15-780 – Mixed integer programming

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Overview

- Introduction to mixed integer programs
- Examples: Sudoku, planning with obstacles
- Solving integer programs with branch and bound
- Extensions

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Introduction

- Recall optimization problem

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & g_i(x) \leq 0 \;\; i = 1, \ldots, m \end{array}$$

  "easy" when f, g_i convex, "hard" otherwise

- But how hard? How do we even go about solving (locally or globally) these problems?

- We've seen how to solve discrete non-convex optimization problems with search, can we apply these same techniques for mathematical optimization?

Mixed integer programs

- A special case of non-convex optimization methods that lends itself to a combination of search and convex optimization

$$\begin{array}{ll} \underset{x,z}{\text{minimize}} & f(x,z) \\ \text{subject to} & g_i(x,z) \leq 0 \ \ i = 1,\ldots,m \end{array}$$

  - $x \in \mathbb{R}^n$, and $z \in \mathbb{Z}^p$ are optimization variables
  - $f : \mathbb{R}^n \times \mathbb{Z}^p \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{Z}^p \to \mathbb{R}$ *convex* objective and constraint functions

- *Not* a convex problem (set of all integers is not convex)

- Note: some ambiguity in naming, some refer to MIPs as only *linear* programs with integer constraints

Mixed binary integer programs

- For this class, we'll focus on a slightly more restricted case

$$
\begin{aligned}
\underset{x,z}{\text{minimize}} \quad & f(x,z) \\
\text{subject to} \quad & g_i(x,z) \leq 0 \quad i = 1,\ldots,m \\
& z_i \in \{0,1\}, \quad i = 1,\ldots,p
\end{aligned}
$$

- Still an extremely power class of problems (i.e., binary integer programing is NP-complete)

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Example: Sudoku

- The ubiquitous Sudoku puzzle

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

- Can be encoded as binary integer program: let $z_{i,j} \in \{0, 1\}^9$ denote the “indicator” of number in the i, j position

$$z_{6,3} = [0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]^T \iff$$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7		3		2				6
	6					2	8	
			4	1	9			5
				8			7	9

- Each square can have only one number

$$\sum_{k=1}^{9} (z_{i,j})_k = 1, \quad i,j = 1,\ldots,9$$

- Every row must contain each number

$$\sum_{j=1}^{9} z_{i,j} = \mathbf{1}, \text{ (all ones vector)} \quad i = 1,\ldots,9$$

- Every column must contain each number

$$\sum_{i=1}^{9} z_{i,j} = \mathbf{1}, \quad j = 1,\ldots,9$$

- Every 3x3 block must contain each number

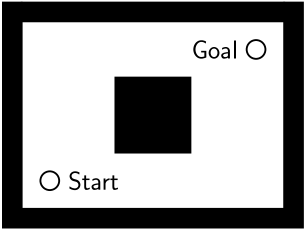
$$\sum_{k,\ell=1}^{3} z_{i+k,j+\ell} = \mathbf{1}, \quad i,j \in \{0,3,6\}$$

- Final optimization problem (note that objective is irrelevant, as we only care about finding a feasible point)

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{i,j=1}^{9} \max_k (z_{i,j})_k \\
\text{subject to} & z_{i,j} \in \{0,1\}^9, \;\; i,j = 1,\ldots,9 \\
& \displaystyle\sum_{k=1}^{9} (z_{i,j})_k = 1, \;\; i,j = 1,\ldots,9 \\
& \displaystyle\sum_{j=1}^{9} z_{i,j} = \mathbf{1}, \;\; i = 1,\ldots,9 \\
& \displaystyle\sum_{i=1}^{9} z_{i,j} = \mathbf{1}, \;\; j = 1,\ldots,9 \\
& \displaystyle\sum_{k,\ell=1}^{3} z_{i+k,j+\ell} = \mathbf{1}, \;\; i,j \in \{0,3,6\}
\end{array}
$$

Example: path planning with obstacles

- Find path from start to goal that avoids obstacles



- Represent path as set of points $x_i \in \mathbb{R}^2$, $i = 1, \ldots, m$ and minimize squared distance between consectutive points

- Obstacle is defined by $a, b \in \mathbb{R}^2$

$$\mathcal{O} = \{x : a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2\}$$

- Constraint that we *not* hit obstacle can be represented as

$$(x_i)_1 \leq a_1 \vee (x_i)_1 \geq b_1 \vee (x_i)_2 \leq a_2 \vee (x_i)_2 \geq b_2, \quad i = 1, \ldots, m$$

- How can we represent this using binary variables?

