Amortized analysis

Amortized analysis lets us consider the runtime behavior of a sequence of operations of an algorithm.

It lets us take a more nuanced view of the runtime of an algorithm: if there's some incredibly rare operation that takes a long time to do, it doesn't make sense to characterize the entire performance of the algorithm by that one operation. By using amortized analysis, we can get a more accurate view of how the algorithm will actually run.

Unbounded arrays

Unbounded arrays are implemented as pointers to `struct uba_headers`:

```
1 struct uba_header {
2    int size;
3    int limit;
4    elem[] data;
5 };
```

We're going to discuss a variation on the UBAs presented in class to give a different example of amortized analysis and discuss

When implementing unbounded arrays on an embedded device, a programmer is concerned that doubling the size of the array when we reach its limit may use precious memory resources too aggressively. So she decides to see if she can increase it by a factor of $\frac{3}{2} = 1.5$ instead, rounding down if the result is not an integral number.

This means that it won't make sense to have fewer than _____2_____ elements in the array, because otherwise you might resize the array and get an array that wasn't any bigger. This would need to be reflected in the data structure invariant!

We're also going to resize the arrays a little bit earlier (which means we may use a bit more memory sometimes!) to make sure that we never end up with a `size` equal to `limit`.
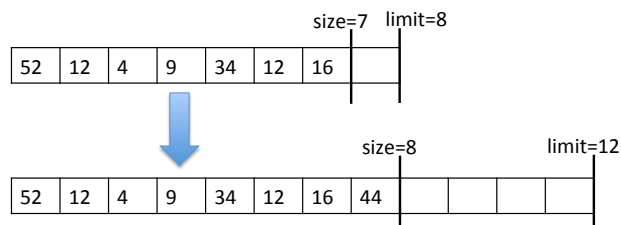
Make sure you know how to write a data structure invariant for this modified UBA:

```
1 bool is_uba(struct uba_header* U) {
2    if (U == NULL) return false;
3    if (!(1 < U->limit)) return false;
4    if (!(0 <= U->size && U->size < U->limit)) return false;
5    //@assert \length(A) == U->limit;
6    return true;
7 }
```

We now carry out the the amortized analysis for this version of unbounded arrays. We'll start right after we've resized the array, when the size of the array is s and the limit of the array is $l = 3s/2$. We'll show that, assuming we start out with some non-zero number of tokens k, the next resizing of the array, from size l to size $l' = 3l/2 = 9s/4$, can be paid for by the cost (in tokens) of the operations that happen before that resize. (Just assume s is divisible by 4 for the purposes of this questions.)

Most of the operations require us to spend 1 token because we write to the U->data array exactly once.

In addition, we need to reserve ____3____ tokens for a total amortized cost of ____4____ tokens.

Starting from a $2/3$ full array, if we write into the old array every time, then after ____$l/3 = s/2$____ insertions we will fill up the old array completely.

At this point, we have a total of ____$k + 3s/2$____ tokens, and we need to copy ____$l = 3s/2$____ tokens into the newly allocated array of size $l' = 3l/2$.

After all those copies, we have ____k____ tokens left. This is no smaller than k, and we have considered the worst case, so we will never run out of tokens.

You can try running through this analysis with other resizing factors:

If we triple the size of the array, we still need to reserve 2 tokens for a total amortized cost of 3 tokens. Starting from a $1/3$ full array, if we write into the old array every time, then after $2l/3 = 2s$ insertions we fill up the array. At this point we have $k + 4s$ tokens, and we need to copy $l = 3s$ tokens to the newly alloccted array of size $3l$. After all these copies we have $k + s$ tokens left. (If we tried to only reserve 1 token, we would end up iwith $k - s$ tokens. If we are allowed to have an amortized cost in fractions of a token, then it would suffice to reserve $1\frac{1}{2}$ tokens for a total amortized cost of $2\frac{1}{2}$ tokens.)

If we resize the array by a factor of $5/4$, we need to reserve 5 tokens for a total amortized cost of 6 tokens. Starting from a $4/5$ full array, if we write into the old array every time, then after $l/5 = s/4$ insertions we fill up the array. At this point we have $k + 5s/4$ tokens, and we need to copy $l = 5s/4$ tokens to the newly allocated array of size $5l/4$. After all these copies we have k tokens left.

If we resize the array by a factor of $4/3$, we need to reserve 4 tokens for a total amortized cost of 5 tokens. Starting from a $3/4$ full array, if we write into the old array every time, then after $l/4 = s/3$ insertions we fill up the array. At this point we have $k + 4s/3$ tokens, and we need to copy $l = 4s/3$ tokens to the newly allocated array of size $4l/3$. After all these copies we have k tokens left.

If we resize the array by a factor of $13/10 = 1.3$, we need to reserve 5 tokens for a total amortized cost of 6 tokens. Starting from a $10/13$ full array, if we write into the old array every time, then after $3l/13 = 3s/10$ insertions we fill up the array. At this point we have $k + 15s/10 = k + 3s/2$ tokens, and we need to copy $l = 13s/10$ tokens to the newly alloccted array of size $13l/10$. After all these copies we have $k + 2s/10 = k + s/5$ tokens left. (If we tried to only reserve 4 tokens instead of 5, we would end up with $k - s/10$ tokens at the end. If we are allowed to reserve fractions of a token, then it would suffice to reserve $4\frac{1}{3}$ tokens for a total amortized cost of $5\frac{1}{3}$ tokens.)