15-213

"The course that gives CMU its Zip!"

Floating Point Arithmetic
Sept. 24, 1998

Topics

- IEEE Floating Point Standard
- Rounding
- Floating Point Operations
- Mathematical properties
- Alpha floating point

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that is true for all argument values
  - Explain why not true

```
int x = …;

float f = …;

double d = …;
```

Assume neither d nor f is NAN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0          ((d*2) < 0.0)`
- `d > f            -f < -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

IEEE Floating Point

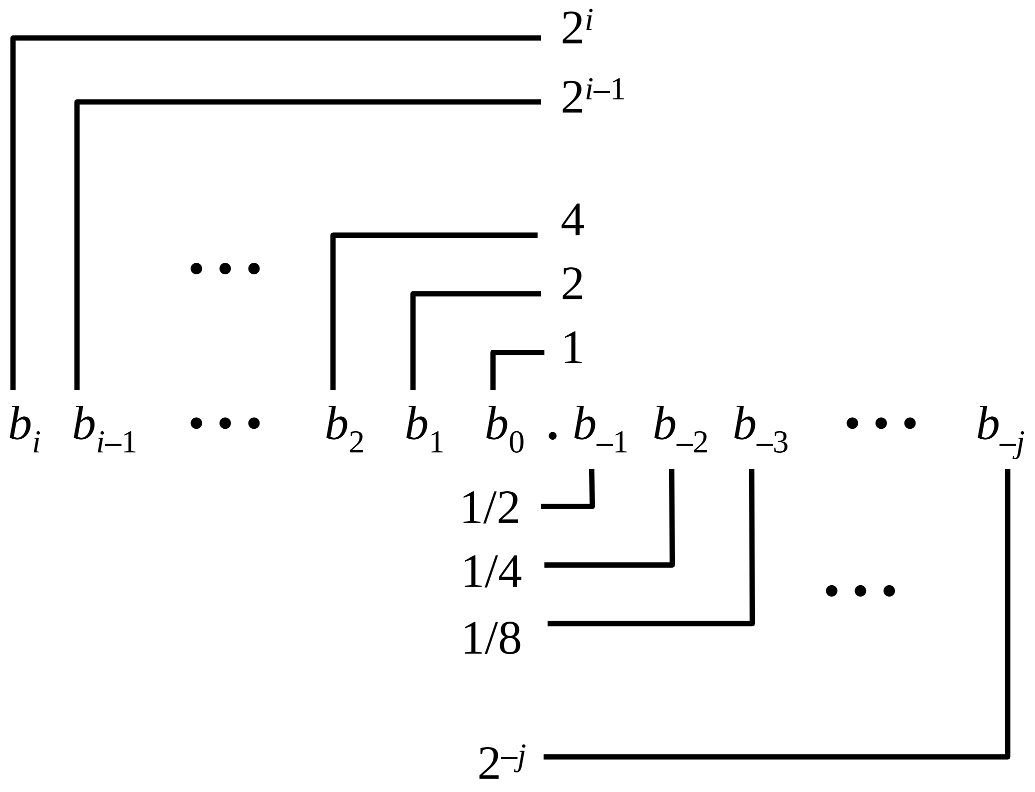
IEEE Standard 754

- **Estabilished in 1985 as uniform standard for floating point arithmetic**
  - Before that, many idiosyncratic formats
- **Supported by all major CPUs**

Driven by Numerical Concerns

- **Nice standards for rounding, overflow, underflow**
- **Hard to make go fast**
  - Numercial analysts predominated over hardware types in defining standard

Fractional Binary Numbers



Representation

- Bits to right of “binary point” represent fractional powers of 2
- Represents rational number:

$$\sum_{k=-j}^{i} b_k \; 2^k$$

Fractional Binary Number Examples

Value	Representation
5-3/4	101.11_2
2-7/8	10.111_2
63/64	0.111111_2

Observation

- **Divide by 2 by shifting right**
- **Numbers of form $0.111111\ldots_2$ just below 1.0**
  - Use notation 1.0 – ε

Limitation

- **Can only exactly represent numbers of the form** $x/2^k$
- **Other numbers have repeating bit representations**

Value	Representation
1/3	$0.0101010101[01]\ldots_2$
1/5	$0.001100110011[0011]\ldots_2$
1/10	$0.0001100110011[0011]\ldots_2$

Floating Point Representation

Numerical Form

- $-1^{s}\ m\ \ 2^{E}$
  - Sign bit *s* determines whether number is negative or positive
  - Mantissa *m* normally a fractional value in range [1.0,2.0).
  - Exponent *E* weights value by power of two

