15-354: Computational Discrete Math — K. Sutner

Assignment 3 — Due: Thursday, September 17, 2009

Problem 1: Lookup Tables and Iteration (50 pts.)

Background

Suppose a function $f : [n] \to [n]$ is given as a lookup table. Assume that n is machine-sized so that $f(x)$ can be determined in $O(1)$ time. In order to determine $f^t(x)$ we can use repeated lookup. However, if we need to determine $f^t(x)$ repeatedly it is better to perform a pre-computation that augments the table for f. The augmented table then allows for speedy lookups of iterated values of the function f.

Task

A. Explain how to augment the table for f so that lookups of $f^t(x)$ are fast.

B. How long does it take in your solution to determine $f^t(x)$?

C. What is the memory requirement for your table?

D. What is the cost of the pre-computation?

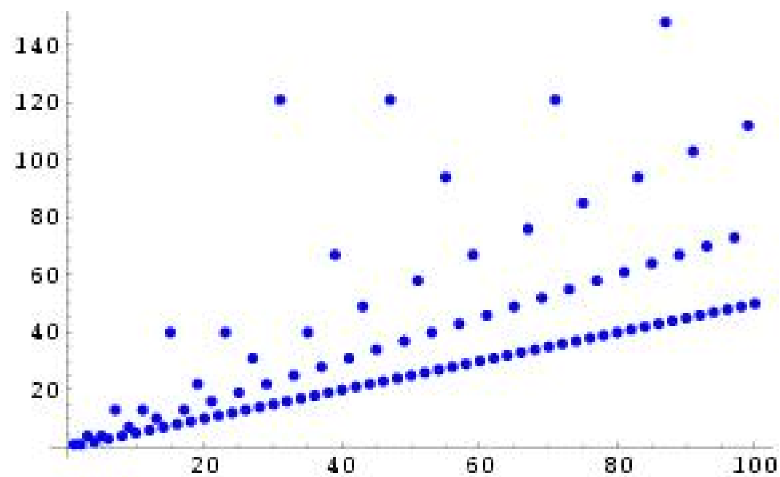
Problem 2: Son of Collatz (50 pts.)

Background

A function defined by an apparently simple recursion can behave in a rather unpredictable way. The Collatz function and the Ackermann function are prime examples. Here is an example that looks somewhat similar. Define a function F on the positive integers by

$$F(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ F(F(3x+1)) & \text{otherwise.} \end{cases}$$

Note the double application of F in the odd case. It is not really clear that this is well-defined, there might be some infinite loop – but there isn't. Here is a plot of F on up to $x = 100$.



Task

A. Determine what the lines in the picture are. More precisely, determine a simple description of the x-coordinates of all the points belonging to a single one of these lines (the y-coordinates are then easy to get).

B. Give a reasonably simple non-recursive description of F.

C. Prove that your description is correct, and conclude that F is really well-defined: for any positive integer x there is exactly one y such that $F(x) = y$.

D. Define $d(x)$ to be the number of recursive calls made in the computation of $F(x)$. For example, for all even x, $d(x) = 0$, $d(1) = 2$ and $d(3) = 4$. Find a simple description of d.

Comment

It is a good idea to try to figure out how this function is related to the Collatz function C. A little experimental computation might also be helpful.