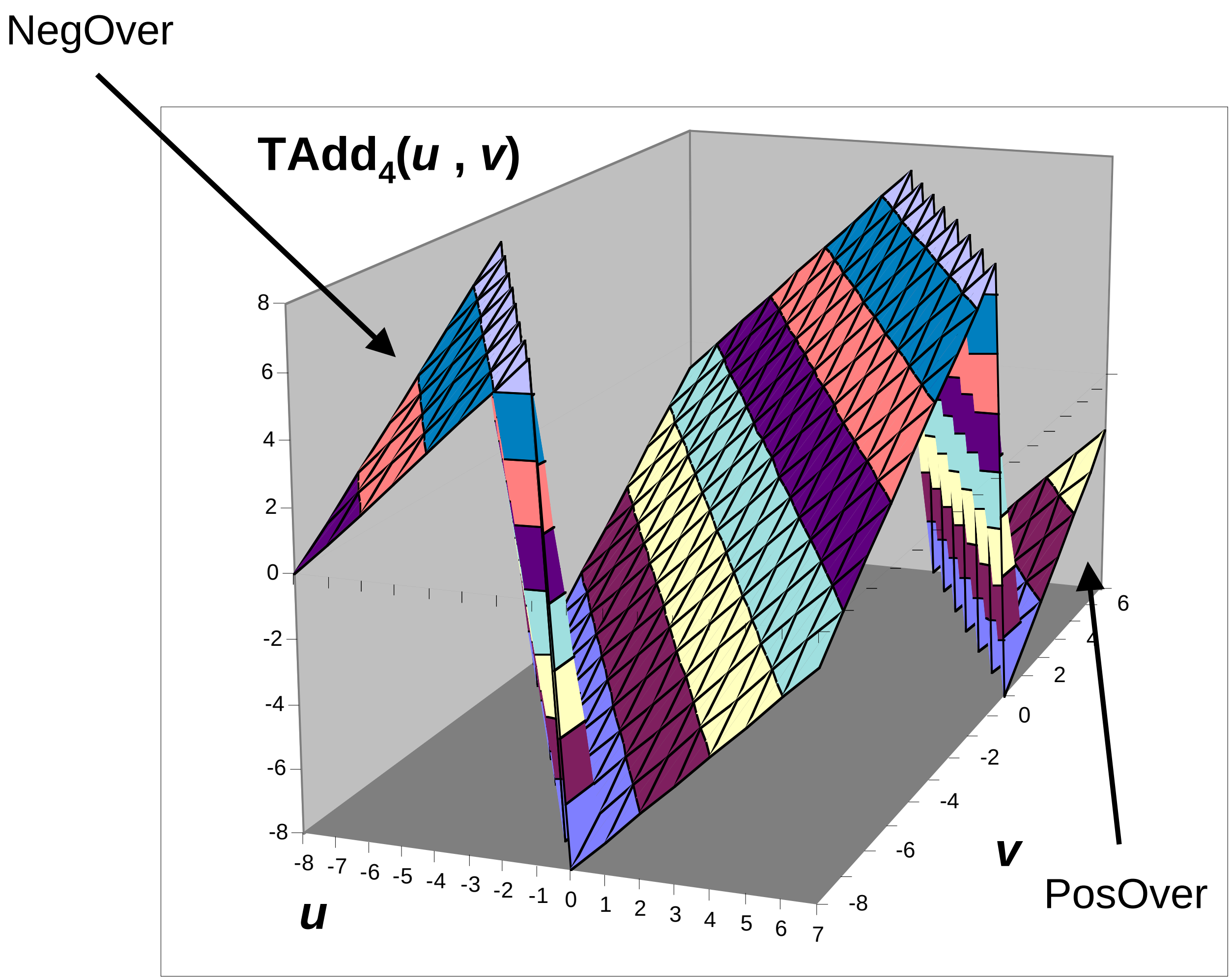
Visualizing 2's Comp. Addition

Values

- **4-bit two's comp.**
- **Range from -8 to +7**

Wraps Around

- **If sum ≥ 2^{w-1}**
  - Becomes negative
  - At most once
- **If sum < -2^{w-1}**
  - Becomes positive
  - At most once

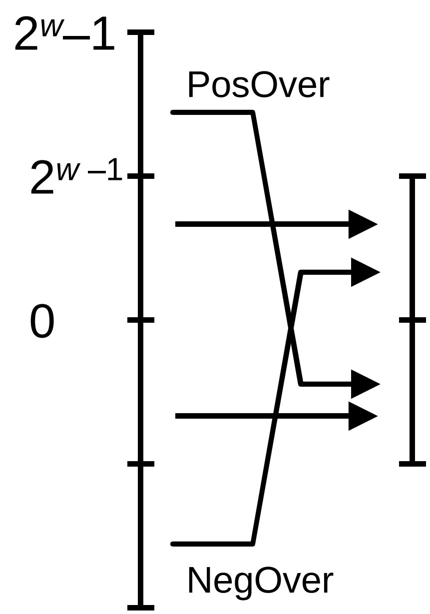


Detecting 2's Comp. Overflow

Task

- **Given** $s = \text{TAdd}_w(u, v)$
- **Determine if** $s = \text{Add}_w(u, v)$
- **Example**

```
int s, u, v;
s = u + v;
```