Encoding

s	exp	significand

- **MSB is sign bit**
- **Exp field encodes *E***
- **Significand field encodes *m***

Sizes

- **Single precision: 8 exp bits, 23 significand bits**
  - 32 bits total
- **Double precision: 11 exp bits, 52 significand bits**
  - 64 bits total

"Normalized" Numeric Values

Condition

- **`exp` ≠ `000…0` and `exp` ≠ `111…1`**

Exponent coded as *biased* value

E* = *Exp* – *Bias

- *Exp* : unsigned value denoted by **`exp`**
- *Bias* : Bias value
  - Single precision: 127
  - Double precision: 1023

Mantissa coded with implied leading 1

$m = 1.\texttt{xxx}\ldots\texttt{x}_2$

- **`xxx…x`**: bits of significand
- Minimum when **`000…0`** (m = 1.0)
- Maximum when **`111…1`** ($m = 2.0 - \varepsilon$)
- Get extra leading bit for "free"

Normalized Encoding Example

Value

`Float F = 15213.0;`

- $15213_{10} = 11101101101101_2 = 1.1101101101101_2$ X 2^{13}

Significand

m = $1.\underline{1101101101101}_2$

sig = $\underline{\mathbf{1101101101101}}\mathbf{0000000000}_2$

Exponent

E = **13**

Bias = **127**

Exp = **140 = 10001100_2**

Floating Point Representation (Class 02):

Hex: **4 6 6 D B 4 0 0**
Binary: **0100 0110 0110 1101 1011 0100 0000 0000**

140: **100 0110 0**

15213: ***1*110 1101 1011 01**

Denormalized Values

Condition

- **`exp` = `000…0`**

Value

- **Exponent value *E* = –*Bias* + 1**
- **Mantissa value *m* =** `0.xxx…x`$_2$
  - **`xxx…x`:** bits of significand

Cases

- **`exp` = `000…0`, `significand` = `000…0`**
  - Represents value 0
  - Note that have distinct values +0 and –0
- **`exp` = `000…0`, `significand` ≠ `000…0`**
  - Numbers very close to 0.0
  - Lose precision as get smaller
  - “Gradual underflow”

Interesting Numbers

Description	**Exp**	**Significand**	**Numeric Value**
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}}$ X $2^{-\{126,1022\}}$
• Single ≈ 1.4 X 10^{-45}			
• Double ≈ 4.9 X 10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon)$ X $2^{-\{126,1022\}}$
• Single ≈ 1.18 X 10^{-38}			
• Double ≈ 2.2 X 10^{-308}			
Smallest Pos. Normalized	00…01	00…00	1.0 X $2^{-\{126,1022\}}$
• Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon)$ X $2^{\{127,1023\}}$
• Single ≈ 3.4 X 10^{38}			
• Double ≈ 1.8 X 10^{308}			

Memory Referencing Bug Example

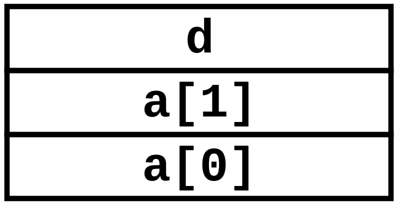
From Class 01

```
main ()
{
  long int a[2];
  double d = 3.14;
  a[2] = 1073741824; /* Out of bounds reference */
  printf("d = %.15g\n", d);
  exit(0);
}
```

	Alpha	MIPS	Sun
`-g`	`5.30498947741318e-315`	`3.1399998664856`	`3.14`
`-O`	`3.14`	`3.14`	`3.14`

Referencing Bug on Alpha

Alpha Stack Frame (-g)



d

a[1]

a[0]

```
long int a[2];
double d = 3.14;
a[2] = 1073741824;
```

Optimized Code

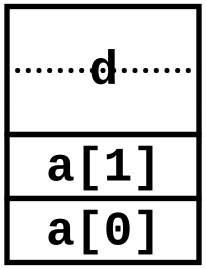
- Double d stored in register
- Unaffected by errant write

Alpha -g

- `1073741824` = `0x40000000` = 2^{30}
- **Overwrites all 8 bytes with value `0x0000000040000000`**
- **Denormalized value** 2^{30} X (smallest denorm 2^{-1074}) **=** 2^{-1044}
- $\approx 5.305 \times 10^{-315}$

Referencing Bug on MIPS

MIPS Stack Frame (-g)



```
long int a[2];
double d = 3.14;
a[2] = 1073741824;
```

