In-Class Handout Version

15-213

"The course that gives CMU its Zip!"

Integers
Sep 4, 2001

Topics

- **Numeric Encodings**
  - Unsigned & Two's complement
- **Programming Implications**
  - C promotion rules
- **Basic operations**
  - Addition, negation, multiplication
- **Programming Implications**
  - Consequences of overflow
  - Using shifts to perform power-of-2 multiply/divide

C Puzzles

- **Taken from Exam #2, CS 347, Spring '97**
- **Assume machine with 32 bit word size, two's complement integers**
- **For each of the following C expressions, either:**
  - Argue that is true for all argument values
  - Give example where not true

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

- `x < 0` ⇒ `((x*2) < 0)`
- `ux >= 0`
- `x & 7 == 7` ⇒ `(x<<30) < 0`
- `ux > -1`
- `x > y` ⇒ `-x < -y`
- `x * x >= 0`
- `x > 0 && y > 0` ⇒ `x + y > 0`
- `x >= 0` ⇒ `-x <= 0`
- `x <= 0` ⇒ `-x >= 0`

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- **C `short` 2 bytes long**

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
y	-15213	C4 93	11000100 10010011

Sign Bit

- **For 2's complement, most significant bit indicates sign**
  - 0 for nonnegative
  - 1 for negative

Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Numeric Ranges

Unsigned Values

- *UMin* = 0
  000…0
- *UMax* = $2^w - 1$
  111…1

Two's Complement Values

- *TMin* = -2^{w-1}
  100…0
- *TMax* = $2^{w-1} - 1$
  011…1

Other Values

- Minus 1
  111…1

Values for *W* = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	16	32	64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

Observations

- |*TMin* | = *TMax* + 1
  - **Asymmetric range**
- *UMax* = 2 * *TMax* + 1

C Programming

- `#include <limits.h>`
  - K&R App. B11
- **Declares constants, e.g.,**
  - `ULONG_MAX`
  - `LONG_MAX`
  - `LONG_MIN`
- **Values platform-specific**

Unsigned & Signed Numeric Values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

Example Values

- *W* = **4**

Equivalence

- **Same encodings for nonnegative values**

Uniqueness

- **Every bit pattern represents unique integer value**
- **Each representable integer has unique bit encoding**

⇒ Can Invert Mappings

- **U2B(*x*) = B2U^{-1}(*x*)**
  - Bit pattern for unsigned integer
- **T2B(*x*) = B2T^{-1}(*x*)**
  - Bit pattern for two's comp integer

Casting Signed to Unsigned

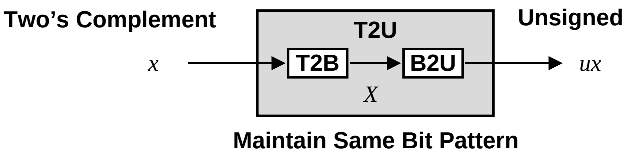
C Allows Conversions from Signed to Unsigned

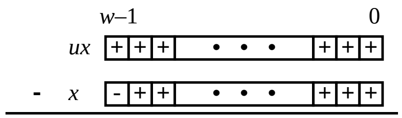
```
short int          x =  15213;
unsigned short int ux = (unsigned short) x;
short int          y  = -15213;
unsigned short int uy = (unsigned short) y;
```

Resulting Value

- **No change in bit representation**
- **Nonnegative values unchanged**
  - *ux* = 15213
- **Negative values change into (large) positive values**
  - *uy* = 50323

Relation Between 2's Comp. & Unsigned





$$+2^{w-1} - -2^{w-1} = 2*2^{w-1} = 2^{w}$$

$$ux = \begin{cases} x & x \ge 0 \\ x + 2^{w} & x < 0 \end{cases}$$

Relation Between Signed & Unsigned

Weight	-15213		50323	
1	1	1	1	1
2	1	2	1	2
4	0	0	0	0
8	0	0	0	0
16	1	16	1	16
32	0	0	0	0
64	0	0	0	0
128	1	128	1	128
256	0	0	0	0
512	0	0	0	0
1024	1	1024	1	1024
2048	0	0	0	0
4096	0	0	0	0
8192	0	0	0	0
16384	1	16384	1	16384
32768	1	*-32768*	1	*32768*
Sum		**-15213**		**50323**