Claim

- **Overflow iff either:**

  $u, v < 0, s \geq 0$ (NegOver)

  $u, v \geq 0, s < 0$ (PosOver)

```
ovf = (u<0 == v<0) && (u<0 != s<0);
```

Proof

- **Easy to see that if** $u \geq 0$ **and** $v < 0$**, then** $TMin_w \leq u + v \leq TMax_w$
- **Symmetrically if** $u < 0$ **and** $v \geq 0$
- **Other cases from analysis of** TAdd

Mathematical Properties of TAdd

Isomorphic Algebra to UAdd

- $TAdd_w(u, v) = U2T(UAdd_w(T2U(u), T2U(v)))$
  - Since both have identical bit patterns

Two's Complement Under TAdd Forms a Group

- **Closed, Commutative, Associative, 0 is additive identity**
- **Every element has additive inverse**

  Let $TComp_w(u) = U2T(UComp_w(T2U(u))$

  $TAdd_w(u, TComp_w(u)) = 0$

$$TComp_w(u) = \begin{cases} -u & u \neq TMin_w \\ TMin_w & u = TMin_w \end{cases}$$

Two's Complement Negation

Mostly like Integer Negation

- TComp(*u*) = *–u*

***T**Min* **is Special Case**

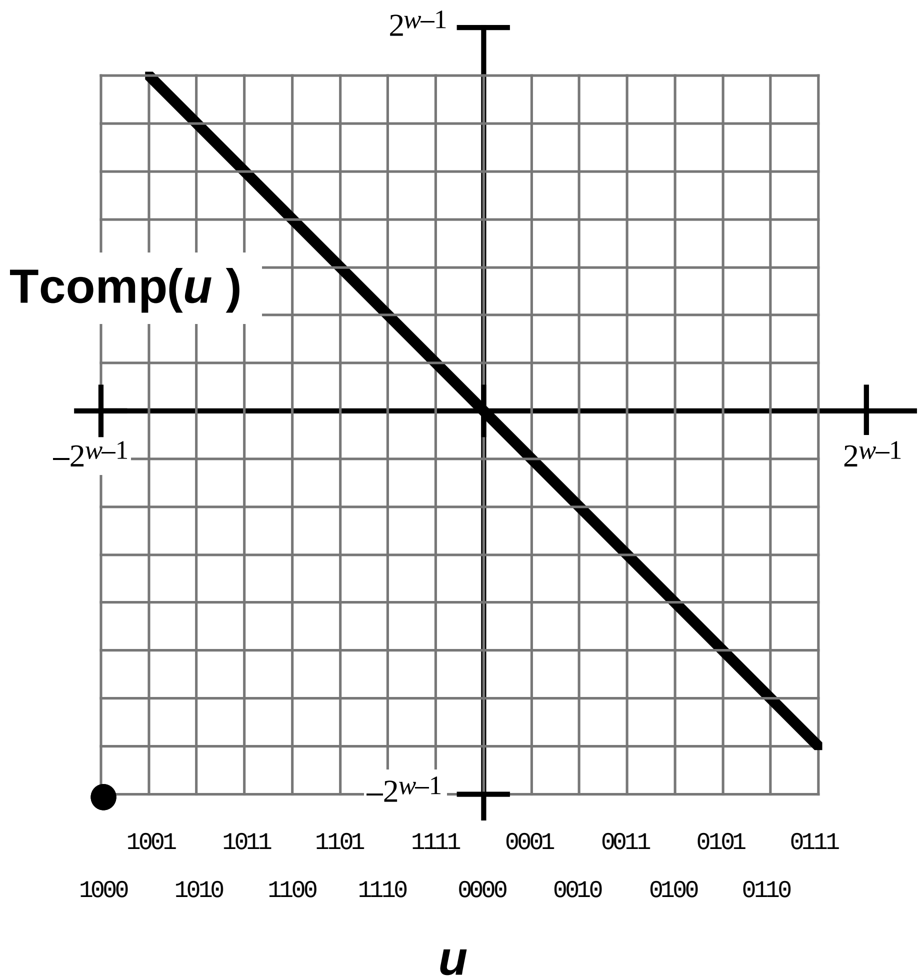
- TComp(*TMin*) = *TMin*

Negation in C is Actually TComp

```
mx = -x
```

- *mx* = TComp(*x*)
- **Computes additive inverse for** TAdd

```
x + -x == 0
```



Negating with Complement & Increment

In C

`~x + 1 == -x`

Complement

- **Observation:** `~x + x == 1111…11`$_2$ `== -1`

x	1	0	0	1	1	1	0	1
+ ~x	0	1	1	0	0	0	1	0
-1	1	1	1	1	1	1	1	1

Increment

- `~x + ~~x~~ + (-~~x~~ + 1)   ==   -~~1~~ + (-x + ~~1~~)`
- `~x + 1   ==   -x`

Warning: Be cautious treating `int`'s as integers

- **OK here: We are using group properties of** TAdd **and** TComp

Comp. & Incr. Examples

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 1001001**1**
y	-15213	C4 93	11000100 10010011