- The trick: "big-M" method

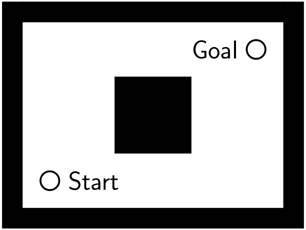
- Let $M \in \mathbb{R}$ be some big number and consider the constraint

$$(x_i)_1 \leq a_1 + zM$$

for $z \in \{0, 1\}$; if $z = 0$, this is the same as the original constraint, but if $z = 1$ then constraint will always be satisfied

- Introduce new variables $z_{i1}, z_{i2}, z_{i3}, z_{i4}$ for each x_i

$$x_i \notin \mathcal{O} \iff \begin{array}{l} (x_i)_1 \leq a_1 + z_{i1}M \\ (x_i)_1 \geq b_1 - z_{i2}M \\ (x_i)_2 \leq a_2 + z_{i3}M \\ (x_i)_2 \geq b_2 - z_{i4}M \\ z_{i1} + z_{i2} + z_{i3} + z_{i4} \leq 3 \end{array}$$



- Final optimization problem

$$
\begin{array}{ll}
\underset{x,z}{\text{minimize}} & \displaystyle\sum_{i=1}^{m-1} \|x_{i+1} - x_i\|_2^2 \\
\text{subject to} & \left.\begin{array}{l}
(x_i)_1 \leq a_1 + z_{i1}M \\
(x_i)_1 \geq b_1 - z_{i2}M \\
(x_i)_2 \leq a_2 + z_{i3}M \\
(x_i)_2 \geq b_2 - z_{i4}M \\
z_{i1} + z_{i2} + z_{i3} + z_{i4} \leq 3
\end{array}\right\} i = 1, \ldots, m \\
& z_{ij} \in \{0, 1\}, \;\; i = 1, \ldots, m, \; j = 1, \ldots, 4 \\
& x_1 = \text{start}, \;\; x_m = \text{goal}
\end{array}
$$

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Solution via enumeration

- Recall that optimization problem

$$\begin{array}{ll} \underset{x,z}{\text{minimize}} & f(x,z) \\ \text{subject to} & g_i(x,z) \leq 0 \quad i = 1, \ldots, m \\ & z_i \in \{0,1\}, \quad i = 1, \ldots, p \end{array}$$

  is easy for a fixed z (then a convex problem)

- So, just enumerate all possible z's and solve optimization problem for each

- 2^p possible assignments, quickly becomes intractable

Branch and bound

- Branch and bound is simply a search algorithm (best-first search) applied to finding the optimal z assignment
- In the worst case, still exponential (have to check every possible assignment)
- In many cases *much* better

Convex relaxations

- The key idea: *convex relaxation* of non-convex constraint

$$
\begin{aligned}
\underset{x,z}{\text{minimize}} \quad & f(x,z) \\
\text{subject to} \quad & g_i(x,z) \leq 0 \quad i = 1,\ldots,m \\
& z_i \in \{0,1\}, \quad i = 1,\ldots,p
\end{aligned}
$$

Convex relaxations

- The key idea: *convex relaxation* of non-convex constraint

$$\begin{aligned}
\underset{x,\bar{z}}{\text{minimize}} \quad & f(x, \bar{z}) \\
\text{subject to} \quad & g_i(x, \bar{z}) \leq 0 \quad i = 1, \ldots, m \\
& \bar{z}_i \in [0, 1], \quad i = 1, \ldots, p
\end{aligned}$$

Convex relaxations

- The key idea: *convex relaxation* of non-convex constraint

$$\begin{aligned} \underset{x,\bar{z}}{\text{minimize}} \quad & f(x, \bar{z}) \\ \text{subject to} \quad & g_i(x, \bar{z}) \le 0 \quad i = 1, \ldots, m \\ & \bar{z}_i \in [0, 1], \quad i = 1, \ldots, p \end{aligned}$$

- Key point: if the optimal solution $\bar{z}^\star$ to the relaxtion *is* integer valued, then it is an optimal solution to the integer program

- Furthermore, all solutions to relaxed problem provide *lower bounds* on optimal objective

$$f(x^\star, \bar{z}^\star) \le f(x^\star, z^\star)$$

Simple branch and bound algorithm

- Idea of approach
    1. Solve relaxed problem
    2. If there are variables $\bar{z}_i^\star$ with non-integral solutions, pick one of the variables and recursively solve each relaxtion with $\bar{z}_i = 0$ and $\bar{z}_i = 1$
    3. Stop when a solution is integral
- By using best-first search (based upon lower bound given by relaxation), we potentially need to search many fewer possibilities than for enumeration

```
function (f, x*, z̄*, C) = Solve-Relaxtion(C)
    // solves relaxation plus constraints in C

q ← Priority-Queue()
q.push(Solve-Relaxtion({}))
while(q not empty):
    (f, x*, z̄*, C) ← q.pop()
    if z̄* integral:
        return (f, x*, z̄*, C)
    else:
        Choose i such that z̄_i non-integral
        q.push(Solve-Relaxtion(C ⋃ {z̄_i = 0}))
        q.push(Solve-Relaxtion(C ⋃ {z̄_i = 1}))
```

