15-354: Computational Discrete Math	**K. Sutner**
Solutions Assignment 3	

Solution: Lookup Tables and Iteration

Part A: Augmented Table

For simplicity assume that $n = 2^k$, the general case is entirely similar. First assume $t < n$. Using the same approach as in fast exponentiation

$$f^t(x) = f^{t_1}(f^{t_2}(\ldots(f^{t_r}(x))\ldots))$$

where the t_i are powers of 2 and $t = \sum t_i$, $r \leq k$. Hence we should precompute $f^{2^i}(x)$ for $0 \leq i < k$. Alas, for arbitrary t this approach fails. The solution is to compute the transient and period for each point. This is easy to do in linear time and linear space; the resulting table has two additional columns. Given transient/period information we can reduce the problem of computing $f^t(x)$ to the problem of computing $f^{t'}(x)$ for some $t' < n$.

Part B: Fast Lookup

Given the precomputed powers-of-two table we can determine $f^t(x)$ in at most k table lookups, perhaps after reducing t as described above.

Part C: Space Requirement

Using a uniform cost function (an integer has size 1) the table has size $O(nk)$.

Part D: Cost Pre-Computation

Initially we are given the table for $f = f^1$. It takes $2n$ lookups in this table to obtain f^2, another $2n$ lookups in this table to obtain f^4 and so on. In total the cost of precomputing the table is $O(nk)$, even if we have to compute transients/periods.

Solution: Son of Collatz

Warm-Up

It's a good idea to start with a little comparison between the F function to the ordinary Collatz function C:

$$\begin{array}{lll} C(1) = 1 & F(x) = 1 & \\ C(x) = x/2 & F(x) = x/2 & x \text{ even} \\ C(x) = (3x+1)/2 & F(x) = F(F(3x+1)) & x \text{ odd} \end{array}$$

This is just the definition except for the easy check $F(1) = F(F(4)) = F(2) = 1$. Hence differences occur only for odd inputs larger than 1. So suppose x is odd, so that $F(x) = F((3x+1)/2)$. But then

$$F(1) = 1$$
$$F(x) = C(x) \qquad x \text{ even}$$
$$F(x) = F(C(x)) \qquad x \neq 1 \text{ odd}$$

In other words, F behaves exactly like C on even numbers and 1, but on odd numbers it contracts a whole sequence of applications of C into just one: $F(x) = C^t(x)/2$ where t is minimal such that $C^t(x)$ is even. Note that this raises the question of whether F is really well-defined: why should the t always exist?

Part A: Lines

Note that the spacing of the x-coordinates seems to double as we go up one line in the picture. The lowest line is easy: it corresponds to even x-coordinates and the points are parameterized as $(2s, s)$, $s \geq 1$. In other words, the starting point is $(2, 1)$ and the slope is $1/2$. (Of course, only the points where this line intersects integer grid points matter.) For the second line consider x-coordinates $4s + 1$, $s \geq 0$. From the Warm-Up section, $F(4s + 1) = C^2(12s + 4) = C(6s + 2) = 3s + 1$, so the slope is $3/4$ in this case. Likewise, for the third line we have $x = 8s + 3$ (check, all the other values are already take care of). Again by the Warm-Up, $F(8s + 3) = 9s + 4$, so the slope is $9/8$.

In general, starting the count from $k = 0$ for simplicity, the kth line has slope $3^k/2^{k+1}$ and x-coordinates $2^{k+1}s + 2^k - 1$ where $s \geq 0$ (except for $k = 0$ where we need $s \geq 1$).

Part B: Description for F

To see why this is true, we use a little parameterization trick. Write $x = 2^a(2b+1)-1$ where $a, b \geq 0$ (the case $a = b = 0$ is of no interest here). When $a = 0$ we get all even numbers, so suppose $a > 0$.

Then $F(2^a(2b + 1) - 1) = F(2^{a-1}(3(2b + 1)) - 1)$, whence

$$F(2^a(2b + 1) - 1) = F((3^a(2b + 1)) - 1) = ((3^a(2b + 1)) - 1)/2.$$

So each line corresponds to the x values obtained from a fixed value of a and b ranging over the natural numbers.

Part C: Correctness

The key is the equation from the last section. Note that for every $x \geq 1$ there is exactly one pair a, b such that $x = x(a, b) = 2^a(2b + 1) - 1$, so we might as well use this special parameterization. Now note that for positive a

$$F(x(a, b)) = F(3x(a, b) + 1) = F(x(a - 1, 3b + 1))$$

In other words, we have dropped from the a-line to the $(a - 1)$-line, albeit at a different x-value. A simple induction on a shows that

$$F(x(a, b)) = 3^a b + (3^a - 1)/2$$

and so F is indeed well-defined.

In fact, we could have written

$$F(x) = x/2 \qquad x \text{ even}$$
$$F(x) = F(3(x + 1)/2) \qquad x \neq 1 \text{ odd}$$

Our original description of F is pure obfuscation!

Part D: Description for d

From the previous analysis we can again use the $x(a, b)$ parameterization to solve this problem:

$$d(x(a, b)) = 2a.$$

As before, this is easy to see by induction on a.