- $uy = y + 2 * 32768 = y + 65536$

Signed vs. Unsigned in C

Constants

- **By default are considered to be signed integers**
- **Unsigned if have "U" as suffix**

```
0U, 4294967259U
```

Casting

- **Explicit casting between signed & unsigned same as U2T and T2U**

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

- **Implicit casting also occurs via assignments and procedure calls**

```
tx = ux;
uy = ty;
```


Casting Surprises

Expression Evaluation

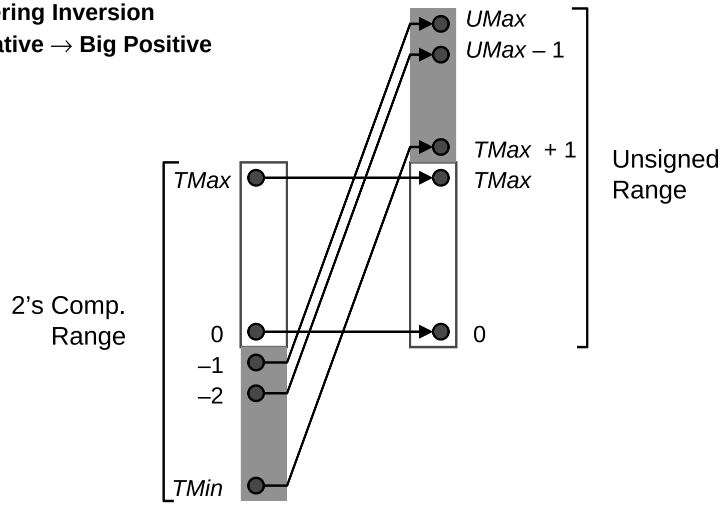
- **If mix unsigned and signed in single expression, signed values implicitly cast to unsigned**
- **Including comparison operations <, >, ==, <=, >=**
- **Examples for *W* = 32**

Constant$_1$	Constant$_2$	Relation	Evaluation
`0`	`0U`		
`-1`	`0`		
`-1`	`0U`		
`2147483647`	`-2147483648`		
`2147483647U`	`-2147483648`		
`-1`	`-2`		
`(unsigned) -1`	`-2`		
`2147483647`	`2147483648U`		
`2147483647`	`(int) 2147483648U`		

Explanation of Casting Surprises

2's Comp. → Unsigned

- **Ordering Inversion**
- **Negative → Big Positive**



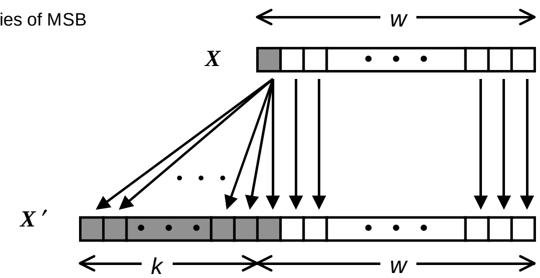
Sign Extension

Task:

- **Given *w*-bit signed integer *x***
- **Convert it to *w*+*k*-bit integer with same value**

Rule:

- **Make *k* copies of sign bit:**
- $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



Sign Extension Example

```
short int x =  15213;
int      ix = (int) x;
short int y = -15213;
int      iy = (int) y;
```