MIPS -g

- **Overwrites lower 4 bytes with value `0x40000000`**
- **Original value `3.14` represented as `0x40091eb851eb851f`**
- **Modified value represented as `0x40091eb840000000`**
- *Exp* = 1024 *E* = 1024–1023 = 1
- **Mantissa difference:** $\mathtt{.0000011eb851f}_{16}$
- **Integer value:** $11eb851f_{16} = 300{,}647{,}711_{10}$
- **Difference =** $2^{1} \times 2^{-52} \times 300{,}647{,}711 \approx 1.34 \times 10^{-7}$
- **Compare to** 3.140000000 – 3.139999866 = 0.000000134

Special Values

Condition

- **`exp` = `111…1`**

Cases

- **`exp` = `111…1`, `significand` = `000…0`**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞
- **`exp` = `111…1`, `significand` ≠ `000…0`**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞
  - No fixed meaning assigned to significand bits

Special Properties of Encoding

FP Zero Same as Integer Zero

- **All bits = 0**

Can (Almost) Use Unsigned Integer Comparison

- **Must first compare sign bits**
- **NaNs problematic**
  - Will be greater than any other values
  - What should comparison yield?
- **Otherwise OK**
  - Denorm vs. normalized
  - Normalized vs. infinity

Floating Point Operations

Conceptual View

- **First compute exact result**
- **Make it fit into desired precision**
  - Possibly overflow if exponent too large
  - Possibly round to fit into significand

Rounding Modes (illustrate with $ rounding)

	$1.40	**$1.60**	**$1.50**	**$2.50**	**–$1.50**
• **Zero**	$1.00	$2.00	$1.00	$2.00	–$1.00
• $-\infty$	$1.00	$2.00	$1.00	$2.00	–$2.00
• $+\infty$	$1.00	$2.00	$2.00	$3.00	–$1.00
• **Nearest Even (default)**	$1.00	$2.00	$2.00	$2.00	–$2.00

A Closer Look at Round-To-Even

Default Rounding Mode

- **Hard to get any other kind without dropping into assembly**
- **All others are statistically biased**
  - Sum of set of positive numbers will consistently be over- or under-estimated

Applying to Other Decimal Places

- **When exactly halfway between two possible values**
  - Round so that least signficant digit is even
- **E.g., round to nearest hundredth**

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

Binary Fractional Numbers

- **“Even” when least significant bit is `0`**
- **Half way when bits to right of rounding position = $\texttt{100}\ldots_2$**

Examples

- **Round to nearest 1/4 (2 bits right of binary point)**

Value	Binary	Rounded	Action	Rounded Value
2-3/32	10.00011_2	10.00_2	(<1/2—down)	2
2-3/16	10.00110_2	10.01_2	(>1/2—up)	2-1/4
2-7/8	10.11100_2	11.00_2	(1/2—up)	3
2-5/8	10.10100_2	10.10_2	(1/2—down)	2-1/2

FP Multiplication

Operands

$(-1)^{s1}\ m1\ \ 2^{E1}$

$(-1)^{s2}\ m2\ \ 2^{E2}$

Exact Result

$(-1)^{s}\ m\ \ 2^{E}$

- **Sign** s**:** $s1$ ^ $s2$
- **Mantissa** m**:** $m1 * m2$
- **Exponent** E**:** $E1 + E2$

Fixing

- **Overflow if** E **out of range**
- **Round** m **to fit significand precision**

Implementation

- **Biggest chore is multiplying mantissas**

FP Addition

Operands

$(-1)^{s1}\ m1\ \ 2^{E1}$

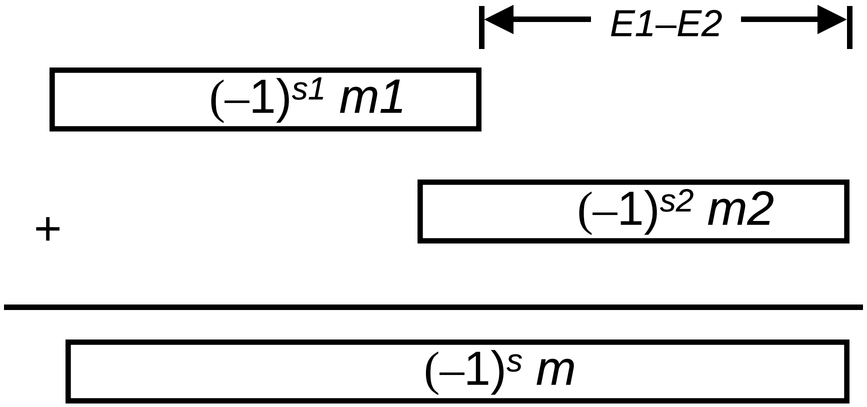
$(-1)^{s2}\ m2\ \ 2^{E2}$

- **Assume** $E1 > E2$

Exact Result

$(-1)^{s}\ m\ \ 2^{E}$

- **Sign** s**, mantissa** m**:**
  - Result of signed align & add
- **Exponent** E**:** $E1 - E2$