TMin

	Decimal	Hex	Binary
TMin	-32768	80 00	10000000 00000000
~TMin	32767	7F FF	01111111 11111111
~TMin+1	-32768	80 00	10000000 00000000

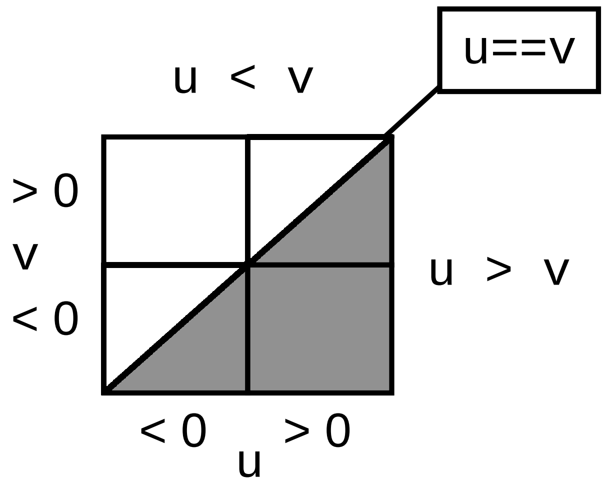
0

	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

Comparing Two's Complement Numbers

Task

- **Given** signed numbers *u*, *v*
- Determine whether or not *u* > *v*
  - Return 1 for numbers in shaded region below



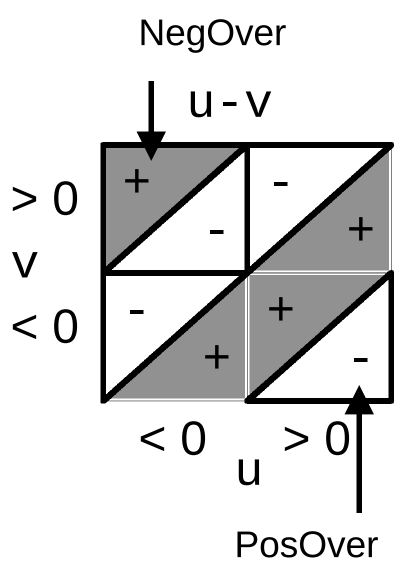
Bad Approach

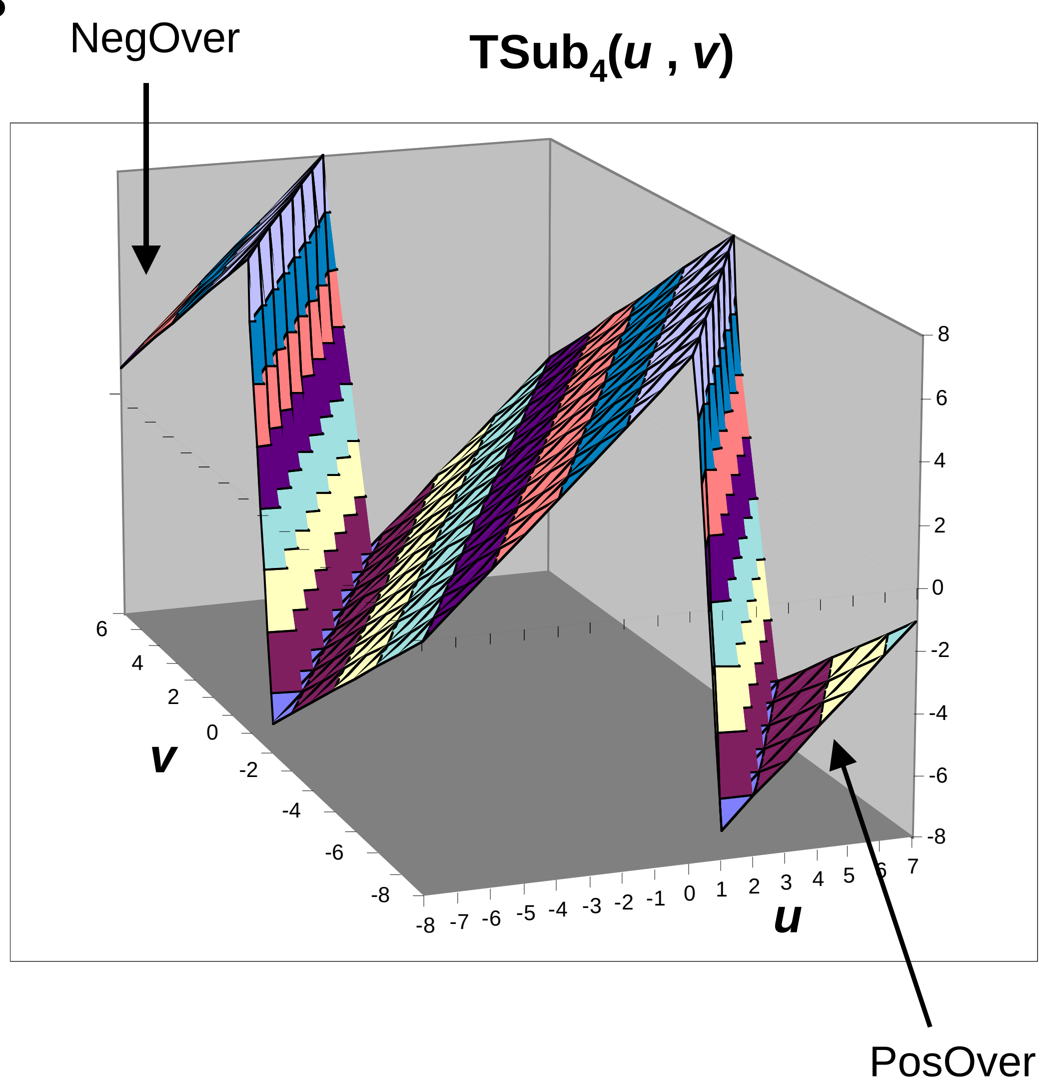
- Test `(u–v) > 0`
  - TSub(*u*,*v*) = TAdd(*u*, TComp(*v*))
- Problem: Thrown off by either Negative or Positive Overflow

