

15-354: Midterm

October 17, 2006

Name:

Andrew ID:

Instructions

- Fill in the box above with your name and your Andrew ID. **Do it, now!**
- Clearly mark your answers in the allocated space. If need be, use the back of a page for scratch space. If you have made a mess, cross out the invalid parts of your solution, and circle the ones that should be graded.
- Scan the test first to make sure that none of the 16 pages are missing. The problems are of varying difficulty and are not necessarily sorted according to increasing difficulty. You might wish to pick off the easy ones first.
- You have 80 minutes. Good luck.

1	15	
2	20	
3	20	
4	20	
5	15	
6	10	
Total	100	

Problem 1: Functions Defined by Equations (15 pts.)

Fred Hacker is highly impressed with the elegant, recursive definition of the Ackermann function. He is trying to come up with some interesting functions of his own. Explain why the following attempts by Fred to define functions f and g fail to work.

A.

$$\begin{aligned}f(0,0) &= 0 \\f(x+1,y) &= f(x,y+1)\end{aligned}$$

B.

$$\begin{aligned}g(x,0) &= x+1 \\g(x,y+1) &= g(x+1,y) \\g(x+1,y+1) &= g(x,g(x,y))\end{aligned}$$

Problem 2: Loop Programs (20 pts.)

Consider the following loop program:

```
// program P:  x --> y
  y = 0;
  do x:
    do x:
      do x:  y++; od
    od
  od
```

1. What function $\mathbb{N} \rightarrow \mathbb{N}$ does P compute?
2. P has loop-depth 3. Find an equivalent loop program of depth 2.

Problem 3: Parity Checker (20 pts.)

For this problem, the input alphabet is $\Sigma = \{a, b, c\}$.

1. Construct the minimal DFA that checks whether a string over Σ contains an even number of a 's, an even number of b 's, and an even number of c 's.

It is probably best to draw a *nice* picture of the transition diagram. Do not forget to explain why your machine is in fact minimal.

Problem 4: Wurzelbrunft vs. Collatz (20 pts.)

Wurzelbrunft claims to have proven the following theorem concerning the classical Collatz function defined by

$$C(x) = \begin{cases} 1 & \text{if } x = 1, \\ x/2 & \text{if } x \text{ even,} \\ (3x + 1)/2 & \text{otherwise.} \end{cases}$$

Wurzelbrunft claims he can prove that

For every $n > 1$ the orbit of n under C contains a point less than n .

The alleged proof is too long to be included here (even in the margin).

- If Wurzelbrunft's proof were correct, why would he become world famous? Explain.

Problem 5: Checking Permutations (15 pts.)

In the following suppose we are given an array F representing a function $f : A \rightarrow A$ where $A = \{0, 1, \dots, n - 1\}$. Thus we can “compute” the function value $y = f(x)$ in $O(1)$ steps by simple lookup.

If you write pseudo-code to describe your algorithms below, make sure to provide ample comments; no credit otherwise. You can use any auxiliary data structure you like, just say clearly what the data structure is.

- A. Give a fast algorithm to check whether f is a permutation (the faster the better). State the running time of your algorithm.

- B. Give an algorithm to check whether f is a permutation using $O(1)$ memory (we are not being charged for F). Explain why your algorithm is correct and state its running time.

Problem 6: Wurtzelbrunft's Group (10 pts.)

After years of painstaking research Prof. Wurtzelbrunft has discovered a group G that has the peculiar property that for all elements x :

$$x \cdot x = 1$$

Note that Wurtzelbrunft is using multiplicative notation. He knows that additive notation should only be used for commutative groups and he has been unable to show that his group is in fact commutative.

- Show that the Wurtzelbrunft group is commutative.