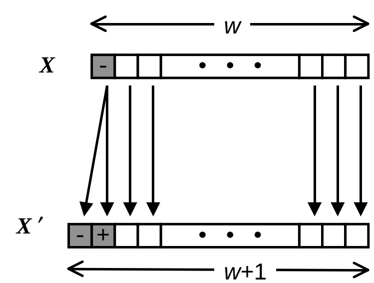
	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 C4 92	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- **Converting from smaller to larger integer data type**
- **C automatically performs sign extension**

Justification For Sign Extension

Prove Correctness by Induction on *k*

- **Induction Step: extending by single bit maintains value**



- **Key observation:** $-2^{w-1} = -2^{w} + 2^{w-1}$
- **Look at weight of upper bits:**

$x \quad -2^{w-1}\, x_{w-1}$

$x' \quad -2^{w}\, x_{w-1} + 2^{w-1}\, x_{w-1} \quad = \quad -2^{w-1}\, x_{w-1}$

Why Should I Use Unsigned?

Don't Use Just Because Number Nonzero

- **C compilers on some machines generate less efficient code**

```
unsigned i;
for (i = 1; i < cnt; i++)
  a[i] += a[i-1];
```

- **Easy to make mistakes**

```
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

Do Use When Performing Modular Arithmetic

- **Multiprecision arithmetic**
- **Other esoteric stuff**

Do Use When Need Extra Bit's Worth of Range

- **Working right up to limit of word size**

Negating with Complement & Increment

In C

`~x + 1 == -x`

Complement

- **Observation: `~x + x == 1111…11`$_2$ `== -1`**

x	1 0 0 1 1 1 0 1
+ ~x	0 1 1 0 0 0 1 0
-1	1 1 1 1 1 1 1 1

Increment

- `~x + ~~x~~ + (-~~x~~ + 1)  ==  ~~-1~~ + (-x + ~~1~~)`
- `~x + 1                   ==  -x`

Warning: Be cautious treating `int`'s as integers

- **OK here**

Comp. & Incr. Examples

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

0

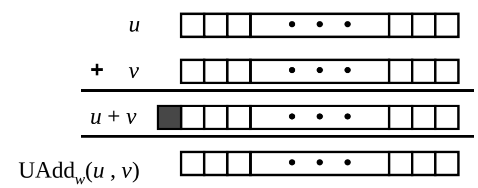
	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



Standard Addition Function

- **Ignores carry output**

Implements Modular Arithmetic

$s \quad = \quad \mathrm{UAdd}_w(u\ ,\ v) \quad = \quad u + v \mod 2^w$

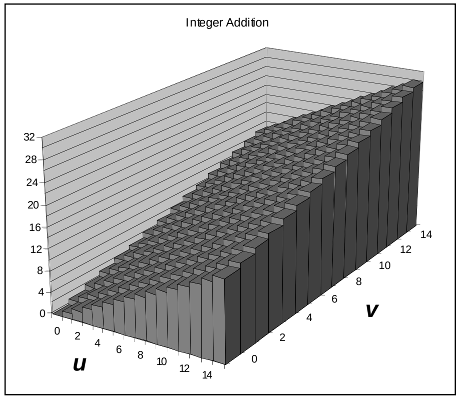
$$UAdd_w(u,v) \quad = \quad \begin{cases} u+v & u+v < 2^w \\ u+v-2^w & u+v \ge 2^w \end{cases}$$

Visualizing Integer Addition