Comparing with TSub

Will Get Wrong Results

- **NegOver:** `u < 0`, `v > 0`
  - but `u-v > 0`
- **PosOver:** `u > 0`, `v < 0`
  - but `u-v < 0`

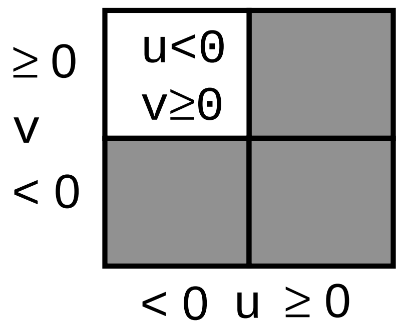




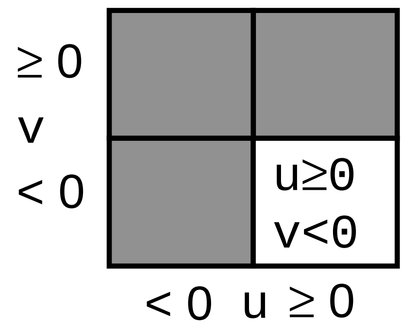
Working Around Overflow Problems

Partition into Three Regions

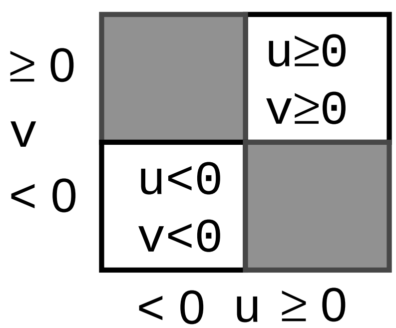
- $u < 0, v \geq 0 \quad \Rightarrow \quad u < v$

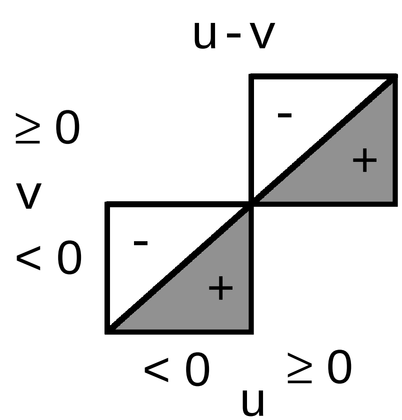


- $u \geq 0, v < 0 \quad \Rightarrow \quad u > v$



- u, v same sign $\Rightarrow$ `u-v` does not overflow
  - **Can safely use test** `(u–v) > 0`





Multiplication

Computing Exact Product of *w*-bit numbers *x*, *y*

- **Either signed or unsigned**

Ranges

- **Unsigned:** $0 \le x * y \le (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
  - Up to $2w$ bits
- **Two's complement min:** $x * y \ge (-2^{w-1}) * (2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}$
  - Up to $2w-1$ bits
- **Two's complement max:** $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
  - Up to $2w$ bits, but only for $TMin_w^2$