- A common modification: in addition to maintaining lower bound from relaxation, maintain an upper bound on optimal objective

- Common method for computing upper bound: round entries in $\bar{z}_i$ to nearest integer, and solve optimization problem with this fixed $\bar{z}$

- (May not produce a feasible solution)

```
function (f, x*, z̄*, C) = Solve-Relaxtion(C)
    // solves relaxation plus constraints in C

q ← Priority-Queue()
q2 ← Priority-Queue()
q.push(Solve-Relaxtion({}))
while(q not empty):
    (f, x*, z̄*, C) ← q.pop()
    q2.push(Solve-Relaxation({z̄ = round(z̄*)}))
    if q2.first() − f < ϵ:
        return q2.pop()
    else:
        Choose i such that z̄_i non-integral
        q.push(Solve-Relaxtion(C ⋃ {z̄_i = 0}))
        q.push(Solve-Relaxtion(C ⋃ {z̄_i = 1}))
```

Simple example (from Boyd and Mattingley)

$$
\begin{aligned}
\underset{z}{\text{minimize}} \quad & 2z_1 + z_2 - 2z_3 \\
\text{subject to} \quad & 0.7z_1 + 0.5z_2 + z3 \geq 1.8 \\
& z_i \in \{0, 1\}, \quad i = 1, 2, 3
\end{aligned}
$$

Search tree

Queue

Simple example (from Boyd and Mattingley)

$$
\begin{array}{ll}
\underset{z}{\text{minimize}} & 2z_1 + z_2 - 2z_3 \\
\text{subject to} & 0.7z_1 + 0.5z_2 + z3 \geq 1.8 \\
 & z_i \in [0,1], \quad i = 1,2,3
\end{array}
$$

Search tree

$$\{\}$$

Queue

$$(-0.143, [0.43, 1, 1], \{\})$$

Simple example (from Boyd and Mattingley)

$$
\begin{array}{ll}
\underset{z}{\text{minimize}} & 2z_1 + z_2 - 2z_3 \\
\text{subject to} & 0.7z_1 + 0.5z_2 + z3 \geq 1.8 \\
& z_i \in [0,1], \quad i = 1,2,3
\end{array}
$$

Search tree

$\{\}$

$z_1 = 0$ $\qquad$ $z_1 = 1$

Queue

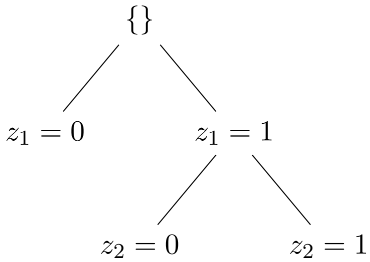
$(0.2, [1, 0.2, 1], \{z_1 = 1\})$

$(\infty, -, \{z_1 = 0\})$

Simple example (from Boyd and Mattingley)

$$
\begin{aligned}
\underset{z}{\text{minimize}} \quad & 2z_1 + z_2 - 2z_3 \\
\text{subject to} \quad & 0.7z_1 + 0.5z_2 + z3 \geq 1.8 \\
& z_i \in [0,1], \quad i = 1,2,3
\end{aligned}
$$

Search tree



Queue

$(1, [1,1,1], \{z_1 = 1, z_2 = 1\})$
$(\infty, -, \{z_1 = 0\})$
$(\infty, -, \{z_1 = 1, z_2 = 0\})$

Sudoku revisited

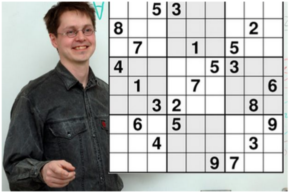
- The hard part with Sudoku is finding puzzles where the initial linear programming relaxation is not already tight

News · Weird News

World's hardest sudoku: Can you solve Dr Arto Inkala's puzzle?

Aug 19, 2010 00:00 | By **Mirror.co.uk** | 0 **Comments**

Could this be the toughest sudoku puzzle ever devised?