Integer Addition

- 4-bit integers *u* and *v*
- Compute true sum $\text{Add}_4(u , v)$
- Values increase linearly with *u* and *v*
- Forms planar surface

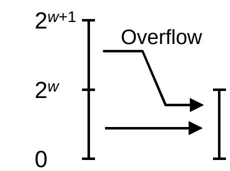
$\text{Add}_4(u , v)$



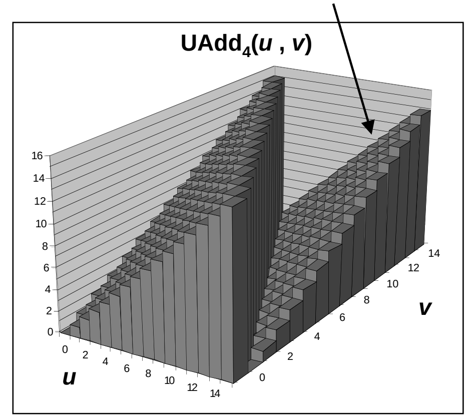
Visualizing Unsigned Addition

Wraps Around

- If true sum ≥ 2^w
- At most once



Overflow



Mathematical Properties

Modular Addition Forms an *Abelian Group*

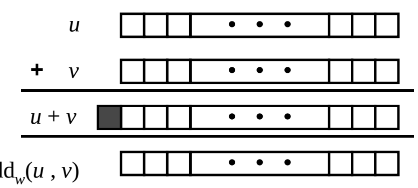
- **Closed under addition**
  $0 \leq \text{UAdd}_w(u , v) \leq 2^w - 1$
- **Commutative**
  $\text{UAdd}_w(u , v) = \text{UAdd}_w(v , u)$
- **Associative**
  $\text{UAdd}_w(t, \text{UAdd}_w(u , v)) = \text{UAdd}_w(\text{UAdd}_w(t, u), v)$
- **0 is additive identity**
  $\text{UAdd}_w(u , 0) = u$
- **Every element has additive inverse**
  – Let $\text{UComp}_w(u) = 2^w - u$
  $\text{UAdd}_w(u , \text{UComp}_w(u)) = 0$

Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



TAdd and UAdd have Identical Bit-Level Behavior

- Signed vs. unsigned addition in C:

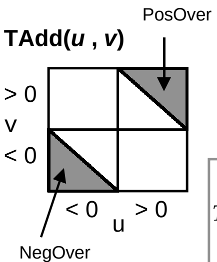
```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
 t = u + v
```

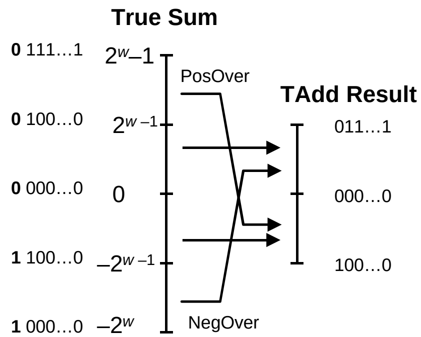
- Will give `s == t`

Characterizing TAdd

Functionality

- **True sum requires** w+1 **bits**
- **Drop off MSB**
- **Treat remaining bits as 2's comp. integer**





$$TAdd_w(u,v) \;=\; \begin{cases} u+v+2^{w-1} & u+v < TMin_w \ \text{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^{w-1} & TMax_w < u+v \ \text{(PosOver)} \end{cases}$$

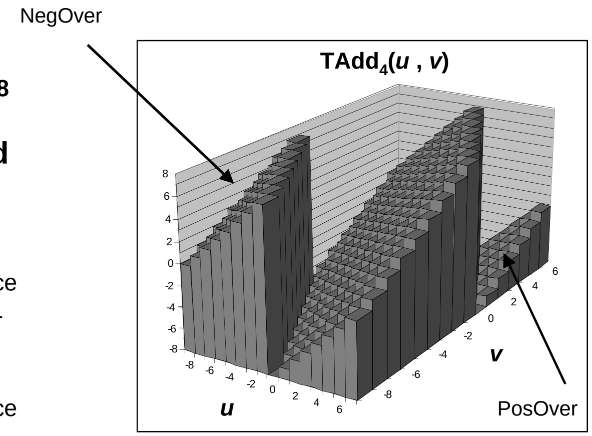
Visualizing 2's Comp. Addition

Values

- **4-bit two's comp.**
- **Range from -8 to +7**

Wraps Around

- **If sum** $\geq$ 2^{w-1}
  - Becomes negative
  - At most once
- **If sum <** -2^{w-1}
  - Becomes positive
  - At most once



Detecting 2's Comp. Overflow

Task

- **Given** s = $\text{TAdd}_w(u , v)$
- **Determine if** s = $\text{Add}_w(u , v)$
- **Example**

