15-850(C) Machine Learning Theory

Assignment 4 Due: November 2, 1995

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1. [Problem 5.1 in KV] Design an algorithm that learns Decision Lists in the Statistical Query model (and prove it).

2. In this problem you will prove that the VC-dimension of the class $\mathcal{H}_n$ of halfspaces in n dimensions is $n+1$. ($\mathcal{H}_n$ is the set of functions $a_1x_1 + \ldots + a_nx_n \geq a_0$, where $a_0, \ldots, a_n$ are real-valued.) Recall that VC-dim($\mathcal{C}$) is the largest d such that there exists a set of d points *shattered* by $\mathcal{C}$ (split in all possible ways by concepts in $\mathcal{C}$).

   We will use the following definition: The *convex hull* of a set of points S is the set of all convex combinations of points in S; this is the set of all points that can be written as $\sum_{x_i \in S} \lambda_i x_i$, where each $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$. It is not hard to see that if a halfspace has all points from a set S on one side, then the entire convex hull of S must be on that side as well.

   (a) **[lower bound]** Prove that VC-dim($\mathcal{H}_n$) $\geq n+1$ by presenting a set of $n+1$ points in n-dimensional space such that one can partition that set with halfspaces in all possible ways. (And, show how one can partition the set in any desired way.)

   (b) **[upper bound part 1]** The following is "Radon's Theorem," from the 1920's.

   Theorem. *Let S be a set of $n+2$ points in n dimensions. Then S can be partitioned into two (disjoint) subsets S_1 and S_2 whose convex hulls intersect.*

   Show that Radon's Theorem implies that the VC-dimension of halfspaces is *at most* $n+1$. Conclude that VC-dim($\mathcal{H}_n$) $= n+1$.

   (c) **[upper bound part 2]** Now we prove Radon's Theorem. We will need the following standard fact from linear algebra. If $x_1, \ldots, x_{n+1}$ are $n+1$ points in n-dimensional space, then they are linearly dependent. That is, there exist real values $\lambda_1, \ldots, \lambda_n$ such that $x_{n+1} = \lambda_1 x_1 + \ldots + \lambda_n x_n$.

   You may now prove Radon's Theorem however you wish. However, as a suggested first step, prove the following. For any set of $n+2$ points $x_1, \ldots, x_{n+2}$ in n-dimensional space, there exist $\lambda_1, \ldots, \lambda_{n+2}$ *not all zero* such that $\sum_i \lambda_i x_i = 0$ and $\sum_i \lambda_i = 0$. (This is called *affine dependence*.) Now, think about the lambdas...