Maintaining Exact Results

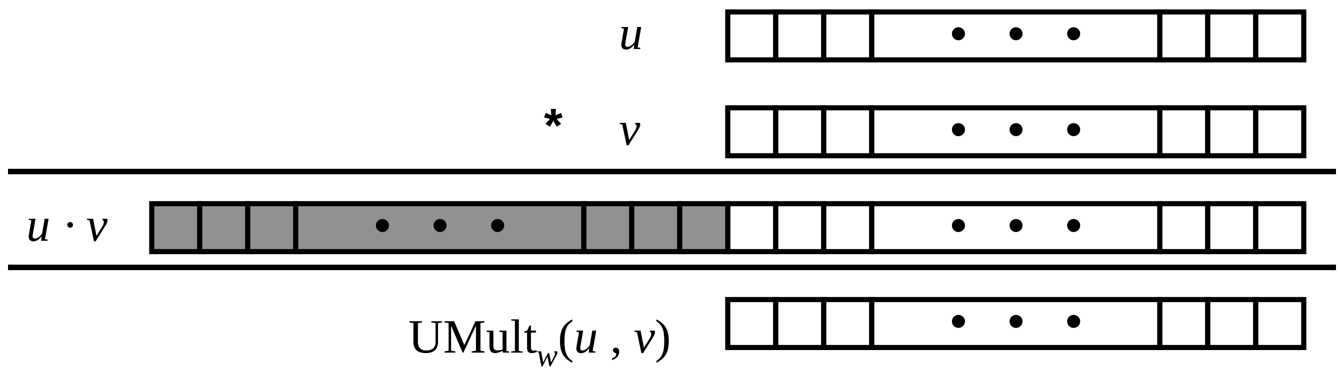
- **Would need to keep expanding word size with each product computed**
- **Done in software by "arbitrary precision" arithmetic packages**
- **Also implemented in Lisp, ML, and other "advanced" languages**

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



Standard Multiplication Function

- **Ignores high order *w* bits**

Implements Modular Arithmetic

$\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$

Unsigned vs. Signed Multiplication

Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

- **Truncates product to *w*-bit number** *up* **=** $\text{UMult}_w(ux, uy)$
- **Simply modular arithmetic**

  $up = ux \cdot uy \mod 2^w$

Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

- **Compute exact product of two *w*-bit numbers *x*, *y***
- **Truncate result to *w*-bit number** *p* = $\text{TMult}_w(x, y)$

Relation

- **Signed multiplication gives same bit-level result as unsigned**
- `up == (unsigned) p`

Multiplication Examples

```
short int x   =  15213;
int     txx   =  ((int) x) * x;
int      xx   =  (int) (x * x);
int     xx2   =  (int) (2 * x * x);
```

```
x   =      15213:                     00111011 01101101
txx =  231435369: 00001101 11001011 01101100 01101001
xx  =      27753: 00000000 00000000 01101100 01101001
xx2 =     -10030: 11111111 11111111 11011000 11010010
```

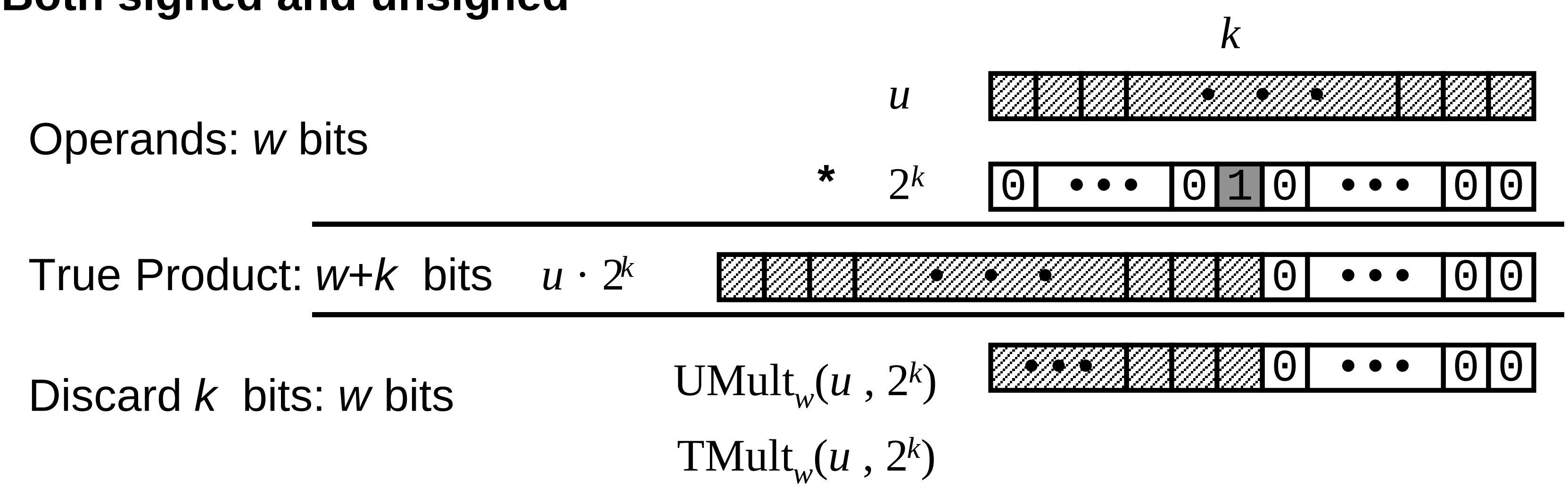
Observations

- **Casting order important**
  - If either operand **`int`**, will perform **`int`** multiplication
  - If both operands **`short int`**, will perform **`short int`** multiplication
- **Really is modular arithmetic**
  - Computes for `xx`: `15213`^{2} mod `65536 = 27753`
  - Computes for `xx2`: `(int) 55506U = -10030`
- **Note that `xx2 == (xx << 1)`**