```
int s, u, v;
s = u + v;
```

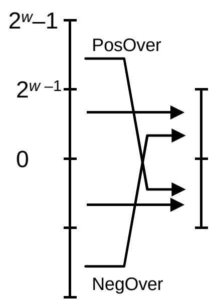
Claim

- **Overflow iff either:**

  $u, v < 0, s \geq 0$ (NegOver)

  $u, v \geq 0, s < 0$ (PosOver)

```
ovf = (u<0 == v<0) && (u<0 != s<0);
```



Mathematical Properties of TAdd

Isomorphic Algebra to UAdd

- $\text{TAdd}_w(u , v)$ = $\text{U2T}(\text{UAdd}_w(\text{T2U}(u), \text{T2U}(v)))$
  - Since both have identical bit patterns

Two's Complement Under TAdd Forms a Group

- **Closed, Commutative, Associative, 0 is additive identity**
- **Every element has additive inverse**

  Let $\text{TComp}_w(u)$ = $\text{U2T}(\text{UComp}_w(\text{T2U}(u)))$

  $\text{TAdd}_w(u , \text{TComp}_w(u))$ = 0

$$TComp_w(u) \;=\; \begin{cases} -u & u \ne TMin_w \\ \boxed{} & u = TMin_w \end{cases}$$

Negating with Complement & Increment

In C

`~x + 1 == -x`

Complement

- Observation: `~x + x == 1111…11`$_2$ `== -1`