Fixing

- **Shift** m **right, increment** E **if** $m \geq 2$
- **Shift** m **left** k **positions, decrement** E **by** k **if** $m < 1$
- **Overflow if** E **out of range**
- **Round** m **to fit significand precision**

Mathematical Properties of FP Add

Compare to those of Abelian Group

• **Closed under addition?**	**YES**
– But may generate infinity or NaN	
• **Commutative?**	**YES**
• **Associative?**	**NO**
– Overflow and inexactness of rounding	
• **0 is additive identity?**	**YES**
• **Every element has additive inverse**	**ALMOST**
– Except for infinities & NaNs	

Montonicity

• $a \leq b$ $a+c \leq b+c$**?**	**ALMOST**
– Except for infinities & NaNs	

Algebraic Properties of FP Mult

Compare to Commutative Ring

- **Closed under multiplication? YES**
  - But may generate infinity or NaN
- **Multiplication Commutative? YES**
- **Multiplication is Associative? NO**
  - Possibility of overflow, inexactness of rounding
- **1 is multiplicative identity? YES**
- **Multiplication distributes over addtion? NO**
  - Possibility of overflow, inexactness of rounding

Montonicity

- $a \leq b$ & $c \geq 0$ $\quad a * c \leq b * c$**? ALMOST**
  - Except for infinities & NaNs

Floating Point in C

C Supports Two Levels

`float`	**single precision**
`double`	**double precision**

Conversions

- **Casting between `int`, `float`, and `double` changes numeric values**
- **`Double` or `float` to `int`**
  - Truncates fractional part
  - Like rounding toward zero
  - Not defined when out of range
    - » Generally saturates to TMin or TMax
- **`int` to `double`**
  - Exact conversion, as long as int has ≤ 54 bit word size
- **`int` to `float`**
  - Will round according to rounding mode

Answers to Floating Point Puzzles

```
int x = …;

float f = …;

double d = …;
```

Assume neither d nor `f` is NAN

`x == (int)(float) x`		**No: 24 bit mantissa**
`x == (int)(double) x`		**Yes: 53 bit mantissa**
`f == (float)(double) f`		**Yes: increases precision**
`d == (float) d`		**No: looses precision**
`f == -(-f);`		**Yes: Just change sign bit**
`2/3 == 2/3.0`		**No: 2/3 == 1**
`d < 0.0`	`((d*2) < 0.0)`	**Yes!**
`d > f`	`-f < -d`	**Yes!**
`d * d >= 0.0`		**Yes!**
`(d+f)-d == f`		**No: Not associative**

Alpha Floating Point

Implemented as Separate Unit

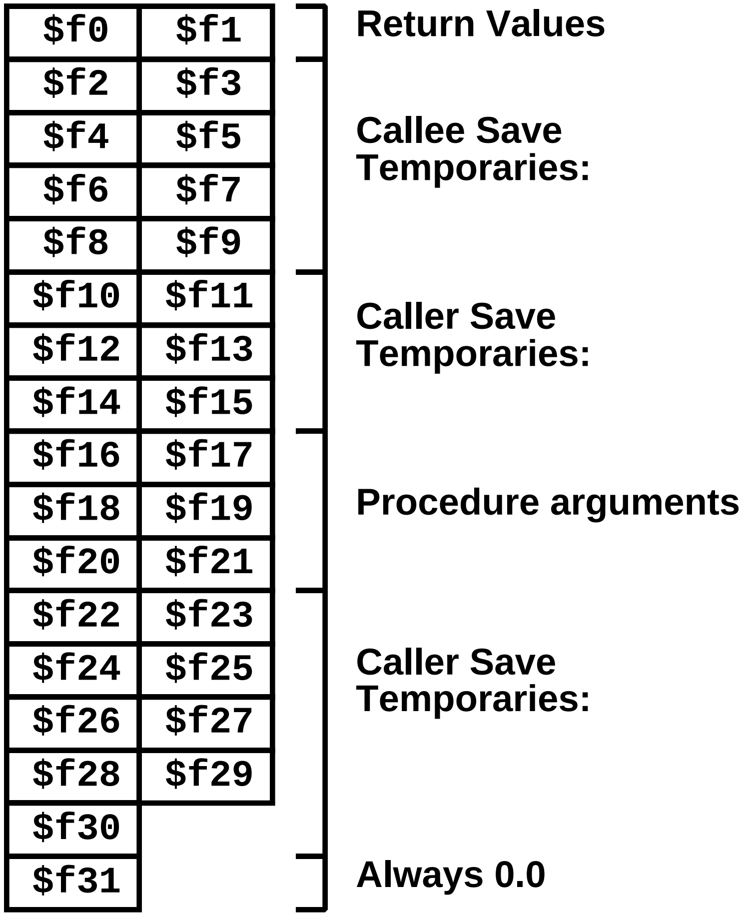
- Hardware to add, multiply, and divide
- Floating point data registers
- Various control & status registers