Power-of-2 Multiply with Shift

Operation

- **u << k gives u * 2^k**
- **Both signed and unsigned**



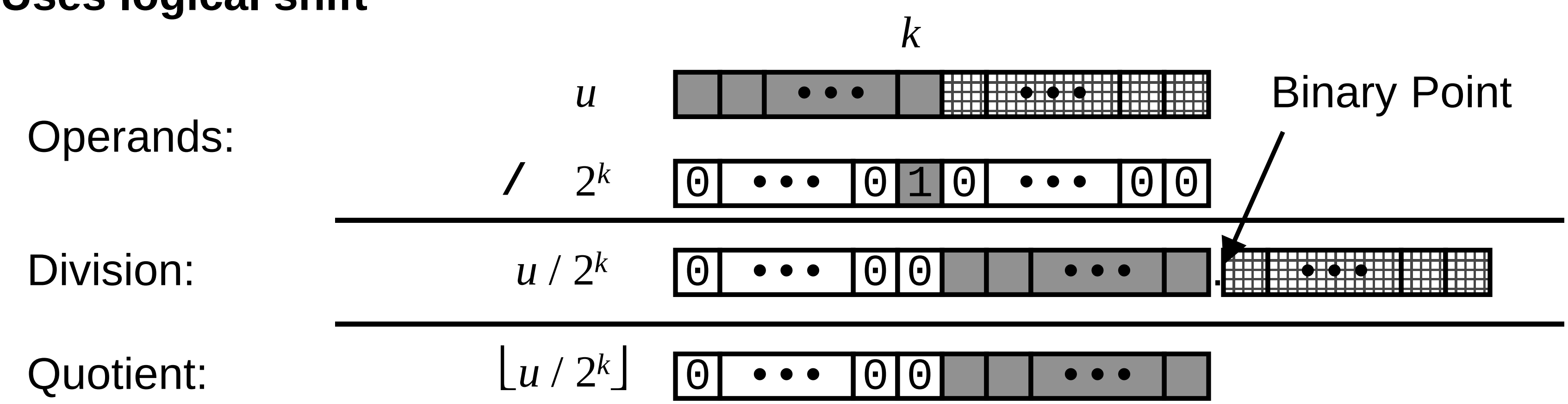
Examples

- `u << 3               ==   u * 8`
- `u << 5 - u << 3      ==   u * 24`
- **Most machines shift and add much faster than multiply**
  - Compiler will generate this code automatically

Unsigned Power-of-2 Divide with Shift

Quotient of Unsigned by Power of 2

- `u >> k` gives $\lfloor u / 2^k \rfloor$
- Uses logical shift

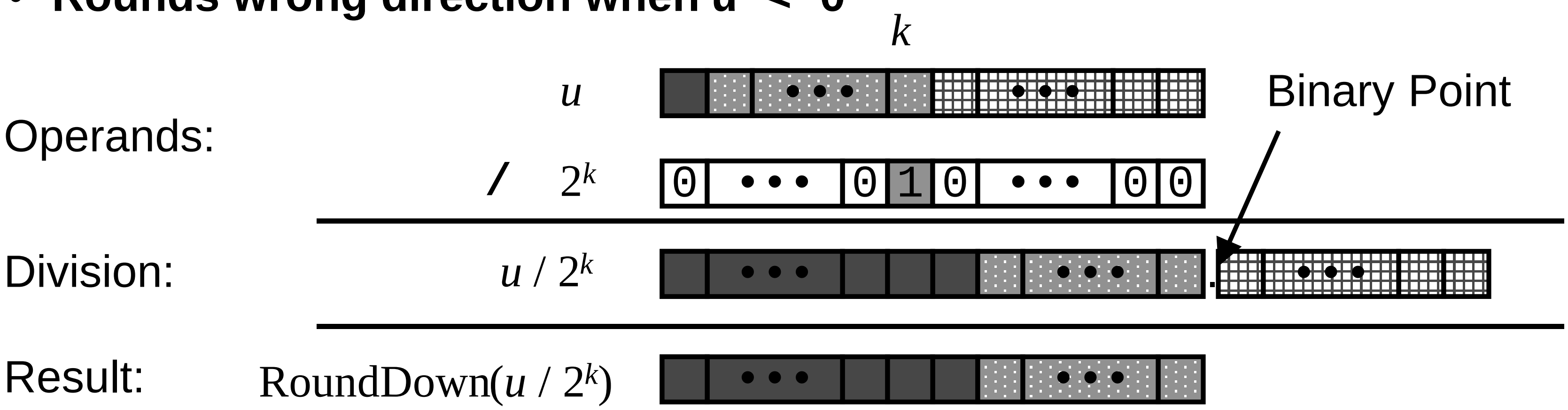


	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	**`0`**`0011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	**`0000`**`0011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	**`00000000`** `00111011`