```
     x  1 0 0 1 1 1 0 1
+   ~x  0 1 1 0 0 0 1 0
-----------------------
    -1  1 1 1 1 1 1 1 1
```

Increment

- `~x +` ~~`x`~~ `+ (-`~~`x`~~ `+ 1)   ==   ` ~~`-1`~~ ` + (-x + ` ~~`1`~~`)`
- `~x + 1   ==   -x`

Warning: Be cautious treating `int`'s as integers

- **OK here: We are using group properties of** TAdd **and** TComp

Multiplication

Computing Exact Product of *w*-bit numbers *x*, *y*

- **Either signed or unsigned**

Ranges

- **Unsigned:** $0 \le x * y \le (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
  - Up to $2w$ bits
- **Two's complement min:** $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
  - Up to $2w-1$ bits
- **Two's complement max:** $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
  - Up to $2w$ bits, but only for $TMin_w^2$

Maintaining Exact Results

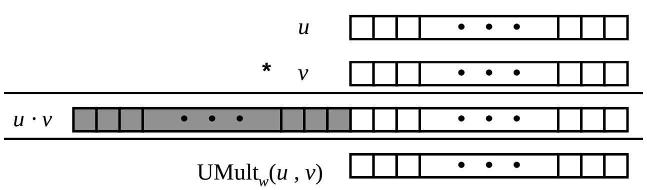
- **Would need to keep expanding word size with each product computed**
- **Done in software by "arbitrary precision" arithmetic packages**
- **Also implemented in Lisp, and other "advanced" languages**

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



Standard Multiplication Function

- **Ignores high order** *w* **bits**

Implements Modular Arithmetic

$\text{UMult}_w(u, v) = u \cdot v \mod 2^w$

Unsigned vs. Signed Multiplication

Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

- **Truncates product to** *w***-bit number** *up* **=** $\text{UMult}_w(ux, uy)$
- **Simply modular arithmetic**

  $up = ux \cdot uy \mod 2^w$

Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

- **Compute exact product of two** *w***-bit numbers** ***x*, *y***
- **Truncate result to** *w***-bit number** *p* = $\text{TMult}_w(x, y)$

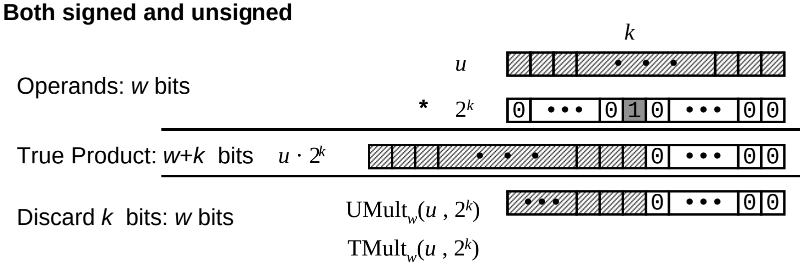
Relation

- **Signed multiplication gives same bit-level result as unsigned**
- `up == (unsigned) p`

Power-of-2 Multiply with Shift

Operation

- **`u << k` gives `u *` 2^k**
- **Both signed and unsigned**



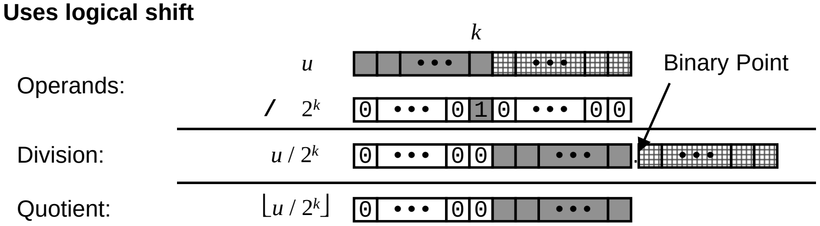
Examples

- **`u << 3          ==   u * 8`**
- **`u << 5 - u << 3  ==   u * 24`**
- **Most machines shift and add much faster than multiply**
  - Compiler will generate this code automatically

Unsigned Power-of-2 Divide with Shift

Quotient of Unsigned by Power of 2

- **`u >> k` gives** $\lfloor$ **`u /`** $2^k \rfloor$
- **Uses logical shift**

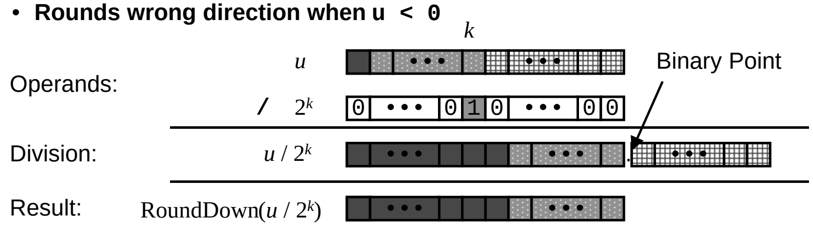


	Division	Computed	Hex	Binary
x	15213	15213	3B 6D	00111011 01101101
x >> 1	7606.5	7606	1D B6	**0**0011101 10110110
x >> 4	950.8125	950	03 B6	**0000**0011 10110110