Floating Point Formats

- S_Floating (C `float`): 32 bits
- T_Floating (C `double`): 64 bits

Floating Point Data Registers

- 32 registers, each 8 bytes
- Labeled `$f0` to `$f31`
- `$f31` is always `0.0`



`$f0`	`$f1`	**Return Values**
`$f2`	`$f3`	**Callee Save Temporaries:**
`$f4`	`$f5`	
`$f6`	`$f7`	
`$f8`	`$f9`	
`$f10`	`$f11`	**Caller Save Temporaries:**
`$f12`	`$f13`	
`$f14`	`$f15`	
`$f16`	`$f17`	**Procedure arguments**
`$f18`	`$f19`	
`$f20`	`$f21`	
`$f22`	`$f23`	**Caller Save Temporaries:**
`$f24`	`$f25`	
`$f26`	`$f27`	
`$f28`	`$f29`	
`$f30`		
`$f31`		**Always 0.0**

Floating Point Code Example

Compute Inner Product of Two Vectors

- **Single precision arithmetic**

```
float inner_prodF
  (float x[], float y[],
   int n)
{
  int i;
  float result = 0.0;
  for (i = 0; i < n; i++) {
    result += x[i] * y[i];
  }
  return result;
}
```

```
     cpys $f31,$f31,$f0 # result = 0.0
     bis $31,$31,$3     # i = 0
     cmplt $31,$18,$1   # 0 < n?
     beq $1,$102        # if not, skip loop
     .align 5
$104:
     s4addq $3,0,$1     # $1 = 4 * i
     addq $1,$16,$2     # $2 = &x[i]
     addq $1,$17,$1     # $1 = &y[i]
     lds $f1,0($2)      # $f1 = x[i]
     lds $f10,0($1)     # $f10 = y[i]
     muls $f1,$f10,$f1  # $f1 = x[i] * y[i]
     adds $f0,$f1,$f0   # result += $f1
     addl $3,1,$3       # i++
     cmplt $3,$18,$1    # i < n?
     bne $1,$104        # if so, loop
$102:
     ret $31,($26),1    # return
```

Numeric Format Conversion

Between Floating Point and Integer Formats

- **Special conversion instructions `cvttq`, `cvtqt`, `cvtts`, `cvtst`, …**
- **Convert source operand in one format to destination in other**
- **Both source & destination must be FP register**
  - Transfer to and from GP registers via memory store/load

C Code

```
float double2float(double d)
{
  return (float) d;
}
```

```
double long2double(long i)
{
  return (double) i;
}
```

Conversion Code

```
cvtts $f16,$f0
```

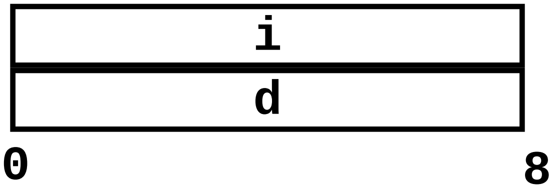
[Convert T_Floating to S_Floating]

```
stq $16,0($30)
ldt $f1,0($30)
cvtqt $f1,$f0
```

[Pass through stack and convert]

Getting FP Bit Pattern

```
double bit2double(long i)
{
  union {
    long i;
    double d;
  } arg;
  arg.i = i;
  return arg.d;
}
```



```
  stq $16,0($30)
  ldt $f0,0($30)
```

```
double long2double(long i)
{
  return (double) i;
}
```

```
  stq $16,0($30)
  ldt $f1,0($30)
  cvtqt $f1,$f0
```

- **Union provides direct access to bit representation of double**
- **`bit2double` generates double with given bit pattern**
  - NOT the same as **`(double) i`**
  - Bypasses rounding step