2's Comp Power-of-2 Divide with Shift

Quotient of Signed by Power of 2

- **`u >> k` gives $\lfloor$ `u / ` 2^k $\rfloor$**
- **Uses arithmetic shift**
- **Rounds wrong direction when `u < 0`**



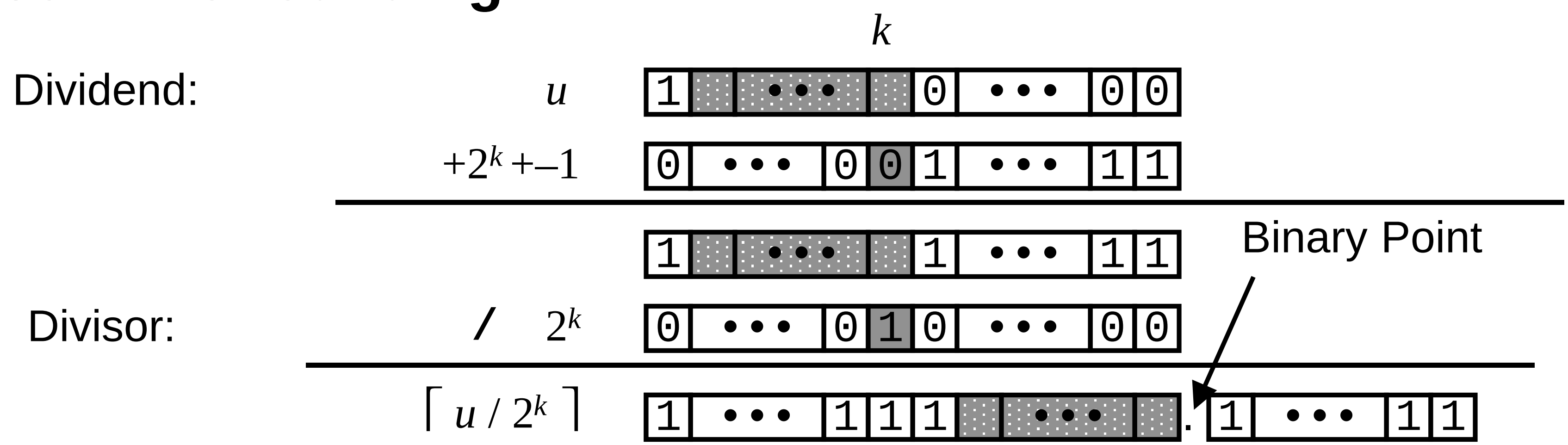
	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	**`1`**`1100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	**`1111`**`1100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	**`11111111`** `11000100`

Correct Power-of-2 Divide

Quotient of Negative Number by Power of 2

- **Want** $\lceil$ **`u / `** 2^k $\rceil$ **(**Round Toward 0)
- **Compute as** $\lfloor$ **`(u+`**2^k**`-1)/ `** 2^k $\rfloor$
  - In C: **`(u + (1<<k)-1) >> k`**
  - Biases dividend toward 0