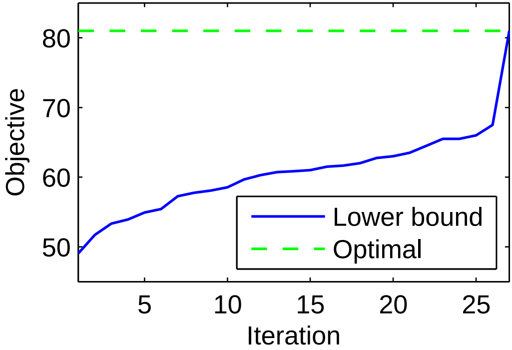


World's Hardest Sudoku

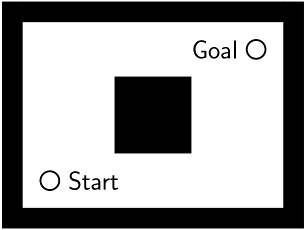
Could this be the toughest sudoku puzzle ever devised?

- Branch and bound solves this problem after 27 steps

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i,j=1}^{9} \max_k (z_{i,j})_k \\
\text{subject to} \quad & z \in \textsf{Valid-Sudoku} \\
& z_{i,j} \in \{0,1\}^9, \quad i,j = 1,\ldots,9
\end{aligned}
$$



Path planning with obstacles



- Final optimization problem

$$
\begin{array}{ll}
\underset{x,z}{\text{minimize}} & \displaystyle\sum_{i=1}^{m-1} \|x_{i+1} - x_i\|_2^2 \\
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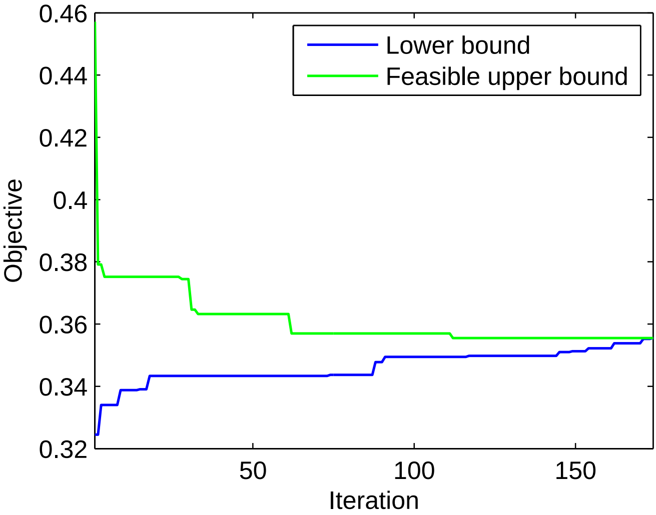
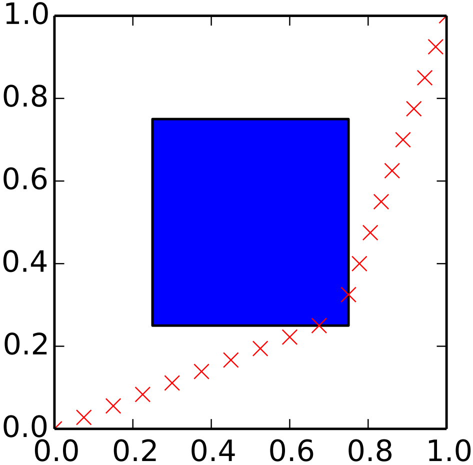
(x_i)_1 \le a_1 + z_{i1} M \\
(x_i)_1 \ge b_1 - z_{i2} M \\
(x_i)_2 \le a_2 + z_{i3} M \\
(x_i)_2 \ge b_2 - z_{i4} M \\
z_{i1} + z_{i2} + z_{i3} + z_{i4} \le 3
\end{array}\right\} i = 1, \ldots, m
\end{array}
$$



Lower bound
Feasible upper bound
Objective
Iteration
0.46
0.44
0.42
0.4
0.38
0.36
0.34
0.32
50
100
150


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

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Extensions to MIP

- How to incorporate actual integer (instead of just binary) constraints?
  - When solution is non-integral, split after adding constraints

$$\{\bar{z}_i \leq \mathrm{floor}(\bar{z}_i^\star)\}, \ \ \{\bar{z}_i \geq \mathrm{ceil}(\bar{z}_i^\star)\}$$

- More advanced splits, addition of "cuts" that rule out non-integer solutions (branch and cut)

- Solve convex problems more efficiently, many solvers can be sped up given a good initial point, and many previous solutions will be good initializations

Take home points

- Integer programs are a power subset of non-convex optimization problems that can solve many problems of interest

- Combining search and numerical optimization techniques, we get an algorithm that solve many problems much more efficiently than the "brute force" approach

- Performance will still be exponential in the worst case, and problem dependent, but can be reasonable for many problems of interest