x >> 8	59.4257813	59	00 3B	**00000000** 00111011

2's Comp Power-of-2 Divide with Shift

Quotient of Signed by Power of 2

- **`u >> k` gives** $\lfloor$ **`u /`** $2^k \rfloor$
- **Uses arithmetic shift**
- **Rounds wrong direction when `u < 0`**



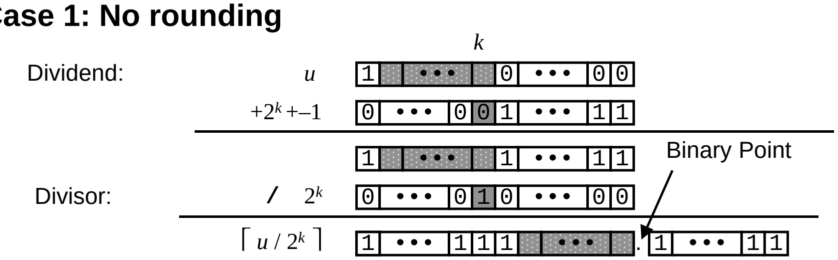
	Division	Computed	Hex	Binary
y	-15213	-15213	C4 93	11000100 10010011
y >> 1	-7606.5	-7607	E2 49	**1**1100010 01001001
y >> 4	-950.8125	-951	FC 49	**1111**1100 01001001
y >> 8	-59.4257813	-60	FF C4	**11111111** 11000100

Correct Power-of-2 Divide

Quotient of Negative Number by Power of 2

- **Want** $\lceil$ **`u /`** $2^k \rceil$ (Round Toward 0)
- **Compute as** $\lfloor$ **`(u+`2^k`-1)/`** $2^k \rfloor$
  - In C: **`(u + (1<<k)-1) >> k`**
  - Biases dividend toward 0

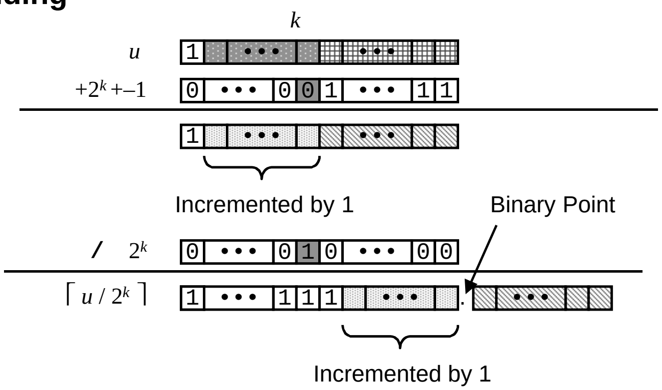
Case 1: No rounding



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

Properties of Unsigned Arithmetic

Unsigned Multiplication with Addition Forms Commutative Ring

- **Addition is commutative group**
- **Closed under multiplication**
  $0 \le \mathrm{UMult}_w(u, v) \le 2^w - 1$
- **Multiplication Commutative**
  $\mathrm{UMult}_w(u, v) = \mathrm{UMult}_w(v, u)$
- **Multiplication is Associative**
  $\mathrm{UMult}_w(t, \mathrm{UMult}_w(u, v)) = \mathrm{UMult}_w(\mathrm{UMult}_w(t, u), v)$
- **1 is multiplicative identity**
  $\mathrm{UMult}_w(u, 1) = u$
- **Multiplication distributes over addtion**
  $\mathrm{UMult}_w(t, \mathrm{UAdd}_w(u, v)) = \mathrm{UAdd}_w(\mathrm{UMult}_w(t, u), \mathrm{UMult}_w(t, v))$

Properties of Two's Comp. Arithmetic

Isomorphic Algebras

- **Unsigned multiplication and addition**
  - Truncating to *w* bits
- **Two's complement multiplication and addition**
  - Truncating to *w* bits

Both Form Rings

- **Isomorphic to ring of integers mod** 2^w

Comparison to Integer Arithmetic

- **Both are rings**
- **Integers obey ordering properties, e.g.,**
  $u > 0 \quad \Rightarrow \quad u + v > v$
  $u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$
- **These properties are not obeyed by two's complement arithmetic**
  TMax **+ 1** **==** *TMin*
  15213 * 30426 == -10030 **(16-bit words)**

C Puzzle Answers

- **Assume machine with 32 bit word size, two's complement integers**
- *TMin* **makes a good counterexample in many cases**

- `x < 0` ⇒ `((x*2) < 0)`
- `ux >= 0`
- `x & 7 == 7` ⇒ `(x<<30) < 0`
- `ux > -1`
- `x > y` ⇒ `-x < -y`
- `x * x >= 0`
- `x > 0 && y > 0` ⇒ `x + y > 0`
- `x >= 0` ⇒ `-x <= 0`
- `x <= 0` ⇒ `-x >= 0`