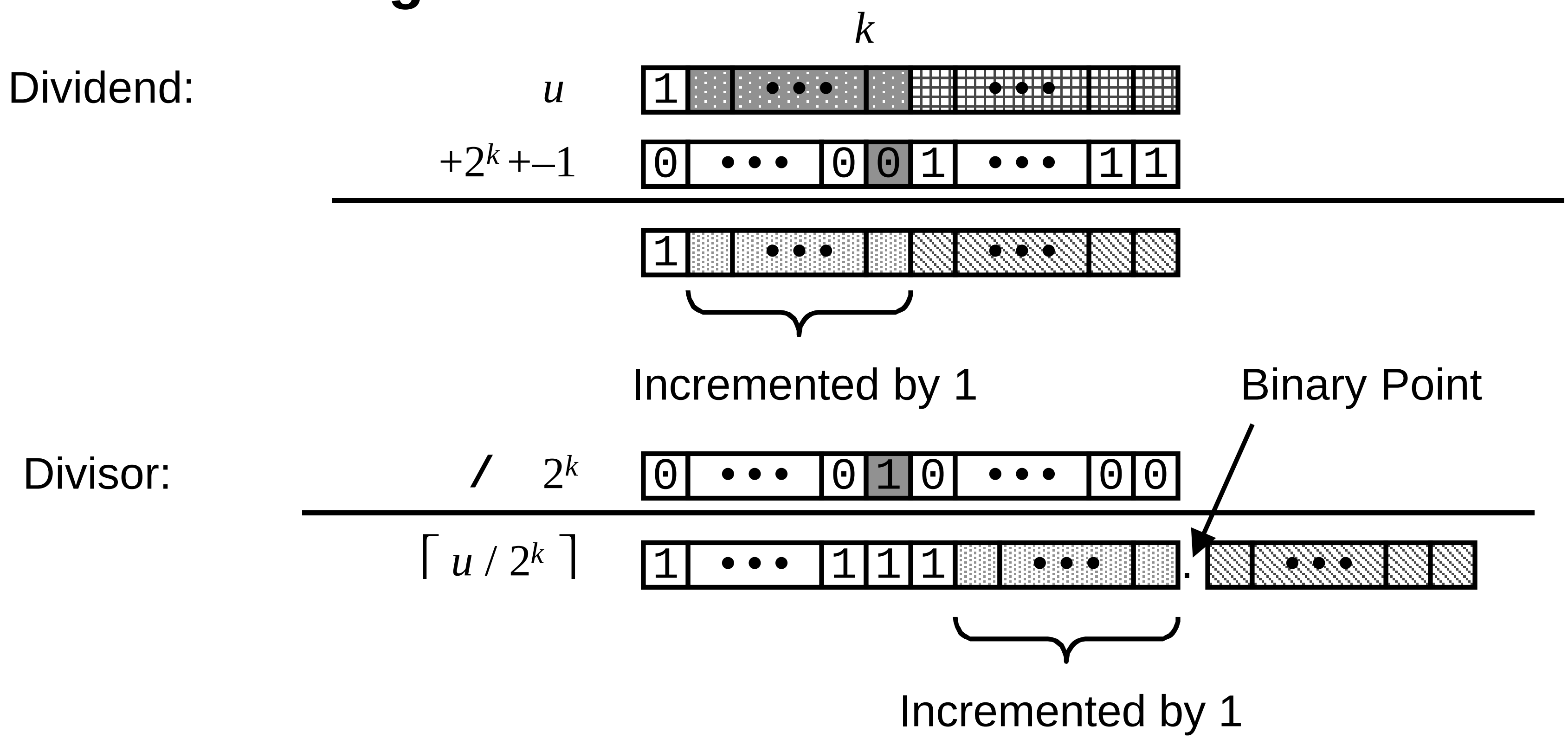
Case 1: No rounding



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

Correct Power-of-2 Divide Examples

	$y/2^k$	Computed	Hex	Binary
y	-15213	-15213	C4 93	11000100 10010011
y+1		-15212	C4 94	11000100 10010*100*
(y+1)>>1	-7606.5	-7606	E2 4A	**1**1100010 010010*10*
y	-15213	-15213	C4 93	11000100 10010011
y+15		-15197	C4 A2	11000100 10*100010*
(y+15)>>4	-950.8125	-950	FC 4A	**1111**1100 010010*10*
y	-15213	-15213	C4 93	11000100 10010011
y+255		-14958	C5 92	1100010*1* *10010010*
(y+255)>>8	-59.4257813	-59	FF C5	**11111111** 1100010*1*

Properties of Unsigned Arithmetic

Unsigned Multiplication with Addition Forms Commutative Ring

- **Addition is commutative group**
- **Closed under multiplication**

  $0 \leq \text{UMult}_w(u, v) \leq 2^w - 1$
- **Multiplication Commutative**

  $\text{UMult}_w(u, v) = \text{UMult}_w(v, u)$
- **Multiplication is Associative**

  $\text{UMult}_w(t, \text{UMult}_w(u, v)) = \text{UMult}_w(\text{UMult}_w(t, u), v)$
- **1 is multiplicative identity**

  $\text{UMult}_w(u, 1) = u$
- **Multiplication distributes over addtion**

  $\text{UMult}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UMult}_w(t, u), \text{UMult}_w(t, v))$

Properties of Two's Comp. Arithmetic

Isomorphic Algebras

- **Unsigned multiplication and addition**
  - Truncating to *w* bits
- **Two's complement multiplication and addition**
  - Truncating to *w* bits

Both Form Rings

- **Isomorphic to ring of integers mod** 2^w

Comparison to Integer Arithmetic

- **Both are rings**
- **Integers obey ordering properties, e.g.,**

$u > 0 \quad \Rightarrow \quad u + v > v$

$u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$

- **These properties are not obeyed by two's complement arithmetic**

TMax **`+ 1`** **`==`** *TMin*

`15213 * 30426 == -10030` **(16-bit words)**

C Puzzle Answers

- **Assume machine with 32 bit word size, two's complement integers**
- *TMin* **makes a good counterexample in many cases**

• **`x < 0`**	⇒	**`((x*2) < 0)`**	**False:**	*TMin*
• **`ux >= 0`**			**True:**	0 = *UMin*
• **`x & 7 == 7`**	⇒	**`(x<<30) < 0`**	**True:**	$x_1 = 1$
• **`ux > -1`**			**False:**	0
• **`x > y`**	⇒	**`-x < -y`**	**False:**	`-1`, *TMin*
• **`x * x >= 0`**			**False:**	`65535`
			`(x*x = -131071)`	
• **`x > 0 && y > 0`**	⇒	**`x + y > 0`**	**False:**	*TMax*, *TMax*
• **`x >= 0`**	⇒	**`-x <= 0`**	**True:**	–*TMax* < 0
• **`x <= 0`**	⇒	**`-x >= 0`**	**False:**	*TMin*